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EMBEDDINGS INTO CATEGORIES WITH FIXED POINTS IN REPRESENTATIONS

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PART A: FIXED POINTS AND EXACT COLIMITS

A,1 Two seemingly unconnected problems concerning functor-categories turn out to have closely related solutions. One, dealing with fixed points of representations of a category K, has been solved by Adámek and Reiterman [1]. A representation, i. e., a functor $F : K \to \mathrm{Set}$

– is non-trivial if $FX \neq \emptyset$ for some $X \in K$;

– is indecomposable if it is non-trivial and, whenever $F = F_1 \vee F_2$, then F_1 or F_2 is trivial;

– has the fixed point property if for each endomorphism $\tau : F \to F$ there exists an object X in K and a point $x \in FX$ with $\tau_X(x) = x$.

Problem 1. *Characterize categories whose all indecomposable representations have the fixed point property.*

This problem was inspired by V. Trnková as part of a broader program of extending set-theoretical properties to functors.

A,2 The solution of Problem 1 involves *quasi-filters* for parallel pairs of morphisms $f_1, f_2 : A \to B$ in K. A quasi-filter is an n-tuple of morphisms $\alpha_0, \ldots, \alpha_{n-1} : B \to C$ in K such that identities of the following form

$$\alpha_0 f_{i_0} = \alpha_1 f_{j_0}\,,$$
$$\alpha_1 f_{i_1} = \alpha_2 f_{j_1}\,,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{n-1} f_{n-1} = \alpha_0 f_{j_n - 1}$$

hold, where i_t, j_t are 1 or 2 and $\sum_{t=0}^{n-1} (i_t - j_t) = \pm 1$.

Definition [1]. A category K is called *quasi-filtered* if every parallel pair of morphisms has a quasi-filter and, given objects M, N, there exists an object X with $\hom(M, X) \neq \emptyset \neq \hom(N, X)$.

Let us recall that a category K is *indecomposable* (connected) if it is not a sum of non-void categories. Each category is a sum of its *components*, i.e. its maximal indecomposable subcategories. Furthermore, a category K is *filtered* if every parallel pair f_1, f_2 has a filter, i.e. a morphism α with $\alpha f_1 = \alpha f_2$, and for objects M, N there is X with $\hom(M, X) \neq \emptyset \neq \hom(N, X)$. Obviously, a category K has all components filtered iff it fulfils the first condition and a weakening of the second: given morphisms $f_1 : A \to M$, $f_2 : A \to N$, there exist morphisms $g_1 : M \to X$, $g_2 : N \to X$ with $g_1 f_1 = g_2 f_2$.

Theorem [1]. *The following conditions on a category K are equivalent*:

(i) *Each indecomposable representation of K has the fixed point property*;

(ii) *K has quasi-filtered components*;

(iii) *K satisfies*

(1) *for each pair of morphisms f_1, f_2 with a common domain there exist morphisms g_1, g_2 with $g_1 f_1 = g_2 f_2$,*

(2) *each parallel pair of morphisms has a quasi-filter.*

A,3 The second problem, solved by Isbell and Mitchell [4], concerns the category Ab^K of functors from a small category K to Ab, the category of Abelian groups. Each of these functors has a colimit, which gives rise to a functor colim: $Ab^K \to Ab$. If colim preserves finite limits, i.e. if finite limits commute with colimits, then colim is *exact*.

Problem 2. *Characterize small categories K for which* colim : $Ab^K \to Ab$ *is exact.*

While analogous problems concerning set-valued functors are rather easy, see [3], the above problem turned out to be very difficult. The solution is in terms of affinization aff K of a small category K : aff K is a category whose objects coincide with those of K. Morphisms from A to B are all formal combinations $\sum_{i=1}^{p} \lambda_i f_i$ of K-morphisms $f_i : A \to B$ such that λ_i are integers with $\sum_{i=1}^{p} \lambda_i = 1$. Composition is given by $(\sum \lambda_i f_i)(\sum \mu_j g_j) = \sum \lambda_i \mu_j (f_i g_j)$.

Theorem [4]. *The following conditions on a small category K are equivalent*:

(i) colim : $Ab^K \to Ab$ *is exact*;

(ii) aff K *has filtered components*;

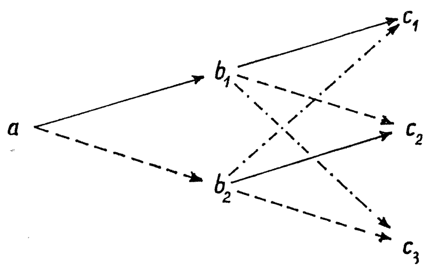
(iii) *K satisfies* (1) *and*

(3) *for every k-tuple of parallel morphisms* $f_1, \ldots, f_k$ *in* K ($k = 2, 3, 4, \ldots$) *there exists a morphism* ϱ *in* aff K *with* $\varrho f_1 = \varrho f_2 = \ldots = \varrho f_k$.

A,4 Denote by (3*) the above condition (3) restricted to $k = 2$. Then (1) + (3*) is easily seen to be equivalent to (1) + (2) (see [5] for a precise proof). Overlooking the distinctness of (3) and (3*), J. Adámek and J. Reiterman made in [1] a remark (insubstantial for their paper) that the solutions of Problem 1 and Problem 2 are the same. J. R. Isbell and B. Mitchell conjectured in [5] that this remark is false, and they asked for an example of a quasifiltered category K with non-filtered aff K. We shall present such an example in Part C.

A,5 Every filtered category K has filtered aff K. A counterexample to the converse implication was exhibited by J. R. Isbell and B. Mitchell: the categoty K_0 of finite ordinals and order preserving injections. They proved that aff K_0 is a filtered category. And K_0 is far from being filtered: it is a *mono-category*, i.e. a category in which each morphism is a mono, equivalently, no pair of disctinct morphisms has a filter.

When trying to find a quasi-filtered category such that aff K is not filtered, it is interesting to observe that the existence of such K implies the existence of a mono-category K^* with the same property. Indeed, define a congruence on $K : \alpha \sim \beta$ iff α, β are parallel and there is γ with $\gamma\alpha = \gamma\beta$. Then the quotient category $K^* = K/\sim$ is evidently a quasifiltered mono-category. It is rather easy to verify that aff K^* is not filtered. This observation led us to the investigation of quasi-filtered mono-categories. We started from a conjecture concerning the following concrete category Ω: It has three objects $A = \{a\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ and five



non-identical morphisms: the two from A to B and the following three $f, g, h : B \to C$:

	b_1	b_2
f	c_1	c_2
g	c_2	c_3
h	c_3	c_1

Conjecture. *The category Ω cannot be fully embedded into any mono-category K such that* aff K *has filtered components.*

Then we started to embed categories into quasi-filtered monocategories. We succeeded in embedding Ω but failed to prove the above conjecture. Therefore, in the present paper, we find a counterexample by a slightly different method: by combinatorial investigation of the embedding of the hom $(B, -)$-image of Ω (see Part C).

A,6 The present paper has three parts. The main theorem, characterizing small categories, embeddable into quasi-filtered mono-categories, is proved in Part B. The proof of necessity is rather easy, sufficiency is proved in several steps:

I. We observe that we can work with subcategories of $S(1-1)$ (the category of sets and one-to-ne maps) with a certain property (Property (4) below).

II. We study a general pair of one-to-one maps $f_1, f_2 : X \to Y$ with property (4). It turns out that the only important case is that of $X = \overline{n-1}$, $Y = \bar{n}$ and $(f_1, f_2) = (\Phi_n, \Psi_n)$ with the following convention:

Convention. *For every natural number n put $\bar{n} = \{0, 1, \ldots, n-1\}$ and define $\Phi_n, \Psi_n : \overline{n-1} \to \bar{n}$ by $\Phi_n(x) = x + 1$ and $\Psi_n(x) = x$.*

III. We find standard quasi-filters for the pairs Φ_n, Ψ_n. Using these, we construct a quasi-filter for general f_1, f_2 with nice properties.

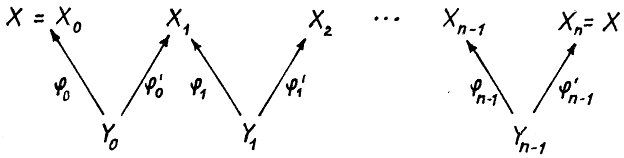
IV. We show that adding formally a "nice" quasi-filter to a parallel pair in K does not spoil Property (4). We show that, analogously, adding nice maps g_1, g_2 for a given f_1, f_2 as in condition (1) does not spoil Property (4).

V. We use IV. sufficiently many times to construct, from a given small category K_n with Property (4), a new category K_{n+1} with Property (4) such that a) each parallel pair in K_n has a quasifilter in K_{n+1} and b) each pair f_1, f_2 in K_n with a common domain has morphisms g_1, g_2 in K_{n+1} with $g_1 f_1 = g_2 f_2$.

Starting with a category $K = K_0$ having Property (4), we obtain a category $K^* = \bigcup_{i=0}^{\infty} K_i$ with quasi-filtered components.

