Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 2, 241–274

Persistent URL: http://dml.cz/dmlcz/101741

A CAUCHY TYPE CONDITION AND INTEGRABILITY

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(Received September 4, 1979)

I. INTRODUCTION

This paper concerns ordered pairs (f, g) of real valued functions on the interval $[0, 1]$ satisfying the weak Cauchy condition: if $\varepsilon > 0$ then there exists a positive number δ such that for each subdivision D of the interval $[0, 1]$ with norm less than δ, each two Riemann approximating sums in terms of D, f, g differ by less than ε:

[1] $$\sum\{Wf[p, q] \, |g(q) - g(p)| : [p, q] \in D\} < \varepsilon$$

where $Wf[p, q]$ denotes the oscillation of f on $[p, q]$;

$$Wf[p, q] = \text{l.u.b.} \{f(x) - f(y) : x \in [p, q], \ y \in [p, q]\} .$$

It is apparent that the existence of the Riemann-Stieltjes integral $\int_0^1 f \, dg$ implies the weak-Cauchy condition for real valued functions (f, g) on $[0, 1]$. This paper presents an example to show that the converse is false, even with f and g continuous.

The following notation is used.

If each of f and g is a real valued function from some subset of the reals and D is a collection of intervals each in the domain of f and of g then

$$O[D, f, g] = \sum\{Wf[p, q] \, |g(q) - g(p)| : [p, q] \text{ in } D\} .$$

The number $O[D, f, g]$ is called *the oscillation of Riemann sums on D* in terms of (f, g).

If D is a collection of sets and H is a number set, then $D . H$ denotes the set $\{C \cap H : C \in D\}$.

If f is a function on the interval $[a, b]$, then $\|f\|$ denotes max $|f(x)|$, x in $[a, b]$.

If g is a step function on the interval $[a, b]$ then $E(g)$ denotes the subdivision of $[a, b]$ each interval of which is a maximal interval having g constant on its interior.

The subdivision $E(g)$ is called *the defining subdivision of g*.

The following theorems are proved.

Theorem 1. *If each of* $[a, b]$ *and* $[u, v]$ *is an interval and* $\varepsilon > 0$, *then there exists an ordered pair* (f, g) *of real valued functions on the interval* $[a, b]$ *such that*

(1) $f(x)$ *is in* $[u, v]$ *and* $g(x)$ *is in* $[u, v]$ *for each* x *in* $[a, b]$,
(2) $\int_a^b f \, dg > 1$,
(3) *if* D *is a nonoverlapping collection of intervals filling up* $[a, b]$ *then* $O[D, f, g] < \varepsilon$,
(4) *each of* f *and* g *is continuous on* $[a, b]$.

Theorem 2. *There exists an ordered pair* (F, G) *of continuous real valued functions on the interval* $[0, 1]$ *such that* $\int_0^1 F \, dG$ *does not exist, but such that the weak-Cauchy condition is satisfied by* F *and* G *on* $[0, 1]$.

Throughout this paper, functions are real valued.

In discussions which follow subdivisions are nondegenerate and of the interval $[a, b]$ and integrals are Riemann-Stieltjes refinement integrals (defined in section IV) on the interval $[a, b]$ unless otherwise stated. An end of the subdivision D means an end of some interval of D.

In this paper it is not necessary to refer to the figures to understand the proofs presented. The figures, however, may be helpful in understanding why such proofs are presented. The discussion on context and related mathematics, in the conclusion, Section IV, is designed to be read before the following proofs as well as at the end of this paper.

It may be of interest that the parts of the proof of Theorem 1 not preceded by an asterisk are sufficient to prove the theorem which results if Property (4) is omitted from the conclusion of Theorem 1.

II. PROOF OF THEOREM 1

The following definitions are used. Figures 1.1, 1.2, and 1.3 furnish a picture of the ideas involved and suggest the construction.

Definition 1. The step function g *has alternating jumps* on the subdivision $E(g) \equiv \equiv [p_i, q_i]$, $i = 1, 2, \ldots, n$ means $n > 2$ and

1. for each i, there exists a number $g(p_i, q_i)$ such that $g(x) = g(p_i, q_i)$ for each x in (p_i, q_i), (i.e., g is constant on (p_i, q_i)).
2. the numbers $[g(p_{i+1}, q_{i+1}) - g(p_i, q_i)]$, $i = 1, 2, \ldots, n - 1$, alternate in sign; the jumps' values alternate in sign.

Definition 2. The ordered step function pair (f, g) is *with coordinated alternating jumps* on the subdivision $E(g)$ means

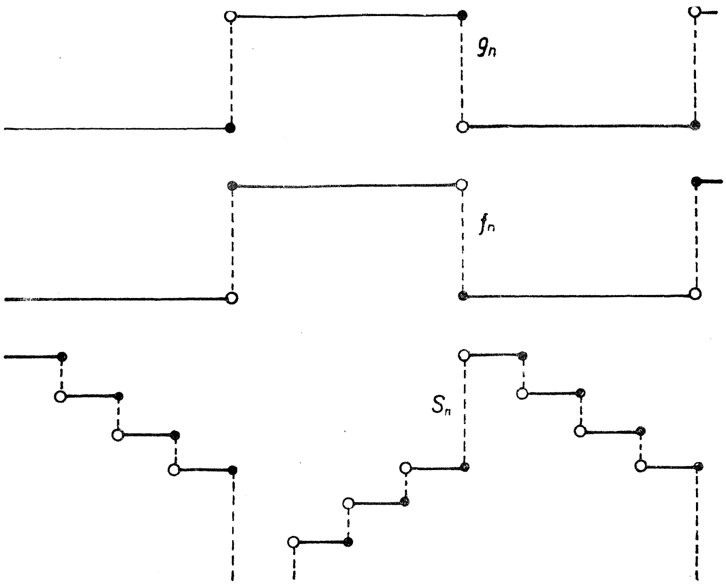


Fig. 1.1.

1. each of f and g has alternating jumps on $E(g)$,
2. g is continuous to the left,
3. f is continuous to the right,
4. if $[p, q]$ is in $E(g)$ then $(g(q) - g(p))(f(q) - f(p)) < 0$.

Definition 3. The step function M is said to *have jumps of constant distance* $K(V)$ on the number set V means there is a number $K(V)$ such that $|M(x+) - M(x-)|$ is either $K(V)$ or 0 for each number x in V.

Definition 4. The statement that the step function S is a *smoothing function* for the step function g means

1. S is continuous to the left,
2. if $[p, q]$ is an interval of $E(g)$ and x is in $[p, q]$ then
   (a) $S(x)$ is between $(g(p) - g(q))$ and 0 or $S(x) = 0$,
   (b) S is nondecreasing on $(p, q]$ if $(g(q) - g(p)) > 0$ and is nonincreasing on $(p, q]$ if $(g(q) - g(p)) < 0$,
   (c) $S(p+) + S(p) = S(q) = 0$ and
   (d) $g + S$ has jumps of constant distance on $[p, q)$, and
3. $E(g + S) . [p, q]$ has at least three intervals for each interval $[p, q]$ of $E(g)$.

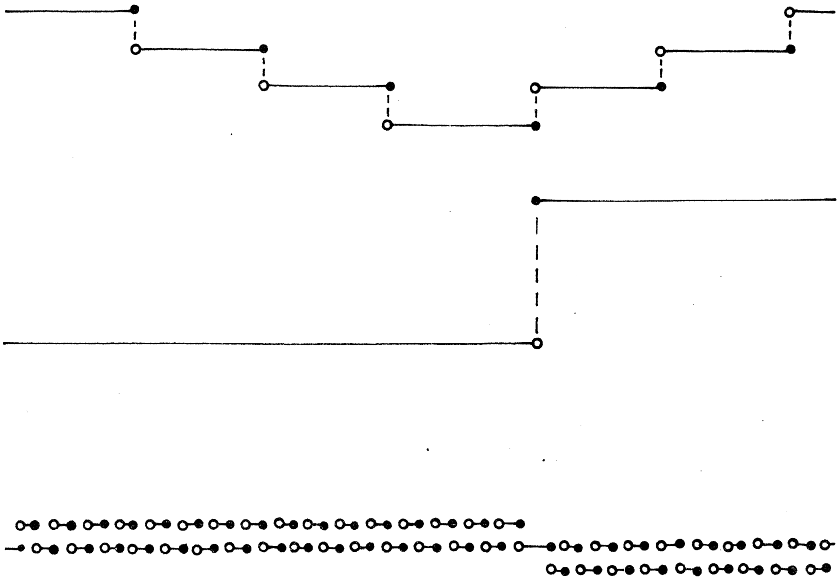


Fig.