PART B: THE EMBEDDING THEOREM

B,1 Let K be a category. By a *morphism chain* on an object X we mean a sequence

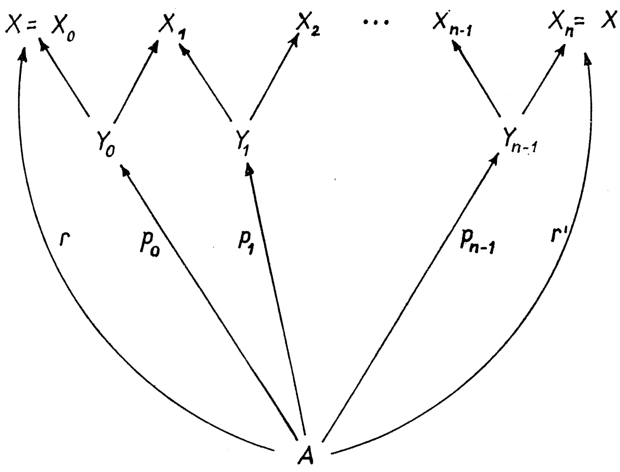


$\Delta = (\varphi_0, \varphi_0', \varphi_1, \varphi_1', \ldots, \varphi_{n-1}, \varphi_{n-1}')$ of morphisms such that range φ_0 = range

$\varphi'_{n-1} = X$ and, for any t, dom $\varphi_t =$ dom φ'_t, range $\varphi'_t =$ range φ_{t+1} (the indices are considered modulo n).

The most important concept of the present paper is the following.

Definition. Let A, X be objects of K and let Δ be a morphism chain on X, $\Delta = (\varphi_t, \varphi'_t)_{t=0}^{n-1}$ with $\varphi_t : Y_t \to X_t$ and $\varphi'_t : Y_t \to X_{t+1}$. We define a graph (a binary relation) $R_\Delta(A, X)$ on the set hom (A, X) as follows: a pair $r, r'(: A \to X)$ is in $R_\Delta(A, X)$ iff $r \neq r'$ and there exist morphisms $p_0 : A \to Y_0, \ldots, p_{n-1} : A \to Y_{n-1}$



such that $r = \varphi_0 p_0$, $r' = \varphi'_{n-1} p_{n-1}$ and, for each $t = 1, \ldots, n-1$, $\varphi_t p_t = \varphi'_{t-1} p_{t-1}$.

Note. An important case is $\Delta = (\varphi_0, \varphi'_0)$. Then $(r, r') \in R_\Delta(A, X)$ iff $r = \varphi_0 p$ and $r' = \varphi'_0 p$ for some $p : A \to X_0$.

We say that a graph R on a set V is *bounded* if there exists a natural number k such that any directed path in R has length $< k$. In other words, (V, R) is bounded iff

a) (V, R) contains no directed cycle;

b) there exists a natural number k such that, given pairwise distinct vertices $v_0, \ldots, v_n \in V$ with $(v_0, v_1) \in R, \ldots, (v_{n-1}, v_n) \in R$, then $n < k$.

Embedding Theorem. *A small mono-category K can be fully embedded into a quasi-filtered mono-category iff each of the relations $R_\Delta(A, X)$ in K is bounded.*

Note. We shall, in fact, prove the equivalence of the following conditions on a mono-category K:

(i) K can be fully embedded into a quasi-filtered mono-category;

(ii) K is isomorphic to a (not necessarily full) subcategory of a quasi-filtered category;
(iii) K is isomorphic to a subcategory of a category with quasi-filtered components;
(iv) Each $R_\Delta(A, X)$ is a bounded relation.

We first prove the necessity of the Embedding Theorem; indeed, we show (iii) → → (iv) then (i) → (ii) → (iii) → (iv) follows. Then we prove the sufficiency, i.e. (iv) → (i); this will prove the equivalence of (i)–(iv).

B,2 Proof of necessity. *We assume that K is a subcategory of a mono-category K, satisfying the conditions (1) and (2). Let A, X be objects and let* $\Delta = (\varphi_0, \varphi_0', \dots \dots, \varphi_s, \varphi_s')$ *be a morphism chain on X. We shall prove that* $R_\Delta(A, X)$ *is bounded by induction on s.*

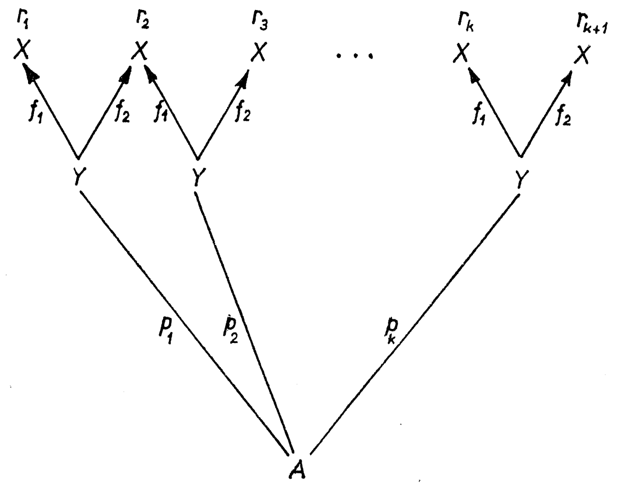
I. The initial step $s = 1$. We have $\Delta = (f_1, f_2)$ for some $f_1, f_2 : Y \to X$ in K. Let $\alpha_0, \dots, \alpha_{n-1} : X \to Z$ be a quasi-filter for f_1, f_2:

$$\alpha_t f_{i_t} = \alpha_{t+1} f_{j_t}, \quad t \in \bar{n},$$

with $\sum_{i=0}^{n-1} (i_t - j_t) = 1$ (or -1, which is irrelevant); we work with the t's modulo n. Put $h(t) = (i_0 - j_0) + (i_1 - j_1) + \dots + (i_{t-1} - j_{t-1})$; notice that $h(n) = 1$ and $h(0) = 0$ (the void sum).

To verify that $R_\Delta(A, X)$ is bounded, let $r_1, r_2, \dots, r_{k+1} : A \to X$ be morphisms with $(r_1, r_2), \dots, (r_k, r_{k+1}) \in R_\Delta(A, X)$; we shall prove that then $k < 2n$.

We have morphisms $p_1, \dots, p_k : A \to Y$ with $r_m = f_1 p_m$ and $r_{m+1} = f_2 p_m$ ($m =$



$= 1, 2, \dots, k$). We shall proceed by contradiction: assume $k \geqq 2n$. Then we can choose m with $n \leqq m \leqq k - n$. Let us prove by induction on $t = 0, \dots, n$ that

$$\alpha_t f_1 p_m = \alpha_0 f_1 p_{m+h(t)} \quad \text{for all} \quad m = t, \quad t + 1, \dots, k - t.$$

Recall that $\alpha_n = \alpha_0$. For $t = 0$, this is trivial. Assuming this for t, we prove it for $t + 1$.

a) Let $i_t = j_t = 1$; then $h(t+1) = h(t)$ and $\alpha_t f_1 = \alpha_{t+1} f_1$, hence $\alpha_{t+1} f_1 p_m = \alpha_t f_1 p_m = \alpha_0 f_1 p_{m+h(t)}$.

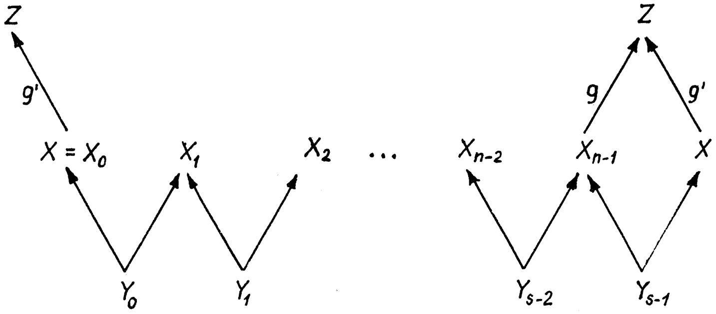
b) Let $i_t = j_t = 2$; then $h(t+1) = h(t)$ and $\alpha_t f_2 = \alpha_{t+1} f_2$, hence $\alpha_{t+1} f_1 p_m = \alpha_{t+1} f_2 p_{m-1} = \alpha_t f_2 p_{m-1} = \alpha_t f_1 p_m = \alpha_0 f_1 p_{m+h(t)}$.

c) Let $i_t = 1$, $j_t = 2$; then $h(t+1) = h(t) - 1$ and $\alpha_t f_1 = \alpha_{t+1} f_2$, hence $\alpha_{t+1} f_1 p_m = \alpha_{t+1} f_2 p_{m-1}$. (Here we use the fact that $m \geqq t + 1$ implies $m - 1 \geqq t$.)

d) Let $i_t = 2$, $j_t = 1$; then $h(t+1) = h(t) + 1$ and $\alpha_t f_2 = \alpha_{t+1} f_1$, hence $\alpha_{t+1} f_1 p_m = \alpha_t f_2 p_m$. (Here we use the fact that $m \leqq k - (t+1)$ implies $m + 1 \leqq k - t$.)

For $t = n$ we get $\alpha_0 f_1 p_m = \alpha_0 f_1 p_{m+1}$. Since $\alpha_0 f_1$ is a mono (indeed, K^* is a mono-category), this implies $p_m = p_{m+1}$. But this cannot occur because $r_m = f_1 p_m$ and $r_{m+1} = f_1 p_{m+1}$ and $(r_m, r_{m+1}) \in R_\Delta(A, X)$, hence, by the definition of $R_\Delta(A, X)$, $r_m \neq r_{m+1}$.

II. The inductive step. Let $\Delta = (\varphi_0, \varphi_0', \varphi_1, \varphi_1', \ldots, \varphi_{s-1}, \varphi_{s-1}')$ with $s > 1$. Use (1) to find g, g' in K with $g\varphi_{s-1} = g'\varphi_{s-1}'(g : X \to_{s-1} Z,\ g' : X \to Z)$.



Define a new morphism chain $\bar{\Delta} = (\psi_0, \psi_0', \ldots, \psi_{s-2}, \psi_{s-2}')$ on Z by

$$\psi_0 = g'\varphi_0\,; \quad \psi_i = \varphi_i \quad \text{for} \quad i = 1, \ldots, s-2\,,$$

$$\psi_i' = \varphi_i' \quad \text{for} \quad i = 0, \ldots, s-3\,; \quad \psi_{s-2}' = g\varphi_{s-2}'\,.$$

Then we have a graph homomorphism from $R_\Delta(A, X)$ into $R_{\bar{\Delta}}(A, Z)$ defined by $r \mapsto g'r$ for any $r : A \to X$. Indeed, given $(r, r') \in R_\Delta(A, X)$ then we have $p_i : A \to Y_i$ with

$$r = \varphi_0 p_0\,, \quad r' = \varphi_{s-1}' p_{s-1} \quad \text{and} \quad \varphi_i' p_i = \varphi_{i+1} p_{i+1}\,.$$

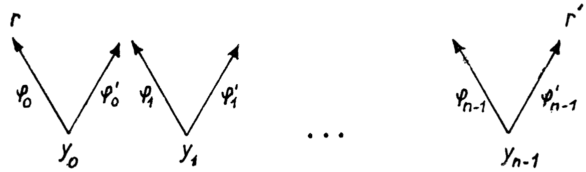
Using the same p_i we see that $(g'r, g'r') \in R_{\bar{\Delta}}(A, Z)$, because

$$g'r = g'\varphi_0 p_0 = \psi_0 p_0\,,$$

$$g'r' = g'\varphi_{s-1}' p_{s-1} = g\varphi_{s-1} p_{s-1} = g\varphi_{s-2}' p_{s-2} = \psi_{s-2}' p_{s-2}\,.$$

This homomorphism is injective because g' is mono (in K^*). Hence, $R_\Delta(A, X)$ is a subgraph of $R_{\bar{\Delta}}(A, Z)$. The latter graph is bounded by the inductive hypothesis. It easily follows that also $R_\Delta(A, X)$ is bounded.

B,3 Now we start proving the sufficiency. First, for concrete categories K we can work with simpler relations than $R_\Delta(A, X)$, not involving A. Let $\Delta = (\varphi_i, \varphi_i')_{i=0}^{n-1}$ be a morphism chain on an object X of a concrete category K, $\varphi_i : Y_i \to X_i$ and $\varphi_i' : Y_i \to X_{i+1}$. Define a *relation* R_Δ on the set X: a pair $(r, r') \in X \times X$ belongs



to R_Δ iff there exist points $y_i \in Y_i$ such that $r = \varphi_0(y_0)$, $r' = \varphi_{n-1}'(y_{n-1})$ and for each $i = 1, \ldots, n-1$, $\varphi_i(y_i) = \varphi_{i-1}'(y_{i-1})$. Recall that $S(1-1)$ denotes the category of sets and one-to-one maps.

Lemma. *The following conditions on a small category K are equivalent*:

(i) *K is a mono-category such that each $R_\Delta(A, X)$ is bounded*;

(ii) *K is isomorphic to a (not necessarily full) subcategory of $S(1-1)$ such that*

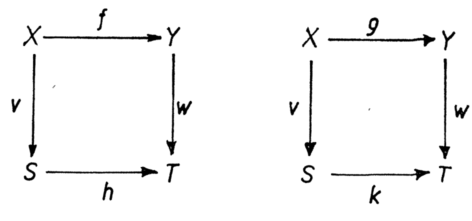
(4) *for every morphism chain Δ on an object X, (X, R_Δ) is bounded.*

Proof. Let $U : K \to \text{Set}$ be the sum of all hom-functors,

$$U = \coprod_{A \in K} \text{hom}(A, (-)).$$

Then U is a one-to-one functor, thus $U(K)$ is a subcategory of Set, isomorphic to K. If (i) holds then $U(K)$ is a subcategory of $S(1-1)$ and (1) and (4) are satisfied. If (ii) holds then clearly (i) does.

B,4 Convention. A pair $f, g : X \to Y$ of maps is said to be a *restriction* of a pair h, $k : S \to T$ if there exist one-to-one maps $v : X \to S$ and $w : Y \to T$ such that the squares below both commute.



The disjoint union of α copies of a set X is denoted by $X^{(\alpha)}$ (thus, $X = X \times I$

for an index set of power α); analogously $f^{(\alpha)} = f \times \mathrm{id}_I$ for a mapping f. Recall Φ_n, Ψ_n in A,6.

Lemma. *Let $f, g : X \to Y$ be a pair of one-to-one mappings such that the graph $R_{(f,g)}$ is bounded. Put $X_0 = \{x \in X;\ f(x) = g(x)\}$, $Y_0 = (Y - (f(X) \cup g(X))) \cup \cup \{f(x);\ x \in X_0\}$, and denote by $f', g' : (X - X_0) \to (Y - Y_0)$ the domain-range restrictions of f, g. Then there exists a natural number n and a cardinal α such that the pair f', g' is a restriction of $\Phi_n^{(\alpha)}, \Psi_n^{(\alpha)} : \overline{n-1}^{(\alpha)} \to \bar{n}^{(\alpha)}$.*

Note. The above number n can be chosen arbitrarily big because, for $m > n$, we see that the pair Φ_n, Ψ_n is a restriction of the pair Φ_m, Ψ_m.

Proof of the lemma. First, since f, g are one-to-one, clearly $f(X - X_0) \sim (Y - Y_0)$ (given $x \in X - X_0$ then $f(x) \in Y_0$ would imply $f(x) = f(x')$ for some $x' \in X_0$ but then $x = x'$ – a contradiction); also $g(X - X_0) \sim (Y - Y_0)$. Hence the restrictions f', g' are correct.

Second, recall that a pair $(a, b) \in Y \times Y$ is an edge in $R_{(f,g)}$ iff there exists $x \in X$ with $a = f(x)$, $b = g(x)$ and $a \neq b$. In other words, edges are exactly the pairs $(f(x), g(x))$ with $x \in X - X_0$. Given two distinct edges (a, b) and (a', b') in $R_{(f,g)}$ then $a \neq a'$ and $b \neq b'$; in other words, if two distinct edges meet then one starts at the end of the other. (Proof: we have $x, x' \in X_0$ with $a = f(x)$, $b = g(x)$ and $a' = f(x')$, $b = g(x')$. Now $a = a'$ implies $x = x'$ since f is one-to-one, and $b = b'$ implies $x = x'$ since g is one-to-one. It follows that $R_{(f,g)}$ is a disjoint union of paths and cycles. Now, $R_{(f,g)}$ is bounded, therefore there are no cycles and there exists n such that all paths in $R_{(f,g)}$ have length smaller than n. We consider an isolated point as a path of length 0. Clearly, all points in Y_0 are isolated.

Thus, there exists a decomposition

$$Y - Y_0 = \bigcup_{i \in I} Y_i$$

such that

a) for every edge $(a, b) \in R_{(f,g)}$ there exists $i \in I$ with $a, b \in Y_i$;

b) the restriction of $R_{(f,g)}$ to Y_i is a path of length $k_i < n$ $(i \in I)$.

Hence, we can write $Y_i = \{y_0^i, y_1^i, \ldots, y_{k_i}^i\}$ where $(y_{t+1}^i, y_t^i) \in R_{(f,g)}$ for all $t \in \bar{k}_i$. Therefore, there exist $x_i^t \in X - X_0$ $(i \in I,\ t \in \bar{k}_i)$ with

$$(*) \qquad f(x_t^i) = y_{t+1}^i \quad \text{and} \quad g(x_t^i) = y_t^i\,.$$

Notice that $X - X_0 = \{x_t^i\}_{i \in I, t \in \bar{k}_i}$ (because $x \in X - X_0$ implies $f(x) \neq g(x)$, hence $(f(x), g(x)) \in R_{(f,g)}$) and $x_t^i \neq x_t^{i'}$ whenever $i \neq i'$ or $t \neq t'$. Also $Y - Y_0 = \{y_t^i\}_{i \in I, t \in \overline{k_i + 1}}$. We remark that the isolated points in $Y - Y_0$, i.e. the points $y \in Y - (f(X) \cup g(X))$ have the form $y = y_0^i$ for some i with $Y_i = \{y_0^i\}$ and for such i there is no x_t^i, of course.

Put $\alpha = \operatorname{card} I$; we shall show that the pair f', g' is a restriction of $\Phi_n \times \mathrm{id}_I$, $\Psi_n \times \mathrm{id}_I : \overline{n-1} \times I \to \bar{n} \times I$, i.e. a restriction of $\Phi_n^{(\alpha)}, \Psi_n^{(\alpha)}$. Define $v : X \to \to \overline{n-1} \times I$ and $w : Y \to \bar{n} \times I$ by

$$v(x_t^i) = (t, i) \quad \text{and} \quad w(y_t^i) = (t, i)\,.$$

Then v, w are one-to-one mappings. By $(*)$ we have $wf' = \Phi_n^{(\alpha)} v$ and $wg' = \Psi_n^{(\alpha)} v$.

B,5 Construction. For every natural number n and for every number $s = 4k + 1$, where $k \geqq n$, we shall construct a quasi-filter

$$\beta_0, \ldots, \beta_{s-1} : \bar{n} \to \bar{s}$$

for $\Phi_n, \Psi_n : \overline{n-1} \to \bar{n}$ in the category $S(1-1)$.