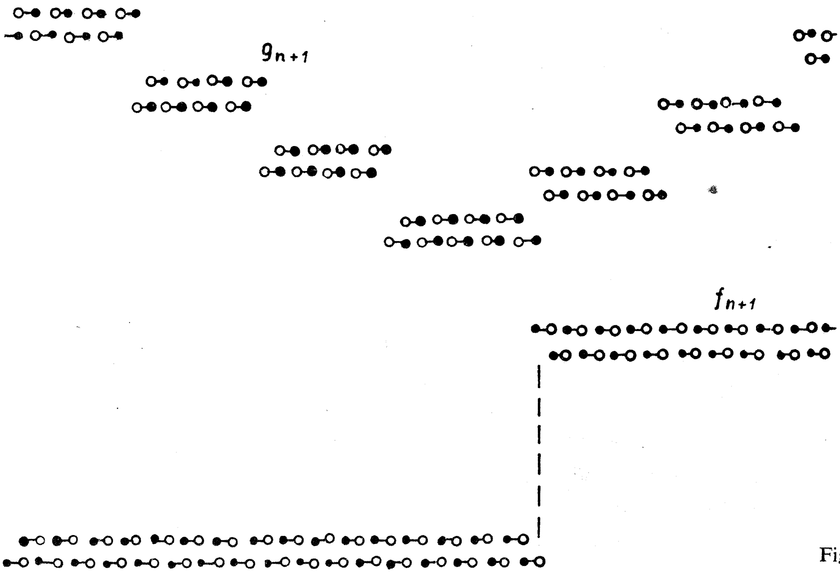
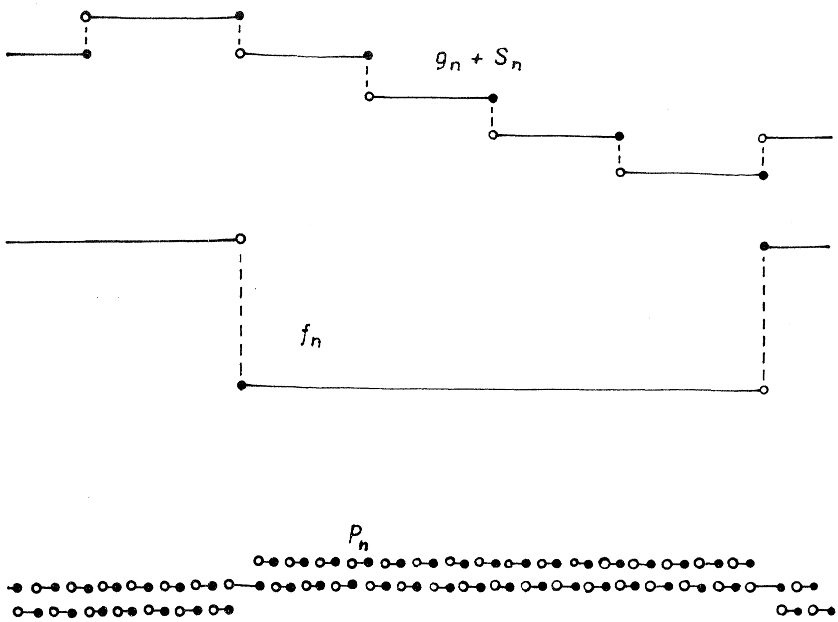
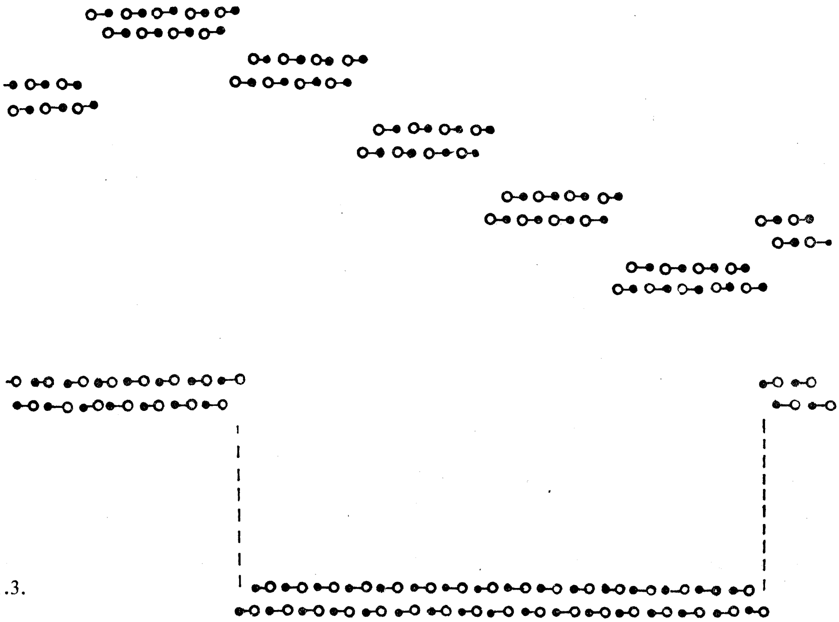


Fig.



1.2.



1.3.

Definition 5. Suppose g is a step function and S is a smoothing function for g. The statement that the step function P is a *perturbation function* for the step function $(g + S)$ means

1. $\{x : x \text{ is an end of } E(g + S) \text{ but not of } E(g)\} \subset \{x : x \text{ is an end of } E(P)\}$,
2. P has jumps of constant distance on each interval of $E(g)$,
3. $(g + S + P)$ has alternating jumps on $E(g + S + P)$,
4. P is continuous to the left and $P(a) = 0$,

5a. if $[p, q]$ and $[q, r]$ are adjacent intervals of $E(g + S)$ such that $[(g + S)(p) - - (g + S)(q)][(g + S)(q) - (g + S)(r)] > 0$ then $P(p +) = 0$, $P(q) \neq 0$ and $(g + S + P)(x)$ is either between $(g + S)(p)$ and $(g + S)(q)$ or is $(g + S)(q)$ for each x in $(p, q]$,

5b. If $[p, q]$ and $[q, r]$ are adjacent intervals of $E(g + S)$ such that $[(g + S)(p) - - (g + S)(q)] [(g + S)(q) - (g + S)(r)] < 0$ or if $[p, q]$ is in $E(g + S)$ and $q = b$ then $P(p +) = P(q) = 0$, $P(x) \neq 0$ for some x in $[p, q]$ and $(g + S + P)(x)$ is either between $(g + S)(p)$ and $(g + S)(q)$ or is $(g + S)(q)$ for each x in $(p, q]$. Note that 5b discusses exactly the ends of $E(g)$ by Definition 4.

Definition 6. If P is a step function, then P_R denotes the step function such that $P_R(x) = P(x +)$ in $[a, b)$ and $P_R(b) = 0$.

Definition 7. Suppose (f, g) is an ordered pair of step functions with coordinated alternating jumps and S is a smoothing function for g and $\varepsilon > 0$. The perturbation function P for $g + S$ *has perturbations evenly distributed within* ε means if $[r,s]$ is an interval of $E(g)$ and $[c_1, d_1]$ and $[c_2, d_2]$ each is a subinterval of $[r, s]$ and each in $E(g + S)$, then

$$|O[E(P) . [c_1, d_1], P_R, P] - O[E(P) . [c_2, d_2], P_R, P]| < \varepsilon .$$

Note. Such a perturbation function is called *evenly distributed within* ε.

In Figure 2, if $\varepsilon > 0$ the part of $g + S + P$ on $[p, d]$ could be considered to have enough jumps to make P not evenly distributed within ε.

The following construction furnishes an ordered function pair (f, g) to prove Theorem 1. It is based on Lemmas 1, 2, and 3, and its plan is to furnish a sequence of step function pairs (f_i, g_i) such that

1. for each i, $i = 1, 2, \ldots, N$, $\int f_i \, dg_i = \int f_{i-1} \, dg_{i-1} + \varepsilon/8$,
2. for each i, $i = 1, 2, \ldots, N$ and each subdivision D of $[a, b]$, $O[D, f_i, g_i] < \varepsilon$.

In the following, for each n, let $E(g_n)$ be denoted by $[p_{ni}, q_{ni}]$, $i = 1, \ldots, t_n$ with the usual ordering. And let E_n denote the set of intervals of $E(g_n)$ which have an end in common with $E(g_{n-1} + S_{n-1})$, $n \geqq 2$.