The maps $\beta_0, \ldots, \beta_{2k}$ are defined by

$$\beta_i(x) = x + i \quad \text{if} \quad x + i \leqq 2k\,, \tag{5}$$
$$\beta_i(x) = x + i + (2k - n + 1) \quad \text{if} \quad x + i > 2k\,;$$

the maps $\beta_{2k+1}, \ldots, \beta_{4k}$ are defined by

$$\beta_i(x) = x - i + 6k - n + 2 \quad \text{if} \quad x - i + 6k - n + 2 > 2k\,, \tag{6}$$
$$\beta_i(x) = x - i + 4k \quad \text{if} \quad x - i + 6k - n + 2 \leqq 2k\,.$$

In other words, the β_i's are described by means of n-tuples $(\beta_i(0), \ldots, \beta_i(n-1))$ as follows:

(7)

a) β_0 $(0, \ldots, n - 1)$
β_1 $(1, \ldots, n)$
...
β_{2k-n+1} $(2k - n + 1, \ldots, 2k)$

b) β_{2k-n+2} $(2k - n + 2. \ldots, 2k,\ 4k - n + 2)$
β_{2k-n+3} $(2k - n + 3, \ldots, 2k,\ 4k - n + 2,\ 4k - n + 3)$
...
β_{2k} $(2k,\ 4k - n + 2, \ldots, 4k)$

c) β_{2k+1} $(4k - n + 1,\ 4k - n + 2, \ldots, 4k)$
β_{2k+2} $(4k - n, \ldots, 4k - 1)$
...
β_{4k-n+1} $(2k + 1, \ldots, 2k + n)$

d) β_{4k-n+2} $(n - 2,\ 2k + 1, \ldots, 2k + n - 1)$
β_{4-n+3} $(n - 3,\ n - 2,\ 2k + 1, \ldots, 2k + n - 2)$
...
β_{4k} $(0, \ldots, n - 2,\ 2k + 1)\,.$

It is a purely routine process to check that

$$\begin{aligned}
\beta_i\Phi_n &= \beta_{i+1}\Psi_n \quad \text{for} \quad i = 0, \dots, 2k - 1\,,\\
\beta_{2k}\Phi_n &= \beta_{2k+1}\Phi_n\,,\\
\beta_i\Psi_n &= \beta_{i+1}\Phi_n \quad \text{for} \quad i = 2k + 1, \dots, 4k - 1\,,\\
\beta_{4k}\Psi_n &= \beta_0\Psi_n\,.
\end{aligned}$$

Thus, $\beta_0, \dots, \beta_{4k}$ is a quasi-filter for Φ_n, Ψ_n.

Lemma. *If $\beta_i(x) = \beta_j(y)$ (for $i, j \in \bar{s}$, $i < j$ and $x, y \in \bar{n}$) then the number $x - y$ is determined by i, j, n and k. More in detail, $\beta_i(x) = \beta_j(y)$ implies just one of the following identities*:

a) $x - y = j - i$ *in the case* $i, j \leqq 2k$,

b) $x - y = i - j$ *in the case* $i, j > 2k$,

c) $x - y = 4k - i + j$ *in the case* $i < n - 1$, $j > 4k - n + 1$,

d) $x - y = 4k + 1 - i + j$ *in the case* $2k - n + 1 < i \leqq 2k$, $2k < j < 2k + n$.

Proof. We have three possibilities.

I. $i, j \leqq 2k$. If the numbers $x + i$, $y + j$ are both smaller or equal to $2k$ or both bigger than $2k$, clearly the case a) takes place. The remaining possibilities cannot occur. E.g. if $x + i \leqq 2k$ but $y + j > 2k$ then $\beta_i(x) = \beta_j(y)$ yields $2k \geqq x + i = y + j + 2k - n + 1 > 4k - n + 1$, a contradiction because $k \geqq n$.

II. $i, j > 2k$. If the numbers $x - i + 6k - n + 2$, $y - j + 6k - n + 2$ are both smaller or equal to $2k$ or both bigger than $2k$ then clearly the case b) takes place. Again, the remaining possibilities cannot occur. E.g., if $x - i + 6k - n + 2 \leqq 2k$ but $y - j + 6k - n + 2 > 2k$ then $\beta_i(x) = x - i + 4k < n - 2 < 2k < y - j + 6k - n + 2 = \beta_j(y)$.

III. $i \leqq 2k$, $j > 2k$. There are four subcases, two of which turn out to be impossible.

α) $x + i \leqq 2k$, $y - j + 6k - n + 2 > 2k$. This is impossible because $\beta_i(x) = x + i \leqq 2k < y - j + 6k - n + 2 = \beta_j(y)$.

β) $x + i \leqq 2k$, $y - j + 6k - n + 2 \leqq 2k$. Then $j \geqq y + 4k - n + 2 > 4k - n + 1$ and also $y - j + 4k \leqq n - 2$. Further, $\beta_i(x) = \beta_j(y)$ yields $x + i = y - j + 4k$. It follows that $x - y = 4k - (i + j)$ and $i \leqq x + i \leqq n - 2 < n - 1$.

γ) $x + i > 2k$, $y - j + 6k - n + 2 > 2k$. Then $i > 2k - x \geqq 2k - n + 1$ and $j < y + 4k - n + 2$; $\beta_i(x) = \beta_j(y)$ yields $x + i + 2k - n + 1 = y - j + 6k - n + 2$, i.e. $x - y = 4k + 1 - i + j$. It follows also $j = 4k + 1 - x + i + y < 4k + 1 - 2k + n - 1 = 2k + n$.

δ) $x + i > 2k$, $y - j + 6k - n + 2 \leqq 2k$. This is impossible because $\beta_i(x) = x + i > 2k > y - j + 4k = \beta_j(y)$.

B,6 Conventions. For a mapping $h : X \to Y$ we denote by $\hat{h}$ the set

$$\hat{h} = \{h(x);\ x \in X\}\ .$$

A path in a graph of length $-n$ $(n = 1, 2, 3, \ldots)$ is simply a path of length n in the graph with the opposite orientation of edges.

Construction. For one-to-one mappings $f, g : X \to Y$ such that $R_{(f,g)}$ is bounded we shall construct a quasi-filter with special properties.

As in B,4, consider $f', g' : (X - X_0) \to (Y - Y_0)$, where $X_0 = \{x \in X;\ f(x) = g(x)\}$, $Y_0 = Y_0^* \cup Y_0^e$, where $Y_0^* = Y - (\hat{f} \cup \hat{g})$, $Y_0^e = f(X_0)$. By B,4, we have one-to-one mappings $v : (X - X_0) \to \overline{n-1} \times I$, $w : (Y - Y_0) \to \bar{n} \times I$ (for some n, I) such that

$$wf' = (\Phi_n \times \mathrm{id}_I)\, v \quad \text{and} \quad wg' = (\Psi_n \times \mathrm{id}_I)\, v\ .$$

By B,5 there is a quasi-filter $\beta_i : \bar{n} \to \bar{s}$ $(i \in \bar{s},\ s = 4k + 1,\ k \geqq n)$ for Φ_n, Ψ_n with the property from Lemma B,5; we have a quasi-filter

$$\beta_i^{(\alpha)} = \beta_i \times \mathrm{id}_I : \bar{n} \times I \to \bar{s} \times I \quad \text{for} \quad \Phi_n^{(\alpha)},\ \Psi_n^{(\alpha)} \quad \text{where} \quad \alpha = \operatorname{card} I\ .$$

Put

$$Z = Y_0^e \cup (Y_0^* \times \bar{s}) \cup (Y - Y_0) \times I$$

(the three sets, union of which is Z, are assumed to be disjoint). Define mappings $\alpha_i : Y \to Z$ $(i \in \bar{s})$ by

$$\begin{aligned} \alpha_i(y) &= y && \text{for} \quad y \in Y_0^e\ , \\ &= (y, i) && \text{for} \quad y \in Y_0^*\ , \\ &= \beta_i^{(\alpha)}\, w(y) && \text{for} \quad y \in Y - Y_0\ . \end{aligned}$$

Then clearly $\alpha_0, \ldots, \alpha_{s-1} : Y \to Z$ is a quasi-filter for f, g. It has the following properties.

(9) Given $i, j \in \bar{s}$, there exists an integer z_{ij} such that arbitrary points $y, y' \in Y - Y_0^e$ with $\alpha_i(y) = \alpha_j(y')$ can be connected by a path of length z_{ij} in the graph $R_{(f,g)}$.

(10) Given $m \in \bar{s}$ then

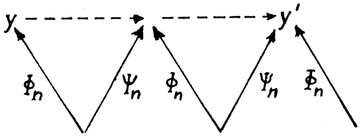
$$\begin{aligned} \hat{\alpha}_m \cap \hat{\alpha}_j &\subset \hat{\alpha}_m \cap \hat{\alpha}_{m+1} && \text{for every} \quad j = m + 1,\ m + 2, \ldots, m + (s - 2n)\ , \\ \hat{\alpha}_j \cap \hat{\alpha}_m &\subset \hat{\alpha}_{m-1} \cap \hat{\alpha}_m && \text{for every} \quad j = m - 1,\ m - 2, \ldots, m - (s - 2n)\ , \end{aligned}$$

where the addition of indices is mod s.

(11) Given $i \in \bar{s}$ and $f_1, f_2 \in \{f, g\}$ such that $\alpha_i f_1 = \alpha_{i+1} f_2$, then $\hat{\alpha}_i \cap \widehat{\alpha_{i+1}} = \widehat{\alpha_i f_i}$. ($i + 1$ means $i + 1$ mod s.)

Proof of properties (9–11). It suffices to show that the quasi-filter $\{\beta_i\}$ for Φ_n, Ψ_n has these properties. Then clearly so does the quasi-filter $\{\beta_i^{(\alpha)}\}$ for $\Phi_n^{(\alpha)}$, $\Psi_n^{(\alpha)}$ and so does any domain-range restriction of $\{\beta_i^{(\alpha)}\}$ as a quasi-filter for any domain-range restriction of $\Phi_n^{(\alpha)}$, $\Psi_n^{(\alpha)}$. And the above constructed quasi-filter differs from a restriction of $\{\beta_i^{(\alpha)}\}$ only at the points of Y_0. Now, the points on Y_0^e do not spoil the properties (9–11): in (9) they are excluded and for (10), (11), we remark that the sets $\alpha_j(Y_0^e)$ $(j \in \bar{s})$ coincide. The points of Y_0^* play no role in (9) and for (10), (11) we remark that the sets $\alpha_j(Y_0^*)$ $(j \in \bar{s})$ are pairwise disjoint.

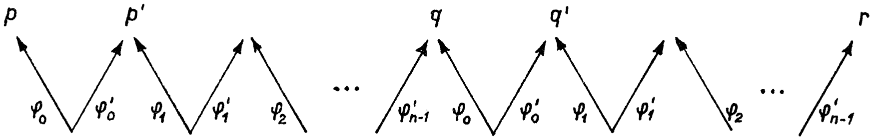
For $\{\beta_i\}$, the property (9) immediately follows from Lemma B,5 and from the observation that, given $y, y' \in \bar{n}$, then there leads exactly one path from y to y' (or, from y' to y) in $R_{(\Phi_n,\Psi_n)}$, the length of which equals $y - y'$. Properties (10), (11) are easy to check by inspection of (7).



B,7 Note. Let $\Delta = (\varphi_0, \varphi_0', \varphi_1, \varphi_1', \ldots, \varphi_{n-1}, \varphi_{n-1}')$ be a morphism chain in the category $S(1-1)$. Then R_Δ is a bounded graph iff R_{Δ_i} is, where $\Delta_i = (\varphi_i, \varphi_i', \varphi_{i+1}, \varphi_{i+1}', \ldots, \varphi_{n-1}, \varphi_{n-1}', \varphi_0, \varphi_0', \ldots, \varphi_{i-1}, \varphi_{i-1}')$ (for any $i = 0, \ldots, n-1$). Indeed, it suffices to show that if R_{Δ_0} is bounded then so is R_{Δ_1}.

This follows from the fact that, given subsequent edges (p, q), (q, r) in R_{Δ_0}, we obtain an edge $(p', q') \in R_{\Delta_1}$ such that

$$p' = \varphi_1' \varphi_1^{-1}(p), \quad q' = \varphi_1' \varphi_1^{-1}(q).$$



Thus, a path of length 2 in R_{Δ_0} induces a path of length 1 in R_{Δ_1}. In the same way, a path of length n in R_{Δ_0} induces a path of length $n-1$ in R_{Δ_1}.

Note. Let $f, g : A \to B$ be one-to-one mappings, let $b \in B_0^e$, where $B_0^e = \{f(a);$ $a \in A,\ f(a) = g(a)\}$. Given a quasi-filter $\alpha_0, \ldots, \alpha_{k-1} : B \to C$ in $S(1-1)$ for f, g then

$$\alpha_0(b) = \alpha_1(b) = \ldots = \alpha_{k-1}(b).$$

Indeed, for each $i \in \bar{k}$ we have $f_1, f_2 \in \{f, g\}$ with $\alpha_i f_1 = \alpha_{i+1} f_2$. Given $a \in A$ with $f(a) = b$, we have $f_1(a) = f_2(a) = b$, hence $\alpha_i(b) = \alpha_{i+1}(b)$.

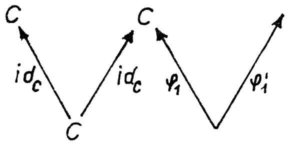
Convention. Given a collection of one-to-one mappings we speak about the category they generate as the least subcategory of $S(1-1)$, containing all these mappings.

Lemma. *Let K be a small subcategory of* $S(1-1)$, *satisfying* (4). *Let* $f, g : A \to B$

be its morphisms. Given a quasi-filter $\alpha_0, \ldots, \alpha_{k-1} : B \to C$ *for* f_1, f_2 *in* $S(1-1)$ *with property* (9) *and such that* $C \notin K$, *then also the category, generated by* $K \cup \cup \{\alpha_0, \ldots, \alpha_{k-1}\}$, *has property* (4).

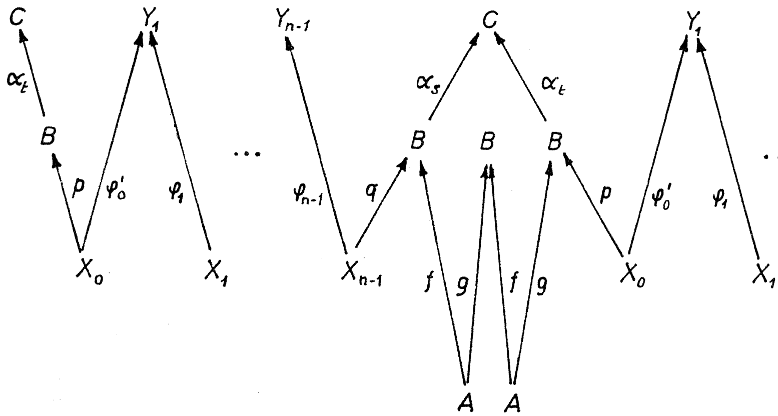
Proof. Denote by L the category generated by $K \cup \{\alpha_t\}_{t \in \bar{k}}$. Its morphisms are those of K, id_C and mappings $\alpha_t p : X \to C$ where $t \in \bar{k}$ and $p : X \to B$ is a morphism of K. Let Δ be a morphism chain in L, $\Delta = (\varphi_0, \varphi'_0, \ldots, \varphi_{n-1}, \varphi'_{n-1})$ with $\varphi_i : X_i \to Y_i$; denote by $b(\Delta)$ the number of those $i = 0, \ldots, n-1$ for which $Y_i = C$. We shall prove that R_Δ is bounded by induction on $b(\Delta)$. If $b(\Delta) = 0$ then Δ lies in K and K has property (4), thus R_Δ is bounded.

Let $b(\Delta) > 0$. By the above note we can assume that $Y_0 = C$. Then either $\varphi_0 = = \mathrm{id}_C$ [which is a trivial case: since $\mathrm{dom}\, \varphi_0 = C = \mathrm{dom}\, \varphi'_0$ we have also $\varphi'_0 = \mathrm{id}_C$ and the chain $\Delta' = (\varphi_1, \varphi'_1, \ldots, \varphi_{n-1}, \varphi'_{n-1})$ has the property that $R_{\Delta'} = R_\Delta$ and



$b(\Delta') < b(\Delta)$] or $\varphi_0 = \alpha_t p$, $t \in \bar{k}$, $p : X_0 \to B$ in K.

In the latter case we have range $\varphi_0 = C =$ range φ'_{n-1}. Then either $\varphi_{n-1} = \mathrm{id}_C$ [a trivial case] or $\varphi'_{n-1} = \alpha_s\, q$, where $s \in \bar{k}$, $q : X_0 \to B$ in K.



Now we use condition (9) for α_s, α_t. Assume e.g. $z_{st} \geqq 0$ and define a new chain Δ' in L: $\Delta' = (p, \varphi'_0, \varphi_1, \varphi'_1, \ldots, \varphi_{n-2}, \varphi'_{n-2}, \varphi_{n-1}, q, f, g, f, g, \ldots, f, g)$ with f, g repeated z_{st}-times. (For $z_{st} < 0$ we would repeat g, f instead of f, g.) Clearly, $b(\Delta') < < b(\Delta)$, and so $R_{\Delta'}$ is bounded.

To prove that also R_Δ is bounded, it clearly suffices to show that, given (z_1, z_2), $(z_2, z_3) \in R_\Delta$, there exists $(z'_1, z'_2) \in R_{\Delta'}$ such that $\alpha_t(z'_1) = z_1$ and $\alpha_t(z'_2) = z_2$.

Since α_t is one-to-one, it follows that each path of length m in R_Δ induces a path of length $m-1$ in $R_{\Delta'}$; hence, if $R_{\Delta'}$ is bounded, so is R_Δ. We have $x_i \in X_i$ with

$$z_1 = \varphi_0(x_0)\,; \quad z_2 = \varphi_{n-1}(x_{n-1}) \quad \text{and} \quad \varphi_i'(x_i) = \varphi_{i+1}(x_{i+1})\,;$$

further we have $\bar{x}_i \in X_i$ with

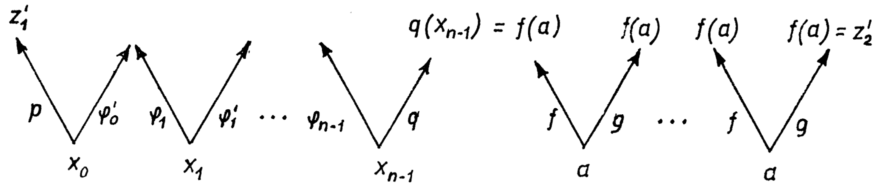
$$z_2 = \varphi_0(\bar{x}_0)\,, \quad z_3 = \varphi_{n-1}'(\bar{x}_{n-1}) \quad \text{and} \quad \varphi_i'(\bar{x}_i) = \varphi_{i+1}(\bar{x}_{i+1})\,.$$

Put $z_1' = p(x_0)$ and $z_2' = p(\bar{x}_0)$. We see that $\alpha_t(z_1') = \alpha_t\, p(x_0) = \varphi_0(x_0) = z_1$ and $\alpha_t(z_2') = z_2$. Let us show that $(z_1', z_2') \in R_{\Delta'}$. Since $z_1 = \alpha_s\, q(x_{n-1}) = \alpha_t\, p(\bar{x}_0)$, we have two possibilities.