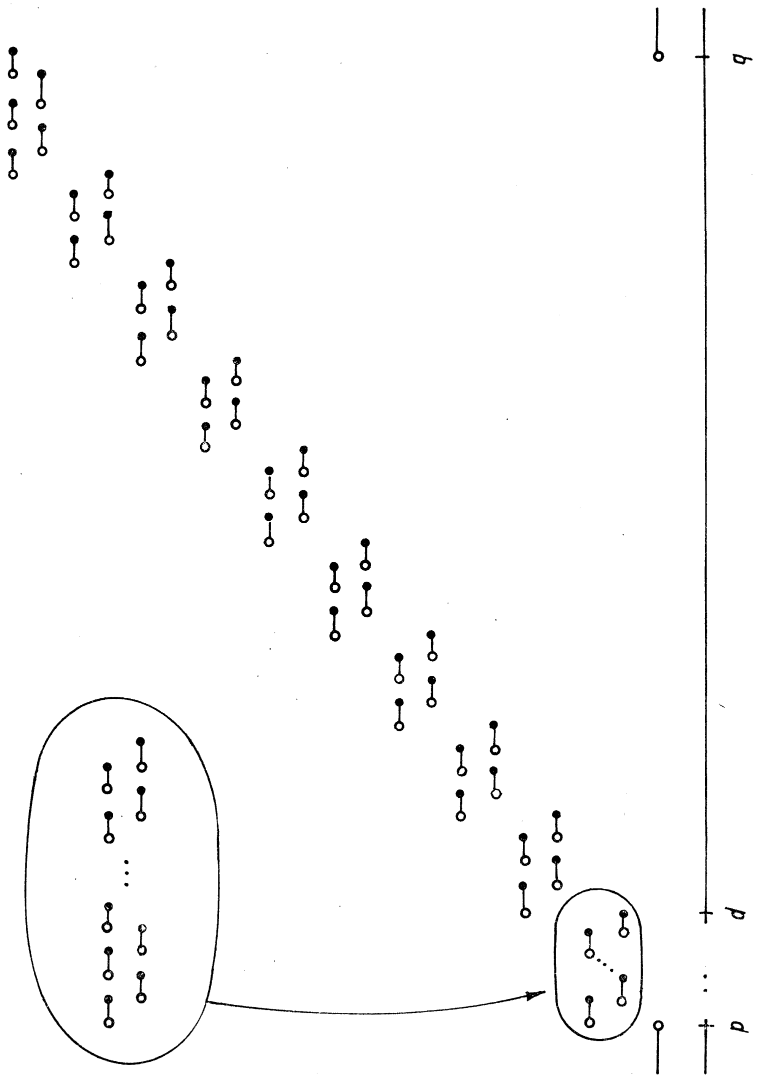


Fig. 2.

Suppose $[u, v]$ is an interval and $\varepsilon > 0$. Note that discussions are on the interval $[a, b]$ unless otherwise stated.

A. Let (f_1, g_1) denote an ordered pair of step functions on the interval $[a, b]$ with coordinated alternating jumps such that

1. $u < f_1(x) < v$ and $u < g_1(x) < v$ for each x in $[a, b]$,

2. $O[E(g_1), f_1, g_1] = \frac{1}{2}\varepsilon$,

3. f_1 and g_1 have jumps of the same constant distance on $[a, b]$,

4. $O[D, f_1, g_1] \leqq \frac{1}{2}\varepsilon$ for each subdivision D, and

5. $\int f_1 \, dg_1 > 0$.

Note that A1 through A5 hold if the f_1 and g_1 of A each has range the two numbers $(u + v)/2$ and $(u + v)/2 + (\varepsilon/2n)^{1/2}$ with n large enough to assure A1 and with $E(g_1)$ a set of n intervals filling up $[a, b]$.

B. Let N denote a positive integer greater than 2 such that $(N - 1)\,\varepsilon > 8$.

C. Construct (f_{n+1}, g_{n+1}) from (f_n, g_n) as follows:

1. let t_n denote the number of intervals in $E(g_n)$,

2. let S_n denote a smoothing function for g_n such that if $[s, t]$ is in $E(g_n)$ then

$$\frac{Wf_n[s, t]\,|g_n(t) - g_n(s)|}{k} < \frac{\varepsilon}{8Nt_n}$$

with k the number of intervals of $E(g_n + S_n)$ in $[s, t]$, and

3. let w_n denote the maximum such k, i.e. the maximum number of intervals of $E(g_n + S_n)$ in an interval of $E(g_n)$.

Lemmas 1 and 2 are used to guarantee that the following parts of the construction are possible. Parts C.4, C.5, C.6 and D follow the lemmas.

Lemma 1. *Suppose (f, g) is an ordered step function pair with coordinated alternating jumps on the interval $[a, b]$ and S is a smoothing function for g. Then if ε is a positive number, there exists a positive number δ such that if P is a perturbation function for $g + S$ with $\|P\| < \delta$ and $[r, s]$ is an interval of $E[g + S + P]$ then $W(f + P_R)\,[r, s]\,|(g + S + P)(s) - (s + S + P)(r)| < \varepsilon$.*

Proof of Lemma 1. Suppose the hypothesis, with $f(x)$ and $g(x)$ in an interval $[u, v]$ for each x in $[a, b]$.

Let P denote a perturbation function for $g + S$. Note that if $[r, s]$ is in $E(g + S + P)$ then either $W(f + P_R)\,[r. s] \leqq \|P\|$ or $|(g + S + P)(s) - (g + S + P)(r)| \leqq \|P\|$ since g is continuous to the left and f is continuous to the right and both are constant in the interior of each interval of $E(g)$. Thus $W(f + P_R)\,[r, s]\,|(g + S + P)(s) - (g + S + P)(r)|$ is the product of two numbers, one not greater than $\|P\|$ and the other not greater than $(v - u) + \|P\|$ and the product is therefore not greater than $\|P\|\,[(v - u) + \|P\|]$.

Thus if $\varepsilon > 0$, then there is a positive number δ such that if P is a perturbation function for $g + S$ with $\|P\| < \delta$ then $W(f + P_R)\,[r, s]\,|(g + S + P)(s) - (g + S + P)(r)| < \varepsilon$ for each interval $[r, s]$ of $E(g + S + P)$.

Lemma 2. *Suppose each of* δ *and* ε *is a positive number. Suppose* (f, g) *is an ordered step function pair with coordinated alternating jumps on the interval* $[a, b]$, *and* S *is a smoothing function for* g. *Suppose* $[p, q]$ *is an interval of* $E(g)$. *Then if* M *is a positive integer, there exists an even integer* $n > M$ *and a perturbation function* P *for* $g + S$ *with perturbations evenly distributed within* ε, *with* n *jumps in* $[p, q]$ *and with* $O[E(P) . [p, q], P_R, P] = Wf[p, q] |g(q) - g(p)|$. *And there exists a positive integer* M *large enough so that if* n *is such a positive integer, and* P *is such a perturbation function then* $|P(x)| < \delta$ *for each* x *in* $[p, q]$ *and* $|O[E(P) . [p, q], P_R, P] - 2 \int_p^q P_R \, dP| < \varepsilon$.

Lemma 2 states for the step functions f and $g + S$ and the interval $[p, q]$ of $E(g)$, that one can choose a perturbation function P for $g + S$ with the number of jumps in $[p, q]$ large enough so that the distance of these jumps which guarantees $O[E(P) . . [p, q], P_R, P] = Wf[p, q] |g(q) - g(p)|$ is small enough to make this oscillation differ from $2 \int_p^q P_R \, dP$ by less than ε. In fact there is such a P which is evenly distributed within ε, and with $\|P\| < \delta$.

Proof of Lemma 2. If n is an even positive integer and P is a perturbation function for $g + S$ with n jumps of constant distance z in the interval $[p, q]$ of $E(g)$, then

[2.1] $$O[E(P) . [p, q], P_R, P] = (n - 1) z^2 .$$

Thus, since there is a positive number z such that $(n - 1) z^2 = Wf[p, q] . . |g(q) - g(p)|$, then there is a perturbation function P for $g + S$ with n jumps in $[p, q]$ such that $O[E(P) . [p, q], P_R, P] = Wf[p, q] |g(q) - g(p)|$. And $z \to 0$ as $n \to \infty$. To insure that P is evenly distributed within ε it is sufficient to choose n large enough so that z is small enough to guarantee that $z^2 < \frac{1}{3}\varepsilon$ and to distribute

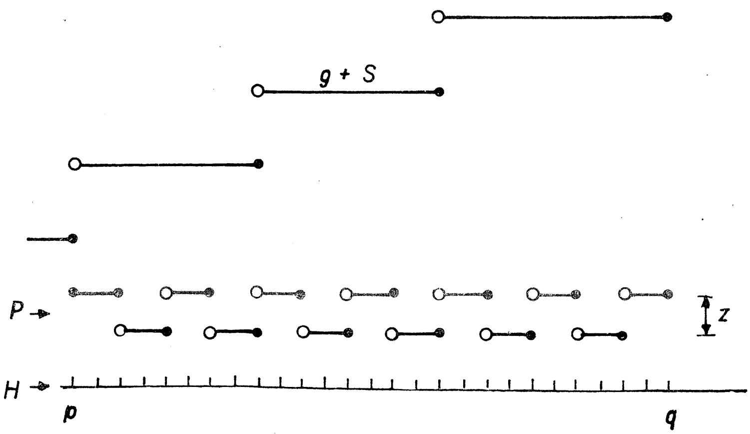


Fig. 3.

the jumps as evenly as possible between intervals of $E(g + S)$, i.e. so that no such interval has more than two more jumps than another. Now suppose H is a refinement of $E(P) . [p, q]$ such that each interval of $E(P)$ contains an end of H in its interior. Let V denote the collection of intervals of H such that an interval $[r, s]$ of H is in V if and only if $(P(s) - P(r)) = z$ and let v denote the number of intervals in V. Note that $v = \frac{1}{2}n$. Let W denote the collection of intervals of H such that an interval $[r, s]$ of H is in W if and only if $P(s) - P(r) = -z$ and let w denote the number of intervals of W. Note that $w = \frac{1}{2}n$. Also note that there exists a number h such that if x is an interval of W, then $P_R(x) = h$ and if x is in an interval of V, then $P_R(x) = h + z$. The number h is either 0 or $-z$. The function P is constant over each interval of H which is neither in V nor in W. See Figure 3. Thus

$$[2.2] \qquad \int_q^q P_R \, dP = \sum\{P_R(r)(P(s) - P(r)) : [r, s] \text{ in } H\} =$$
$$= \frac{n}{2}(z)(h + z) + \frac{n}{2}(-z)h = \frac{n}{2}z^2 .$$

Thus by [2.1] and [2.2] $2 \int_q^p P_R \, dP \to O[E(P) . [p, q], P_R, P]$ as $z \to 0$ and $n \to \infty$ and since $z \to 0$ as $n \to \infty$, Lemma 2 is true.

Parts C.4 and C.5 of the construction depend on Lemmas 1 and 2 and are now presented.

By Lemma 1, there exists a positive number δ such that if P_n is a perturbation function for $g_n + S_n$ with $\|P_n\| < \delta$ and $[s, t]$ is in $E(g_n + S_n + P_n)$ then

$$W(f_n + P_{nR})[s, t] \, |(g_n + S_n + P_n)(t) - (g_n + S_n + P_n)(s)| < \frac{\varepsilon}{16 w_n t_n N} .$$

4. Note Lemma 2 and let P_n denote a perturbation function for $g_n + S_n$ such that
   a. $\|P_n\|$ is less than such a number δ and $3\|P_n\| < |(g_n + S_n)(x+) - (g_n + S_n)(x)|$ for each discontinuity x of $g_n + S_n$.
   b. $O[E(P_n) . [p_{ni}, q_{ni}], P_{nR}, P_n] = Wf_n[p_{ni}, q_{ni}] \, |g_n(q_{ni}) - g_n(p_{ni})|$, $i = 1, 2, \ldots t_n$,
   c. P_n is evenly distributed within $\varepsilon/(8 w_n t_n N)$,
   d. $|O[E(P_n) . [p_{ni}, q_{ni}], P_{nR}, P_n] - 2 \int_{p_{ni}}^{q_{ni}} P_{nR} \, dP_n| < \varepsilon/16 t_n$ for each i, $i = 1, 2, \ldots$ $\ldots, t_n$.

5. Note that if (f, g) is an ordered pair of step functions with coordinated alternating jumps, S is a smoothing function for g, and P is a perturbation function for $g + S$, then
   a. $(f + P_R, g + S + P)$ is a pair of step functions with coordinated alternating jumps,
   b. $E(g + S + P)$ is a refinement of $E(g)$,
   c. If p is an end of $E(g)$ then $(g + S + P)(p) = g(p)$ and $(f + P_R)(p) = f(p)$.

The number $P(p)$ is 0 since each end p of $E(g)$ is either the number a with $P(p) = 0$ by 4 of Definition 5 or is an end with the properties of the end q mentioned in 5.b of Definition 5, by the note at the end of Definition 5.

6. Let $g_{n+1} \equiv g_n + S_n + P_n$ and $f_{n+1} \equiv f_n + P_{nR}$.

*D. Let (f, g) denote the ordered function pair such that for each interval $[p, q]$ of $E(g_N)$ the ordinate $f(x) = f_N(x)$ if x is in $[p, \frac{1}{2}(p + q)]$ and $f(x) = f_N(q)\, 2((q - x) : : (q - p))\, (f_N(p) - f_N(q))$ if x is in $[\frac{1}{2}(p + q), q]$ and such that for each interval $[p, q]$ of $E(g_N)$ the ordinate $g(x) = g_N(p) + 2((x - p)/(q - p))\, (g_N(q) - g_N(p))$ if x is in $[p, \frac{1}{2}(p + q)]$ and $g(x) = g_N(x)$ if x is in $[\frac{1}{2}(p + q), q]$.

This ends the construction. Lemmas 3, 4, 5, and 6 show that the ordered function pair (f, g) satisfies the conclusion of Theorem 1. In the proofs that follow recall that part C.5.c of the construction implies that $g_n(p) = g_N(p)$ and $f_n(p) = f_N(p)$ for each n, $n = 1, 2, \ldots, N$, for each end p of $E(g_n)$. Lemma 3 is used to show that $\int f_N \, \mathrm{d}g_N > 1$. This in turn is used to show that $\int f \, \mathrm{d}g > 1$.

Lemma 3. *Suppose (f, g) is an ordered pair of step functions with coordinated alternating jumps, S is a smoothing function for g and P is a perturbation function for $g + S$. Then $\int f \, \mathrm{d}g = \int f \, d(g + S)$ and $\int (f + P_R) \, \mathrm{d}(g + S + P) = \int f \, \mathrm{d}g + \int P_R \, \mathrm{d}P$, with $[a, b]$ the range of integration.*

Proof of Lemma 3. The following are used:

[3.1] $$\int f \, \mathrm{d}(g + S + P) = \int f \, \mathrm{d}g$$

and

[3.2] $$\int P_R \, \mathrm{d}(g + S + P) = \int P_R \, \mathrm{d}P\,.$$

To show that $\int f \, \mathrm{d}g = \int f \, \mathrm{d}(g + S) = \int f \, \mathrm{d}(g + S + P)$, let E denote a refinement of $E(g + S + P)$ such that no interval of E is in $E(g + S + P)$ and note that each of the three integral expressions denotes the number $\sum\{f(p)\,(g(q) - g(p)) : [p, q] \in \in E$, p an end of $E(g + S + P)\}$ since for each interval $[r, s]$ in $E(g)$ the function f is constant on $[r, s)$ and

$$(g + S + P)(r) = (g + S)(r) = g(r)$$
$$\text{and } (g + S + P)(s -) = (g + S)(s -) = g(s -).$$

To similarly prove [3.2] note where $P_R(x) = 0$.

Then let E denote a refinement of $E(g + S + P)$ such that no interval of E is in $E(g + S + P)$ and note that each of the two integral expressions denotes the number $\sum\{P_R(p)\,(P(q) - P(p)) : [p, q]$ in E, p an end of $E(g + S + P)\}$ since for each interval $[r, s]$ of E either P_R is the constant 0 on $[r, s]$ or $[P(s) - P(r)] = [(g + S + + P)(s) - (g + S + P)(r)]$. This statement holds since $P(r)$ is 0 at each of the numbers over which $g + S$ is discontinuous, by Definition 5, part 5.a and 5.b and Definition 6.

The following shows that $\int f_N \, dg_N > 1$. For each $n < N$

$$O[E(g_{n+1}), f_{n+1}, g_{n+1}] \geqq \sum_{i=1}^{t_n} O[E(P_n) . [p_{ni}, q_{ni}], P_{nR}, P_n]$$

since by Definition 5 if $[r, s]$ is in $E(g_{n+1})$ then it is in $E(P_n) . [p_{ni}, q_{ni}]$ for some i, and

$$Wf_{n+1}[r, s] \geqq WP_{nR}[r, s] \quad \text{and} \quad |g_{n+1}(s) - g_{n+1}(r)| \geqq |P_n(s) - P_n(r)| .$$

Thus if $\sum_{i=1}^{t_n} O[E(P_n) . [p_{ni}, q_{ni}], P_{nR}, P_n] > \varepsilon/2$ then $\sum_{i=1}^{t_{n+1}} Wf_{n+1}[p_{n+1\,i} q_{n+1\,i}] \cdot$ $\cdot |g_{n+1}(q_{n+1\,i}) - g_{n+1}(p_{n+1\,i})| > \varepsilon/2$ since this sum is $O[E(g_{n+1}), f_{n+1}, g_{n+1}]$ by definition, and therefore by C.4.b

$$\sum_{i=1}^{t_{n+1}} O[E(P_{n+1}) . [p_{n+1\,i}, q_{n+1\,i}], P_{n+1\,R}, P_{n+1}] > \varepsilon/2 .$$

Thus since A.2 holds, then, for each n, $\sum_{i=1}^{t_n} O[E(P_n) . [p_{ni}, q_{ni}], P_{nR}, P_n] > \varepsilon/2$ by induction.