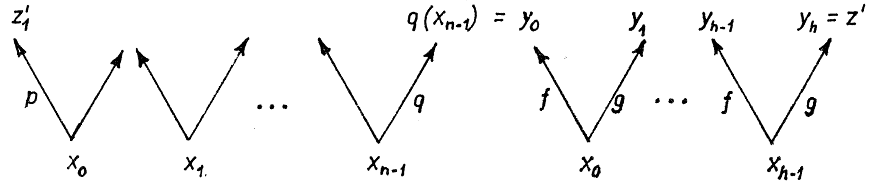
I. $q(x_{n-1}) \in B^e = \{f(a);\ a \in A,\ f(a) = g(a)\}$. By the above note,

$$\alpha_t\, q(x_{n-1}) = \alpha_s\, q(x_{n-1}) \quad [= \alpha_t\, p(\bar{x}_0)]\,.$$

Since α_t is one-to-one, $q(x_{n-1}) = p(\bar{x}_0)$ $[= f(a) = g(a)$ for some $a \in A]$. Then $(z_1', z_2') \in R_{\Delta'}$, as suggested by the following figure:



II. $q(x_{n-1}) \notin B_0^e$ hence $p(\bar{x}_0) \notin B^e$, because if $p(\bar{x}_0) \in B_0^e$ then $\alpha_s\, p(\bar{x}_0) = \alpha_t\, p(\bar{x}_0) = \alpha_s\, q(x_{n-1})$ would imply $q(x_{n-1}) = p(\bar{x}_0) \in B_0^e$. Then we can use condition (9): since $\alpha_s\, q(x_{n-1}) = \alpha_t\, p(\bar{x}_0)$, there exists a path of length z_{st} from $q(x_{n-1})$ to $p(\bar{x}_0) = z_2'$ in $R_{(f,g)}$, say $(y_0, \ldots, y_h)$ with $y_0 = q(x_{n-1})$, $y_h = z_2'$ and $(y_i, y_{i+1}) \in R_{(f,g)}$, i.e. $y_i = f(x_i^*)$, $y_{i+1} = g(x_i^*)$ for suitable $x_i^* \in A$. Then $(z_1', z_2') \in R_{\Delta'}$, as suggested by the following figure:



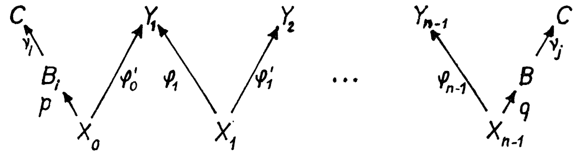
B,8 Lemma. *Let K be a small subcategory of $S(1-1)$, satisfying (4). Given objects B_1, B_2 in K, let $C = B_1 \vee B_2$ be their sum (disjoint union) with $C \notin K$; let $v_i : B_i \to C$, $i = 1, 2$, be the canonical injections. Then the category generated by $K \cup \{v_1, v_2\}$ has property (4) as well.*

Proof. Denote by L the category generated by $K \cup \{v_1, v_2\}$. Its morphisms are those of K, id_C and mappings $v_i p : X \to C$ where $i = 1, 2$, and $p : X \to B_i$ is a morphism of K. Let Δ be a morphism chain in L, $\Delta = (\varphi_0, \varphi'_0, \ldots, \varphi_{n-1}, \varphi'_{n-1})$ with $\varphi_i : X_i \to Y_i$. Denote by $b(\Delta)$ the number of those $i = 0, \ldots, n-1$ for which $Y_i = C$. We shall prove that R_Δ is bounded by induction in $b(\Delta)$. This is clear for $b(\Delta) = 0$. Let $b(\Delta) > 0$ and let e.g. $Y_0 = C$, i.e. C is the range of φ_0, φ'_{n-1}. The case that φ'_{n-1} is id_C is trivial. Thus we may assume that

$$\varphi_0 = v_i p\,, \quad \varphi'_{n-1} = v_j q\,, \quad \text{where} \quad i, j = 1 \text{ or } 2\,, \quad p, q \in K\,.$$

If $i \neq j$ then $\hat{\varphi}_0 \cap \hat{\varphi}'_{n-1} = \emptyset$, hence R_Δ contains no path of length 2, and so R_Δ is bounded. If $i = j$, put

$$\Delta' = (p, \varphi'_0, \varphi_1, \varphi'_1, \ldots, \varphi_{n-2}, \varphi'_{n-2}, \varphi_{n-1}, q)\,.$$



Then Δ' is a morphism chain with $b(\Delta') < b(\Delta)$, hence $R_{\Delta'}$ is bounded. It follows that R_Δ is bounded because for each edge $(z_1, z_2) \in R_\Delta$ there exists edge an $(z'_1, z'_2) \in$ $\in R_{\Delta'}$ with $v_i(z'_1) = z_1$ and $v_i(z'_2) = z_2$. Hence, each path of length m in R_Δ induces a path of length m in $R_{\Delta'}$.

B,9 The proof of sufficiency of B,1. By Lemma B,3 we are to prove that every small subcategory K of $S(1-1)$ with property (4) can be fully embedded into a quasi-filtered subcategory of $S(1-1)$. Put $K = K_0$ and define categories $K_1, K_2, K_3, \ldots$ by induction as follows.

Given K_m, construct K_{m+1} in two steps. First choose a well-ordering of the set of all parallel pairs of morphisms in K_m. We get a collection $\{(f^i, g^i);\ i < \gamma\}$ (γ an ordinal), $f^i, g^i : A^i \to B^i$ and for each of them we find a quasi-filter

$$\alpha^i_0, \ldots, \alpha^i_{n_i-1} : B^i \to C^i$$

in $S(1-1)$ which has the property (9) and such that $C^i \notin K_m \cup \{C^{i'}\}_{i'<i}$ (this is possible by Construction B,6). Using Lemma B,7 inductively (with $i < \gamma$) we see that the category L_m, generated by $K_m \cup \{\alpha^i_t;\ i < \gamma \text{ and } t \in \bar{n}_i\}$, has property (4) as well. For the second step choose a well-ordering of the set of all pairs of objects in K_m. We get a collection $\{(B^j_1, B^j_2);\ j < \delta\}$ (δ an ordinal). For each j choose a disjoint union $C^j = B^j_1 \vee B^j_2$ with canonical $v^j_t : B^j_t \to C^j$ $(t = 1, 2)$ so that $C^j \notin L_m \cup$ $\cup \{C^{j'}\}_{j'<j}$. Using Lemma B,8 inductively (with $j < \delta$) we see that the category K_{m+1}, generated by $L_m \cup \{v^j_t\}_{j<\delta, t=1,2}$, has property (4) as well. This category K_{m+1} contains K_m as a full subcategory and has the property that

a) each parallel pair of morphisms in K_m has a quasi-filter in K_{m+1},

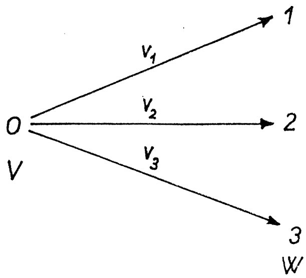
b) given objects B_1, B_2 in K_m, there exists an object C in K_{m+1} and morphisms from B_1 to C and from B_2 to C.

It follows that the category $K^* = \bigcup_{m=0}^{\infty} K_m$ is quasi-filtered. It is a subcategory of $S(1-1)$ and, on the other hand, K_0 is its full subcategory.

Example. The category Ω of A,5 is easily seen to have property (4).

PART C: A COUNTEREXAMPLE

C,1 Construction. Let $K = K_0$ be the concrete category with two objects $V = \{0\}$ and $W = \{1, 2, 3\}$ and with three non-identity morphisms $v_1, v_2, v_3 : V \to W$, $v_i(0) = i$.



We construct a category $K^* = \bigcup_{m=0}^{\infty} K_m$ as in B,9, only restricting somehow the quasi-filters used. Recall that, for each quasi-filter $\alpha_0, \ldots, \alpha_{s-1}$ for $f, g \in K^*$ added when passing from K_m to K_{m+1} we have $s = 4k + 1$, $k \geqq n$, where n is a natural number depending on f, g while k can be chosen arbitrarily. Thus, we shall suppose that $s/3n$ is an integer and

$$(*) \qquad \frac{s}{3n} \geqq \frac{1}{1 - \frac{a_{m+1}}{a_m}},$$

where $\{a_m\}_{m=0}^{\infty}$ is a strictly decreasing sequence of real numbers such that

$$a_0 = 2\,, \quad a_m > \frac{3}{2} \quad \text{and} \quad \frac{1}{1 - \frac{a_{m+1}}{a_m}} \text{ is a natural number for each } m\,.$$

Remember also properties (9), (10), (11).

C,2 Theorem. *The above category K^* is quasi-filtered but* aff K^* *has not filtered components.*

Proof. I. It suffices to prove that for arbitrary pairwise distinct K^*-morphisms $h_1, \ldots, h_r : W \to X$ we have

(a) $$\left|\bigcup_{i=1}^{r} \hat{h}_i\right| > \tfrac{3}{2}r\,.$$

Then there exists no aff K^*-morphism $\varrho : W \to X$ with $\varrho v_1 = \varrho v_2 = \varrho v_3$. Indeed, suppose that such ϱ exists. We can choose it in the form

(b) $$\varrho = (f_1 + \ldots + f_{t+1}) - (g_1 + \ldots + g_t)$$

with t the least possible number. Then clearly $f_i \neq g_j$ for all i, j (else we choose $\varrho' = \varrho - f_i + g_j$). Now, let $h_1, \ldots, h_r$ be the list of all distinct morphisms among the f_i's and g_j's. For each $x \in \bigcup_{i=1}^{r} \hat{h}_i$ there exist distinct $p, q \in \{1, \ldots, r\}$ with $x \in \hat{h}_p \cap \hat{h}_q$. (Proof: given $x \in \hat{h}_p$ with, say, $h_p = f_i$, there exists $z \in \{1, 2, 3\}$ with $x \in \widehat{f_i v_z}$ since $W = \hat{v}_1 \cup \hat{v}_2 \cup \hat{v}_3$. Assume e.g. $z = 1$. Recalling $\varrho v_1 = \varrho v_2$ we see that either $f_i v_1 = g_j v_1$ for some j, or $f_i v_1 = f_u v_1$ for some u. We have either $x \in \hat{g}_j$ and $f_i \neq g_j$, or $x \in \hat{f}_u$ and $f_i \neq f_u$, for f is a mono.)

Since each $\hat{h}_i$ has power 3, it immediately follows that

$$\left|\bigcup_{i=1}^{r} \hat{h}_i\right| \leqq \tfrac{3}{2}r\,.$$

This contradicts (a).

II. To prove (a) we shall verify that for every K^*-morphism $g : Y \to X$ with $X \in K_m$ the following conditions hold:

(c_m) Given $h : W \to X$ in K^* with $|\hat{h} \cap \hat{g}| > 1$, there exists $h_1 : W \to Y$ in K^* such that $h = gh_1$;

(d_m) given distinct $h_1, \ldots, h_r : W \to X$ in K^* with $|\hat{h}_i \cap \hat{g}| \leqq 1$ for each i, then $\left|\bigcup_{i=1}^{r} \hat{h}_i - \hat{g}\right| \geqq a_m r$.

Then (a) is proved as follows: given pairwise distinct $h_1, \ldots, h_r : W \to X$ then $|\hat{h}_i \cap \hat{h}_r| \leqq 1$ (for, if $|\hat{h}_i \cap \hat{h}_r| > 1$, then, by ($c_m$), there would exist $h'_r : W \to W$ with $h_r = h_i h'_r$; since hom $(W, W) = \mathrm{id}_W$, $h_r = h_i$ – a contradiction). Thus, by (d_m),

$$\left|\bigcup_{i=1}^{r-1} \hat{h}_i - h_r\right| \geqq a_m(r-1)\,;$$

therefore

$$\left|\bigcup_{i=1}^{r} \hat{h}_i\right| = \left|\bigcup_{i=1}^{r-1} \hat{h}_i - \hat{h}_r\right| + |\hat{h}_r| \geqq a_m(r-1) + |\hat{h}_r|\,.$$

As $a_m > \frac{3}{2}$ and $|\hat{h}_r| = 3$, (a) follows.

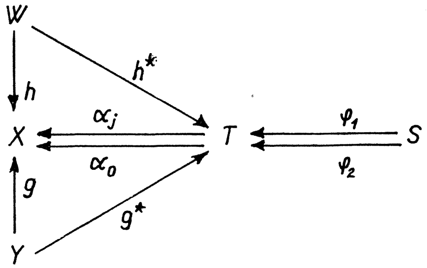
We prove (c_m) and (d_m) by induction on m.

III. The proof of c_0 and d_0 is clear (the only morphisms to W in K are those in K_0).

IV. Assuming (c_m), (d_m) we shall prove (c_{m+1}), (d_{m+1}). This is clear if $X \in K_m$. If $X \in K_{m+1} - K_m$, there are two possibilities: X is either the range of a quasi-filter of a parallel pair of morphisms in K_m, or it is a sum of two objects from K_m like in B,8. In the former case, denoted here by V, all K^*-morphisms into X factor through the maps of the quasi-filter. In the latter case, denoted here by VI, they factor through the new summand injections.

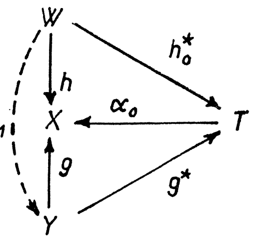
V. Let $\varphi_1, \varphi_2 : S \to T$ be morphisms in K_m with a quasi-filter $\alpha_0, \ldots, \alpha_{s-1} : T \to X$ in K_{m+1} such that g, h and $h_1, \ldots, h_r$ factor through the α_j's.

V, 1. The proof of (c_{m+1}). Without loss of generality, g factors through α_0, i.e. $g = \alpha_0 g^*$ for some $g^* : Y \to T$ in K^*. We know that h factors through some α_j, $h = \alpha_j h_j^*$.



We shall prove that also h factors through α_0. As $|\hat{h} \cap \hat{g}| \geqq 2$, also $|\hat{\alpha}_j \cap \hat{\alpha}_0| \geqq 2$. By (10) we have either $\hat{\alpha}_0 \cap \hat{\alpha}_j \subset \hat{\alpha}_{j-1} \cap \hat{\alpha}_j$ or $\hat{\alpha}_j \cap \hat{\alpha}_0 \subset \hat{\alpha}_j \cap \hat{\alpha}_{j+1}$. In the former case, consider the equality $\alpha_j \varphi_p = \alpha_{j-1} \varphi_q$, which must hold for some $p, q \in \{1, 2\}$ because α_j, α_{j-1} are neighbours in a quasi-filter for φ_1, φ_2. Since by (11) $\widehat{\alpha_j \varphi_p} = \hat{\alpha}_{j-1} \cap \hat{\alpha}_j$, we see that $|\hat{h} \cap \widehat{\alpha_j \varphi_p}| \geqq 2$, i.e. $|\widehat{\alpha_j h_j^*} \cap \widehat{\alpha_j \varphi_p}| \geqq 2$ and, since α_j is $1-1$, $|\hat{h}_j^* \cap \hat{\varphi}_p| \geqq 2$. Thus, we can use (c_m) on h_j^* and φ_p to obtain $h_j^{**} : W \to S$ in K^* with $h_j^* = \varphi_p h_j^{**}$. Put $h_{j-1}^* = \varphi_q h_j^{**}$. Then $h = \alpha_{j-1} h_{j-1}^*$ and we can repeat this procedure until we get $h_0^* : W \to T$ in K^* with $h = \alpha_0 h_0^*$. In the case $\hat{\alpha}_j \cap \hat{\alpha}_0 \subset \hat{\alpha}_j \cap \hat{\alpha}_{j+1}$ we proceed analogously, this time considering the equality $\alpha_j \varphi_p = \alpha_{j+1} \varphi_q$.

Now we use (c_m) on g^*, h_0^*: since α_0 is $1-1$, $|\hat{h}_0^* \cap \hat{g}| \geqq 2$ and $T \in K_m$.



Hence, there exists $h_1 : W \to Y$ in K with $h_0^* = g^* h_1$. We get $h = g h_1$.

V, 2. The proof of (d_{m+1}). Without loss of generality, again $g = \alpha_0 g^*$. Define sets $H_0, \ldots, H_{s-1} \subset \{1, \ldots, r\}$ by

$$H_0 = \{i;\ h_i = \alpha_0 h_i' \text{ for some } h_i' \text{ in } K^*\},$$

$$H_j = \{i;\ i \notin H_0 \cup \ldots \cup H_{j-1} \text{ and } h_i = \alpha_j h_i' \text{ for some } h_i' \text{ in } K\}$$

for $j = 1, \ldots, s - 1$. Further, for each $j = 0, \ldots, s - 3n$ put $\tilde{H}_j = \bigcup_{t=j}^{j+3n-1} H_t$. Remember that $s/3n$ is an integer and observe that the sets $\tilde{H}_0, \tilde{H}_{3n}, \tilde{H}_{6n}, \ldots, \tilde{H}_{s-3n}$ are pairwise disjoint, their number is $s/3n$; their union is $\{1, \ldots, r\}$. Thus, if $\tilde{H}_{j_0}$ is the one of them with the minimal cardinality, then

$$\text{(e)} \qquad |\tilde{H}_{j_0}| \leqq \frac{r}{\dfrac{s}{3n}} \leqq r\left(1 - \frac{a_{m+1}}{a_m}\right)$$

(for the latter inequality, see $(*)$). To prove (d_{m+1}) we put $L = \{1, \ldots, n\} - \tilde{H}_{j_0}$ and verify that

$$\text{(f)} \qquad \left|\bigcup_{i \in L} \hat{h}_i - \hat{g}\right| \geqq a_m |L|.$$