Therefore, by C.4.d of the construction

$$2 \int_a^b P_{nR} \, dP_n > \varepsilon/2 - \varepsilon/16 .$$

And since by Lemma 3

$$\int_a^b f_N \, dg_N = \int f_1 \, dg_1 + \sum_{i=1}^{N-1} \int P_{iR} \, dP_i$$

then $\int f_N \, dg_N > \int f_1 \, dg_1 + (N - 1) \varepsilon/8 > 1$ by A5 and B of the construction. The integral $\int f \, dg > 1$ then follows from $\int f \, dg = \int f_N \, dg_N$ which occurs since if D is a subdivision with each end of $E(g_N)$ an end of D and with the midpoints of intervals of $E(g_N)$ being ends of D then

$$\sum\{f(x) (g(s) - g(r)) : [r, s] \in D, \ x \in [r, s]\} = \int f \, dg =$$

$$= \sum\{f_N(x) (g_N(s) - g_N(r)) : [r, s] \in D, \ x \in [r, s]\} = \int f_N \, dg_N .$$

To show that (f, g) is sufficient to prove Theorem 1 it is necessary to show that if A is a subdivision, then $O[A, f, g] < \varepsilon$. To show this, the inequality, $O[A, f_N, g_N] <$ $< \varepsilon$ for each subdivision A, is used. The following is a proof of this inequality, using Lemmas 4 and 5.

Suppose A is a subdivision.

Definition. Let H denote the subdivision such that x is an end of H if and only if one of the following is true.

1. For some n, $n = 1, 2, \ldots, N-1$, there exists an interval $[p, q]$ of $E(S_n)$ such that (p, q) contains an end of A which is not an end of $E(g_n)$ and x is an end of $E(g_{n+1})$ in $(p, q]$.
2. x is an end of $E(g_1)$.

Definition. Let d denote a positive number which is less than half the shortest length of an interval in $E(g_N)$ and less than half the shortest positive distance from an end of A to an end of H.

Definition. Let G denote the subdivision such that t is an end of G if and only if one of the following is true.

1. t is an end of H which is also an end of A in which case let t' denote t.
2. There is an interval $[r, s]$ of A such that t is an end of H in (r, s) and $[r, s]$ has as a subset some interval $[\gamma, \delta]$ of H such that

$$Wf_N[\gamma, \delta]\,|g_N(\delta) - g_N(\gamma)| \geqq Wf_N[r, s]\,|g_N(r) - g_N(s)|\,.$$

In this case let t' denote t.

3. There is an interval $[r, s]$ of A which contains no such interval $[\gamma, \delta]$ of H as a subset and $t + d$ is an end x of H in (r, s) and not an end of A. In this case let x' denote $x - d = t$.

Note that if j denotes an end of H then j' denotes an end of G.

Lemma 4. $O[G, f_N, g_N] \geqq O[A, f_N, g_N]$.

To prove Lemma 4, it is sufficient to show that if $[r, s]$ is an interval of A then one of the following three statements is true:

(1) there exists an interval $[\gamma, \delta]$ of G not in A such that γ is in $[r, s)$ and

$$Wf_N[\gamma, \delta]\,|g_N(\delta) - g_N(\gamma)| \geqq Wf_N[r, s]\,|g_N(s) - g_N(r)|$$

or

(2) $[r, s]$ is in G or

(3) $Wf_N[r, s]\,|g_N(s) - g_N(r)| = 0$.

Suppose $[r, s]$ is an interval of A. Let $E(S_0)$ denote the degenerate set $\{[a, b]\}$, $E(P_0) \equiv E(g_1)$ and n denote the largest non-negative integer α such that $[r, s]$ is a subset of some interval of $E(S_\alpha)$. Let $[w, x]$ denote the interval of $E(S_n)$ having $[r, s]$ as a subset.

It is shown that one of the three statements is true, first in case $[r, s]$ is not a subinterval of an interval of $E(g_{n+1})$ and then in the contrary case.

Suppose $[r, s]$ is not a subinterval of an interval of $E(g_{n+1})$. Let $[i, j]$ denote the interval of $E(g_{n+1})$ such that $[i, j)$ contains r. Then $s > j$, since $[r, s]$ is not a subinterval of $[i, j]$ by supposition.

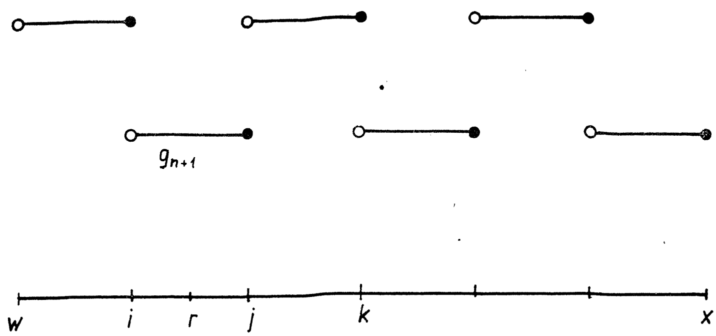


Fig. 4.1.

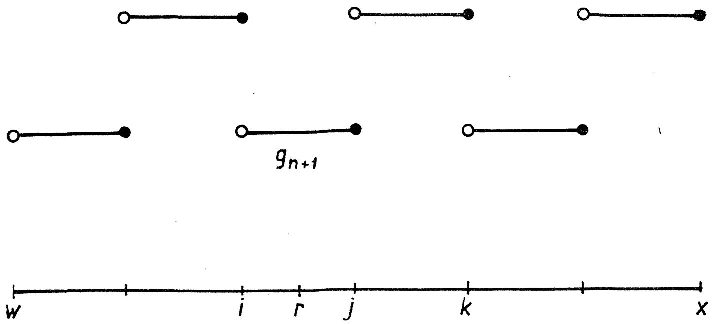


Fig. 4.2.

Let $[j, k]$ denote the interval of $E(g_{n+1})$ with left end j. Since each interval of $E(g_{n+1})$ is a subset of some interval of $E(S_n)$ and since $[r, s]$ is a subset of $[w, x]$ and $s > j$, then $[j, k]$ is a subset of $[w, x]$. Note that j is an end of H by the definition of H and that $g_N(j') = g_N(j)$ since either $j' = j$ or $j' = j - d$ by the definition of G. Also $g_{n+1}(j) = g_N(j)$ by C.5.c of the construction.

Now suppose

$$[4.1] \qquad g_N(j) < g_N(i).$$

Note that $g_{n+1}(j) = g_N(j)$ and $g_{n+1}(i) = g_N(i)$ by C.5.c of the construction. If α is in $[i, j)$, then $g_N(j) \leqq g_N(\alpha) \leqq g_N(i)$ by definition 5.a and 5.b, definition 4, and C.6 of the construction. Thus since r is in $[i, j)$ then

$$[4.2] \qquad g_N(j) \leqq g_N(r).$$

This supposition is used throughout the proof. It is noted that if $g_N(j) > g_N(i)$ then a similar proof suffices. Figure 4.1 or Figure 4.2 furnishes a drawing. The part of

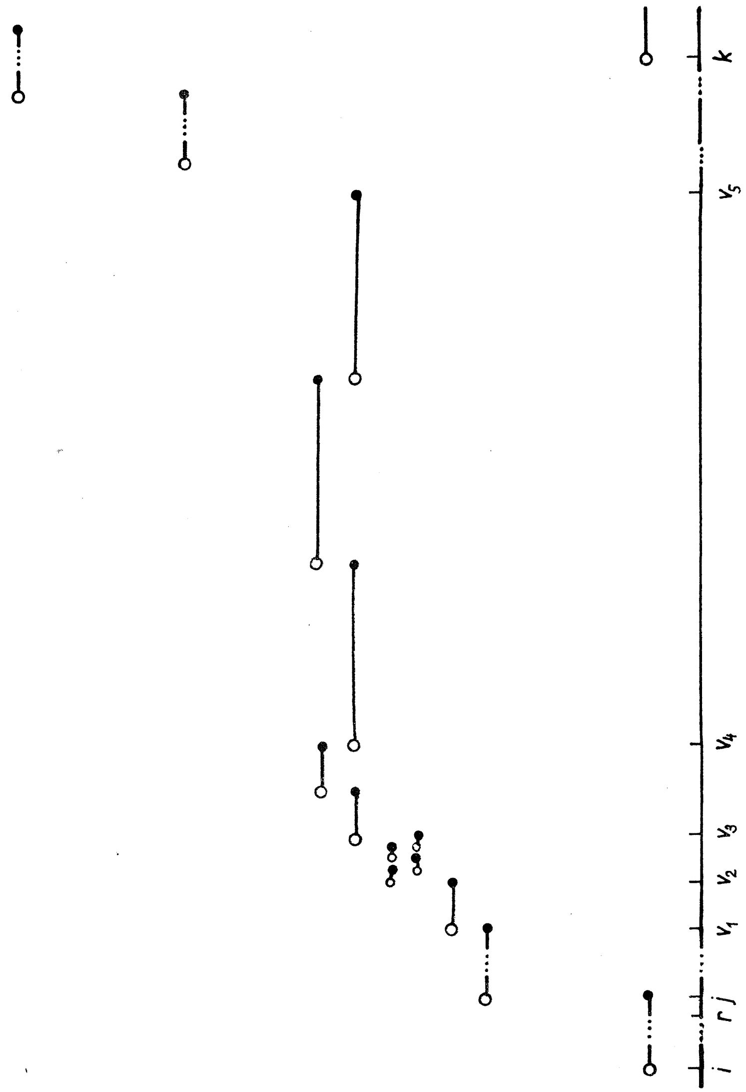


Fig. 4.3.

Figure 4.1 which relates to the rest of the proof and the part of Figure 4.2 which relates to the rest of the proof are the same. Thus, it doesn't matter which figure is in mind. In fact, if $r = i = w$, then a third picture would be more appropriate.

Suppose $g_N(r) \neq g_N(s)$. The proof that either statement 1 or statement 2 is ture is divided into the following three.

Case 1. $g_N(s) > g_N(r)$ and s is in $[j, k]$.
Case 2. $g_N(s) > g_N(r)$ and $k < s$.
Case 3. $g_N(s) < g_N(r)$.

Suppose Case 1. Figure 4.3 is presented to help in the understanding of the proof in Case 1. On the interval $[i, j]$ the function g_{n+1} is pictured. On $[j, v_1]$ the function $g_{n+1} + S_{n+1}$ is pictured. On $[v_1, v_2]$ the function $g_{n+2} + S_{n+2}$ is pictured. On $[v_2, v_3]$ the function g_{n+3} and on $[v_4, v_5]$ the function g_{n+2}.

Suppose there is no interval $[\gamma, \delta]$ of G with the properties to assure statement 1. The proof that this supposition is false depends on the fact that if $[j, l]$ denotes the interval of H with left end j then $g_N(l) \geqq g_N(s)$ which is least apparent in case the following supposition holds. In each other case it is readily apparent as will be shown.

Suppose for some interval $[y_{n+1}, z_{n+1}]$ of $E(S_{n+1})$ the number s is in the leftmost number set of $E(g_{n+2}) \cdot (y_{n+1}, z_{n+1}]$ and let t denote the largest integer h such that for some interval $[y_h, z_h]$ of $E(S_h)$ the number s is in the leftmost number set of $E(g_{h+1}) \cdot (y_h, z_h]$. For each h, $h = n + 1, \ldots, t$, the collection $E(g_{h+1}) \cdot (y_h, z_h]$ has leftmost member $(p, q]$ the common part of $(y_h, z_h]$ and the first (leftmost) interval of $E(g_{h+1})$ which intersects $(y_h, z_h]$. All the rest of the number sets of $E(g_{h+1}) \cdot (y_h, z_h]$ are intervals of $E(g_{h+1})$ and are thus closed on each end. Included here is the assumption that an S_{n+1} and a g_{n+2} exist.

Let $[y_h, z_h]$ denote the interval of $E(S_h)$ such that s is in $(y_h, z_h]$ for each h, $h = n, \ldots, t + 1$, $h < N$. Let $[j_h, k_h]$ denote the leftmost interval of $E(g_{h+1})$ in $[y_h, z_h]$, $h = n + 1, \ldots, t + 1$, $h < N$. Let $[j, k] \equiv [j_n, k_n]$. Either $t = N - 1$ or for some interval $[y_{t+1}, z_{t+1}]$ of $E(S_{t+1})$, the number s is in a number set of $E(g_{t+2}) \cdot (y_{t+1}, z_{t+1}]$ other than the leftmost number set of $E(g_{t+2}) \cdot (y_{t+1}, z_{t+1}]$.

Suppose the latter. With the following two statements, mathematical induction yields that $g_{t+2}(k_{t+1}) \geqq g_{t+2}(u)$ for each u in $(y_{t+1}, z_{t+1}]$.

First, since $g_{n+1}(j) < g_{n+1}(i)$ by (4.1), then $g_{n+1}(k_n) > g_{n+1}(j_n)$ and thus $g_{n+1}(k_n) \geqq g_{n+1}(u)$ for each u in $(y_n, z_n] = (w, x]$ because on $(w, x]$ the step function g_{n+1} has alternating jumps of constant distance by C.5.a of the construction and the fact that P_n has jumps of constant distance on $[w, x]$ by part 2 of Definition 5.

Second, if for some h, $h = n, \ldots, t$, $g_{h+1}(k_h) \geqq g_{h+1}(u)$ for each u in $(y_h, z_h]$ then $g_{h+2}(k_{h+1}) \geqq g_{h+2}(u)$ for each u in $(y_{h+1}, z_{h+1}]$.

To show that this second statement holds, suppose h is a positive integer $h = n, \ldots, t$ such that $g_{h+1}(k_h) \geqq g_{h+1}(u)$ for each u in $(y_h, z_h]$. Thus $g_{h+1}(k_h) \geqq g_{h+1}(j_h)$ since g_{h+1} has alternating jumps on $E(g_{h+1})$ by C.5.a of the construction and since g_{h+1} is not constant on $(y_h, z_h]$ by Definition 5.

Therefore S_{h+1} is nondecreasing on $(j_h, k_h]$ by part 2.b of Definition 4. Let $(\theta_i)_{i=1}^{v}$ denote the ends of $E(S_{h+1})$ in $[j_h, k_h]$ and let $[\theta_0, \theta_1]$ denote the interval of $E(S_{h+1})$ with right end $\theta_1 = j_h$. Such a $[\theta_0, \theta_1]$ exists because $\theta_1 \neq a$ since $a \leqq r < j \leqq \theta_1$. Since g_{h+1} has alternating jumps on the subdivision $E(g_{h+1})$, then by Definition 4,

$$(g_{h+1} + S_{h+1})(\theta_0) > (g_{h+1} + S_{h+1})(\theta_1).$$

Thus by Definition 5 part 5.b, $P_{h+1}(\theta_1) = 0$ since $[\theta_0, \theta_1]$, $[\theta_1, \theta_2]$ satisfy the conditions of 5.b.

Now note that $[j_{h+1}, k_{h+1}]$ is the leftmost interval of $E(g_{h+2}) \cdot [\theta_i, \theta_{i+1}]$ for some i, $i = 1, \ldots, v - 1$ with $j_{h+1} = \theta_i$. Suppose this $i = 1$ and $j_{h+1} = \theta_1$. The interval pair $[\theta_0, \theta_1]\,[\theta_1, \theta_2]$ satisfies the conditions of 5.b of Definition 5. So $P_{h+1}(\theta_1) = 0$ and $g_{h+1}(\theta_1) = g_{h+1}(j_{h+1}) = g_{h+2}(j_{h+1})$, by the definition of g_{h+2}. Also since $[\theta_1, \theta_2]\,[\theta_2, \theta_3]$ satisfies the conditions of 5.a of Definition 5, then $g_{h+2}(k_{h+1}) > > g_{h+2}(j_{h+1}) = (g_{h+1} + S_{n+1})(\theta_1)$, since k_{h+1} is in $[\theta_1, \theta_2]$ and since S_{h+1} is nondecreasing on $[j_h, k_h]$ with $(g_{h+1} + S_{h+1})(\theta_1) < (g_{h+1} + S_{h+1})(\theta_2)$.