Then we obtain (d_{m+1}) because

$$\left|\bigcup_{i=1}^{r} \hat{h}_i - \hat{g}\right| \geqq a_m|L| = a_m(r - |\tilde{H}_{j_0}|) \geqq a_m\left(r - r\left(1 - \frac{a_{m+1}}{a_m}\right)\right) = r a_{m+1}.$$

Define sets $L_0, \ldots, L_{s-3n} \subset L$ by the following rule: for $j = 0, \ldots, j_0 - 1$,

$$L_j = \{i \in L;\ h_i = \alpha_j h_i^* \text{ for some } h_i^* \text{ in } K^*\} - \bigcup_{i=0}^{j-1} L_i;$$

for $j = j_0, \ldots, s - 3n - 1$,

$$L_j = \{i \in L;\ h_i = \alpha_{s-j+j_0-1} h_i^* \text{ for some } h_i^* \text{ in } K^*\} - \bigcup_{i=0}^{j-1} L_i.$$

By induction on $t = 0, \ldots, s - 3n - 1$ we prove for $\hat{L}_j = \bigcup_{i \in L_j} \hat{h}_i$ that

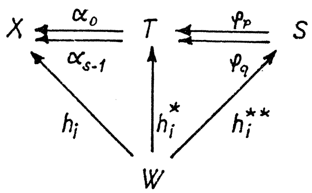
$$(\text{g}_t) \qquad \left|\bigcup_{j=0}^{t} \hat{L}_j - \hat{g}\right| \geqq a_m \sum_{j=0}^{t} |L_j|.$$

This will prove (f), for $t = s - 3n - 1$ yields $\bigcup_{j=0}^{t} \hat{L}_j = \bigcup_{i \in L} \hat{h}_i$.

V, 2.1. $t = 0$. We are to show that $|\hat{L}_0 - \hat{g}| \geqq a_m |L_0|$. First, let $j_0 \neq 0$; then we can use (d_m) on g^* and h_i^*, $i \in L_0$: since $1 \geqq |\hat{g} \cap \hat{h}_i| = |\widehat{\alpha_0 g^*} \cap \widehat{\alpha_0 h_i^*}| = |\hat{g}^* \cap \hat{h}_i^*|$, we get

$$|\hat{L}_0 - \hat{g}| = \left|\bigcup_{i \in L_0} \widehat{\alpha_0 h_i^*} - \widehat{\alpha_0 g^*}\right| = \bigcup_{i \in L_0} |\hat{h}_i^* - \hat{g}^*| \geqq a_m |L_0|.$$

Second, let $j_0 = 0$. Then $L_0 = \{i \in L;\ h_i = \alpha_{s-1} h_i^*\}$. Since α_0, α_{s-1} are neighbours in the quasi-filter for φ_1, φ_2, we have $\alpha_0 \varphi_p = \alpha_{s-1}\varphi_q$ for some $p, q \in \{1, 2\}$. For each $i \in L_0$ we have $|\hat{h}_i^* \cap \hat{\varphi}_q| \leqq 1$. (Proof: assume the contrary for some $i \in L_0$;



then by (c_m) there exists $h_i^{**} : W \to S$ in K^* with $h_i^* = \varphi_q h_i^{**}$ and so $h_i = \alpha_0(\varphi_p h_i^{**})$, which implies $i \in H_0 \subset \tilde{H}_0$, a contradiction to $i \in L$.)

We can use (d_m) on h_i^*, φ_q $(i \in L_0)$ to obtain

$$\Big|\bigcup_{i\in L_0} \hat{h}_i - \hat{\varphi}_q\Big| \geqq a_m |L_0| .$$

Now, since g factors trough α_0 and each h_i $(i \in L_0)$ factors through α_{s-1}, we have $\bigcup_{i\in L_0} \hat{h}_i - \hat{g} \supset \bigcup_{i\in L_0} \hat{h}_i - (\hat{\alpha}_{s-1} \cap \hat{\alpha}_0)$ and, by (11), $\hat{\alpha}_{s-1} \cap \hat{\alpha}_0 \subset \widehat{\alpha_{s-1}\varphi_q}$. Since α_{s-1} is $1-1$, we get

$$|\hat{L}_0 - \hat{g}| = \Big|\bigcup_{i\in L_0} \hat{h}_i - \hat{g}\Big| \geqq \Big|\bigcup_{i\in L_0} \alpha_{s-1} h_i^* - \widehat{\alpha_{s-1}\varphi_q}\Big| =$$

$$= \Big|\bigcup_{i\in L_0} \hat{h}_i^* - \hat{\varphi}_q\Big| \geqq a_m |L_0| .$$

V, 2.2. $0 < t < j_0$ and (g_{t-1}) holds. We shall verify that

$$\text{(h)} \qquad |\hat{L}_t - (\hat{\alpha}_{t-1} \cap \hat{\alpha}_t)| \geqq a_m |L_t| .$$

Then (g_{m+1}) follows, because, by (10), $\hat{\alpha}_j \cap \hat{\alpha}_t \subset \hat{\alpha}_{t-1} \cap \hat{\alpha}_t$ for $j = 0, \ldots, t-1$ (indeed, as $j_0 \leqq s - 3n < s - 2n$, all j's with $0 < j < t(<j_0)$ belong to $\{t-1, t-2, \ldots, t-s-2n \pmod s\}$. Now, g and h_i $(i \in L_0 \cup \ldots \cup L_{t-1})$ factor through $\alpha_0, \ldots, \alpha_{t-1}$ and each h_i $(i \in L_t)$ factors through α_t, hence we see that

$$\hat{L}_t - \Big(\bigcup_{j=0}^{t-1} L_j \cup \hat{g}\Big) \supset \hat{L}_t - \bigcup_{j=0}^{t-1} \hat{\alpha}_j \cap \hat{\alpha}_t \supset \hat{L}_t - (\hat{\alpha}_{t-1} \cap \hat{\alpha}_t) .$$

Thus, using (h) and (g_{t-1}) we get (g_t).

To prove (h) we again use the fact that $\alpha_{t-1}\varphi_p = \alpha_t \varphi_q$ for some $p, q \in \{1, 2\}$. By the definition of L_t, no h_i $(i \in L_t)$ factors through α_{t-1}; applying (c_m) we easily see that then $|\hat{h}_i^* \cap \varphi_q| \leqq 1$, $i \in L_t$. Hence, we can use (d_m) to obtain

$$\Big|\bigcup_{i\in L_t} \hat{h}_i - \widehat{\alpha_t\varphi_q}\Big| = \Big|\bigcup_{i\in L_t} \widehat{\alpha_t h_i^*} - \widehat{\alpha_t\varphi_q}\Big| = \Big|\bigcup_{i\in L_t} \hat{h}_i - \hat{\varphi}_q\Big| \geqq a_m |L_t| .$$

Now, $\widehat{\alpha_t \varphi_q} = \hat{\alpha}_{t-1} \cap \hat{\alpha}_t$ holds by (11); and this yields (h).

V. 2.3. $j_0 \leqq t \leqq s - 3n - 1$ and (g_{t-1}) holds. This is analogous to V, 2.2: g and h_i $(i \in L_0 \cup \ldots \cup L_{t-1})$ factor trough $\alpha_0, \ldots, \alpha_{j_0-1}, \alpha_{s-1}, \alpha_{s-2}, \ldots, \alpha_{s-t+j_0}$. As $j_0 \leqq s - 3n < s - 2n$, all j's with $0 \leqq j \leqq j_0 - 1$ and all j's with $s - 1 \geqq j \geqq \geqq s - t + j_0$ belong to $\{(s - t + j_0 - 1) + 1, \; (s - t + j_0 - 1) + 2, \ldots \ldots, (s - t + j_0 - 1) + (s - 2n)\}$, we can apply (10) to obtain

$$\hat{\alpha}_j \cap \hat{\alpha}_{s-t+j_0-1} \subset \hat{\alpha}_{s-t+j_0-1} \cap \hat{\alpha}_{s-t+j_0-1+1}$$

for all these j's. Thus, it suffices to prove

$$\text{(h}') \qquad |\hat{L}_t - (\hat{\alpha}_{s-t+j-1} \cap \hat{\alpha}_{s-t+j})| \geqq a_m |L_t| \,,$$

which is done similarly to (h) above.

VI. Let X be the sum of B_1, B_2 in K with injections $v_i : B \to X$ so that g, h and all $h_1, \ldots, h_r$ factor through v_1, v_2. Then (c_{m+1}) is clear: since $\hat{g} \cap \hat{h} \neq \emptyset$, both g and h must factor through the same $v_i : g = v_i g'$, $h = v_i h'$. Apply (c_m) to g', h'.

(d_{m+1}) is also clear: assume $g = v_1 g'$ and let A be the set of all $i \in \{1, \ldots, r\}$ with $h_i = v_1 h'_i$; applying (d_m) first to g', h'_i $(i \in A)$ we get $|\bigcup_{i \in A} \hat{h}'_i - \hat{g}| \geqq a_m |A|$. For each $j \notin A$ we have $h_j = v_2 h'_j$ and, by (d_m), $|\bigcup_{j \notin A} \hat{h}'_j| \geqq a_m(r - |A|)$. We get

$$|\bigcup_{i=1}^{r} \hat{h}_i - \hat{g}| = |\bigcup_{i \in A} \hat{h}'_i - \hat{g}| + |\bigcup_{j \notin A} \hat{h}'_j| \geqq a_m |A| + a_m(r - |A|) = a_m r \,.$$

This concludes the proof of C,2.

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