Now suppose this $i \neq 1$ and $j_{h+1} \neq \theta_1$. Then $g_{h+2}(\theta_i) = g_{h+2}(j_{h+1}) \leqq \leqq (g_{h+1} + S_{h+1})(\theta_i)$ since $[\theta_{i-2}, \theta_{i-1}]\,[\theta_{i-1}, \theta_i]$ satisfies either part 5.a or part 5.b of Definition 5. Also $(g_{h+1} + S_{h+1})(\theta_i) < g_{h+2}(k_{h+1})$ since $[\theta_{i-1}, \theta_i]\,[\theta_i, \theta_{i+1}]$ satisfies either part 5.a or 5.b of Definition 5. Thus $g_{h+2}(k_{h+1}) - g_{h+2}(j_{h+1}) > 0$ and $g_{h+2}(k_{h+1}) \geqq g_{h+2}(u)$ for each u in $(y_{h+1}, z_{h+1}]$ because the step function g_{h+2} has alternating jumps by C.5.a of the construction and since P_{h+1} has jumps of constant distance on $[y_{h+1}, z_{h+1}]$ by part 2 of Definition 5. Thus

$$[4.3] \qquad g_{t+2}(k_{t+1}) \geqq g_{t+2}(s)\,.$$

Recall that r and s are consecutive ends of A. Thus by the definition of H, a number x in (j, k) is in H if and only if for some positive integer τ, $n + 1 < \tau < t + 1$, x is an end of $E(g_{\tau+1})$ in $(y_\tau, z_\tau]$. But for each τ, the number s is in the leftmost interval $[j_\tau, k_\tau]$ of $E(g_{\tau+1})$ in $[y_\tau, z_\tau]$ and thus no end x of $E(g_{\tau+1})$ in $(y_\tau, z_\tau]$ preceeds k_τ. Thus since $k_{n+1} > k_{n+2} > \ldots > k_{t+1}$, then either k_{t+1} is the right end of the interval of H whose left end is j or $s = k_t = z_{t+1}$ in which case s is the leftmost end of $E(g_{t+1})$ in $(y_t, z_t]$ and thus s is the right end of the interval of H whose left end is j, by the definition of H.

This is true since either $z_{t+1} = s = k_t$ or s is not an end of $E(g_{t+1})$ since s is in the leftmost interval $[j_t, k_t]$ of $E(g_{t+1})$ in $[y_t, z_t]$ and $s \neq y_t$ by the definition of t. Since r is in $[i, j)$ and $s > j$, then either $[j', k'_{t+1}]$ or $[j', s]$ is a member of G not in A with j' in $[r, s)$.

If $t = N - 1$, then by a similar argument k_{N-1} or s is the right end of the interval of H whose left end is j and $g_N(k_{N-1}) \geqq g_N(s)$ which is equation [4.3] with $t = N - 2$. And neither $[j', k'_{N-1}]$ nor $[j', s]$ is in A since $[i, j)$ contains r. Also j' is in $[r, s)$.

Similarly, if for some interval $[y_{n+1}, z_{n+1}]$ of $E(S_{n+1})$, the number s is in an interval of $E(g_{n+2})$ in $[y_{n+1}, z_{n+1}]$ other than the first interval $[j_{n+1}, k_{n+1}]$ of $E(g_{n+2})$ in $[y_{n+1}, z_{n+1}]$, then $[j, k_{n+1}]$ is in H with

$$[4.4] \qquad g_{n+2}(k_{n+1}) \geqq g_{n+2}(s)$$

or $[j, s]$ is in H (if $s = k$ and thus is an end of $E(g_{n+1})$). So either $[j', k'_{n+1}]$ is in G, not in A with $g_{n+2}(k'_{n+1}) \geqq g_{n+2}(s)$ or $[j', s]$ is in G not in A.

Finally, if $n + 1 = N$, and there is no S_{n+1}, then $[j, k]$ is in H and $[j', k']$ is in G.

Thus in Case 1, let $[j, l]$ denote the interval of H with left end j and note that

$$[4.5] \qquad g_N(l') = g_N(l) \geqq g_N(s)$$

by [4.3] and [4.4] since by C.5.c of the construction $g_h(l) = g_N(l) = g_N(l')$, $g_N(s) = g_h(s)$ for each h, $h = 1, 2, \ldots, N$. Also $[j', l']$ is not in A. Now

$$Wf_N[r, s] = |f_N(j) - f_N(i)| = Wf_N[j', l'],$$

by Definitions 5 and 6 and by A.3 and C.5.c of the construction. Also $g_N(j') \leqq g_N(r)$ by [4.2] and $g_N(l') \geqq g_N(s)$ by [4.5].

Thus $|g_N(l') - g_N(j')| \geqq |g_N(s) - g_N(r)|$ and $Wf_N[j', l']\,|g_N(l') - g_N(j')| \geqq Wf_N[r, s]\,|g_N(s) - g_N(r)|$. And $[j', l']$ is in G not in A with j' in $[r, s)$. Thus $[j', l']$ is an interval $[\gamma, \delta]$ with the necessary properties to insure statement 1.

So statement 1 holds for Case 1.

Suppose Case 2. The following shows that $r \neq w$. Suppose $r = w$. Either $g_{n+1}(w) < g_{n+1}(x)$ or $g_{n+1}(w) > g_{n+1}(x)$ by Definition 4. By Definition 5 parts 5.a and 5.b if $g_{n+1}(w) < g_{n+1}(x)$ then $g_{n+1}(w) < g_{n+1}(\mu)$ for each number μ in $(w, x]$. Since $r = w = i$, then $g_{n+1}(i) < g_{n+1}(j)$ which contradicts [4.1]. Also by Definition 5 parts 5.a and 5.b if $g_{n+1}(w) > g_{n+1}(x)$ then $g_{n+1}(w) > g_{n+1}(\mu)$ for each number μ in $(w, x]$. Since $r = w$, then $g_{n+1}(r) > g_{n+1}(s)$ which implies Case 2 does not hold. So $r \neq w$.

Suppose statement 2 is not true. If $s \neq x$, then $Wf_N[j, k] = Wf_N[r, s]$ by Definitions 5 and 6 and thus $Wf_N[j, k]\,|g_N(k) - g_N(j)| \geqq Wf_N[r, s]\,|g_N(s) - g_N(r)|$ since $|g_N(k) - g_N(j)| \geqq |g_N(s) - g_N(r)|$ by Definition 5.

So $[j, k]$ is an interval $[\gamma, \delta]$ with the necessary properties to insure statement 1 is true for Case 2.

If $s = x$ then since $r \neq w$, the rightmost interval of $E(g_{n+1})$ in $[w, x]$ is an interval $[\gamma, \delta]$ of H with the necessary properties to insure statement 1, by Definitions 5 and 6.

Thus either statement 1 or statement 2 is true for case 2.

Suppose Case 3. Suppose s is an end of $E(g_n)$. Let $[t, x]$ denote the righmost interval of $E(g_{n+1})$ in $[w, x]$. Since $g_N(s) < g_N(r)$, then by Definitions 4 and 5, $g_N(x) = \min\{g_N(\mu) : \mu \in (w, x]\}$. Thus S_{n+1} is nonincreasing on $[t, x]$. If r is in $[t, x]$ then let l denote the right end of the interval of $E(S_{n+1})$ which contains r. Note that $[l, x]$ is an interval $[\gamma, \delta]$ with the necessary properties to insure that statement 1 holds since $g_N(l) \geqq g_N(r)$ by Definition 5 with S_{n+1} nonincreasing on $[t, x]$.

If r is in $(w, t]$ then $[t, x] = [t, s]$ is an interval $[\gamma, \delta]$ with the necessary properties to insure that statement 1 is true since $g_N(t) = \max\{g_N(\mu) : \mu \in (w, x]\}$ by Definitions 4 and 5.

If $r = w$ then $[w, x] = [r, s]$ is in G and statement 2 is true.

Now suppose s is not an end of $E(g_n)$. Let $[l, h]$ denote the interval of $E(S_{n+1})$ which contains r. If $[l, h]$ is the rightmost interval of $E(S_{n+1})$ in $E(S_{n+1}) \cdot [i, j]$ then $[j, k]$ is in H since $g_N(s) < g_N(r)$ which implies $k < s$. Thus $[j, k] = [j', k']$ is an interval $[\gamma, \delta]$ with the necessary properties to insure statement 1.

Suppose $[l, h]$ is not the rightmost interval of $E(S_{n+1})$ in $E(S_{n+1}) \cdot [i, j]$. Then $[h, j]$ is an interval $[\gamma, \delta]$ with the necessary properties to insure statement 1.

Thus in Case 3 either statement 1 or statement 2 is true.

So if $[r, s]$ is not a subinterval of an interval of $E(g_{n+1})$, then one of the three statements is true, assuming $g_N(j) < g_N(i)$. A similar proof suffices if $g_N(j) > g_N(i)$.

Now suppose $[r, s]$ is a subinterval of the interval $[i, j]$ of $E(g_{n+1})$ and

$$[4.1] \qquad g_N(j) < g_N(i) .$$

Either $[r, s]$ is not a subinterval of an interval of $E(S_{n+1})$ or $n + 1 = N$ by the definition of n. Suppose $[r, s]$ is not a subinterval of an interval of $E(S_{n+1})$. Then $g_N(s) < g_N(r)$ by [4.1] and Definitions 4 and 5. Let $[P, Q]$ denote the first (leftmost) interval of $E(S_{n+1})$ containing r. The word "first" (or leftmost) is used in case r is an end of an interval of $E(S_{n+1})$.

The following is similar to the proof in case $[r, s]$ is not a subinterval of an interval of $E(g_{n+1})$ and Case 1 is assumed.

Suppose there is no interval $[\gamma, \delta]$ of H with the necessary properties to assure statement 1. Suppose for some interval $[y_{n+1}, z_{n+1}]$ of $E(S_{n+1})$ the number s is in the leftmost number set of $E(g_{n+2}) \cdot (y_{n+1}, z_{n+1}]$ and let t denote the largest integer such that for some interval $[y_t, z_t]$ of $E(S_t)$ the number s is in the leftmost number set of $E(g_{t+1}) \cdot (y_t, z_t]$.

Let $[y_h, z_h]$ denote the interval of $E(S_h)$ such that s is in $(y_h, z_h]$ for each h, $h = n, \ldots, t + 1$, $h < N$. Let $[i_h, j_h]$ denote the leftmost interval of $E(g_{h+1})$ in $[y_h, z_h]$, $h = n + 1, \ldots, t + 1$, $h < N$. Let $[i, j] \equiv [i_n, j_n]$. Either $t = N - 1$ or for some interval $[y_{t+1}, z_{t+1}]$ of $E(S_{t+1})$, the number s is in a number set of $E(g_{t+2}) \cdot (y_{t+1}, z_{t+1}]$ other than the leftmost number set of $E(g_{t+2}) \cdot (y_{t+1}, z_{t+1}]$.

Suppose the latter.

With the following two statements mathematical induction yields that $g_{t+2}(j_{t+1}) \leqq g_{t+2}(\mu)$ for each μ in $(y_{t+1}, z_{t+1}]$.

First, note that by 4.1 $g_{n+1}(j) < g_{n+1}(i)$, and thus $g_{n+1}(j) \leqq g_{n+1}(\mu)$ for each μ in $(y_n, z_n] = (w, x]$ because on $(w, x]$ the step function g_{n+1} has alternating jumps of constant distance by C.5.a of the construction and the fact that P_n has jumps of constant distance on $[w, x]$ by part 2 of Definition 5.

Second, if for some h, $h = n, \ldots, t$, $g_{h+1}(j_h) \leqq g_{h+1}(\mu)$ for each μ in $(y_h, z_h]$ then $g_{h+2}(j_{h+1}) \leqq g_{h+2}(\mu)$ for each μ in $(y_{h+1}, z_{h+1}]$.

To show that this second statement holds suppose h is a positive integer $h = n, \ldots, t$ such that $g_{h+1}(j_h) \leqq g_{h+1}(\mu)$ for each μ in $(y_h, z_h]$. Thus $g_{h+1}(j_h) \leqq g_{h+1}(i_h)$ since g_{h+1} has alternating jumps on $E(g_{h+1})$ by C.5.a of the construction and since g_{h+1} is not constant on $(y_h, z_h]$ by Definition 5.

Therefore S_{h+1} is nonincreasing on $(j_h, k_h]$ by part 2.b of Definition 4. Let $(\theta_i)_{i=1}^{v}$ denote the ends of $E(S_{h+1})$ in $[j_h, k_h]$ and let $[\theta_0, \theta_1]$ denote the interval of $E(S_{h+1})$ with right end $\theta_1 = j_h$. Since g_{h+1} has alternating jumps on the subdivision $E(g_{h+1})$, then by Definition 4, $(g_{h+1} + S_{h+1})(\theta_0) < (g_{h+1} + S_{h+1})(\theta_1)$. Thus by Definition 5 part 5.b, $P_{h+1}(\theta_1) = 0$ since $[\theta_0, \theta_1]$, $[\theta_1, \theta_2]$ satisfies the conditions of 5.b.

Now note that $[i_{h+1}, j_{h+1}]$ is the leftmost interval of $E(g_{h+2}) \cdot [\theta_i, \theta_{i+1}]$ for some i, $i = 1, 2, \ldots, v - 1$ with $i_{h+1} = \theta_i$. Suppose this $i = 1$ and $i_{h+1} = \theta_1$. The interval pair $[\theta_0, \theta_1]$ $[\theta_1, \theta_2]$ satisfies the conditions of 5.b of Definition 5. So $P_{h+1}(\theta_1) = 0$, and

$$g_{h+1}(\theta_1) = g_{h+1}(i_{h+1}) = g_{h+2}(i_{h+1}),$$

by the definition of g_{h+2}. Also since $[\theta_1, \theta_2]$, $[\theta_2, \theta_3]$ satisfies the conditions of 5.a of Definition 5, then

$$g_{h+2}(j_{h+1}) < g_{h+2}(i_{h+1}) = (g_{h+1} + S_{h+1})(\theta_1)$$

since j_{h+1} is in $[\theta_1, \theta_2]$ and since S_{h+1} is nonincreasing on $[i_h, j_h]$ with

$$(g_{h+1} + S_{h+1})(\theta_1) > (g_{h+1} + S_{h+1})(\theta_2).$$

Now suppose this $i \neq 1$ and $i_{h+1} \neq \theta_1$. Then

$$g_{h+2}(\theta_i) = g_{h+2}(i_{h+1}) \geqq (g_{h+1} + S_{h+1})(\theta_i)$$

since $[\theta_{i-2}, \theta_{i-1}]$ $[\theta_{i-1}, \theta_i]$ satisfies either part 5.a or part 5.b of Definition 5. Also $(g_{h+1} + S_{h+1})(\theta_i) > g_{h+2}(j_{h+1})$ since $[\theta_{i-1}, \theta_i]$ $[\theta_i, \theta_{i+1}]$ satisfies either part 5.a or 5.b of Definition 5. Thus $g_{h+2}(j_{h+1}) - g_{h+2}(i_{h+1}) < 0$ and $g_{h+2}(i_{h+1}) \leqq$ $\leqq g_{h+2}(\mu)$ for each μ in $(y_{h+1}, z_{h+1}]$ because the step function g_{h+2} has alternating jumps by C.5.a of the construction and since P_{h+1} has jumps of constant distance on $[y_{h+1}, z_{h+1}]$ by part 2 of Definition 5. Thus,

$$g_{t+2}(j_{t+1}) = \min \{g_{t+2}(\mu) : \mu \text{ in } (y_{t+1}, z_{t+1}]\}$$

by Definition 5. Thus

[4.6] $$g_{t+2}(j_{t+1}) \leqq g_{t+2}(s).$$

And either j_{t+1} or s is the right end of the interval of H whose left end is Q by the definition of H. The interval $[Q, s]$ is in H if $[y_{t+1}, z_{t+1}]$ is the rihgtmost interval of $E(S_{t+1}) \cdot [i_t, j_t]$ and $s = j_t$. Also Q is in $[r, s)$.

If $t = N - 1$, then by a similar argument j_{N-1} or s is the right end of the interval of H whose left end is Q and $g_N(j_{N-1}) \leqq g_N(s)$ which is equation [4.6] with $t =$ $= N - 2$. Also Q is in $[r, s)$ since r is in $[P, Q]$.

Similarly, if for some interval $[y_{n+1}, z_{n+1}]$ of $E(S_{N+1})$, the number s is in an interval of $E(g_{n+2})$ in $[y_{n+1}, z_{n+1}]$ other than the first interval $[i_{n+1}, j_{n+1}]$ of $E(g_{n+2})$ in $[y_{n+1}, z_{n+1}]$, then $[Q, j_{n+1}]$ is in H with

[4.7] $$g_{n+2}(j_{n+1}) \geqq g_{n+2}(s)$$

or $[Q, s]$ is in H if $[y_{n+1}, z_{n+1}]$ is the rightmost interval of $E(S_{n+1}) \cdot [i, j]$ and $s=j$.

Thus if $[r, s]$ is a subinterval of the interval $[i, j]$ of $E(g_{n+1})$ and $[r, s]$ is not a subinterval of an interval of $E(S_{n+1})$ let $[Q, l]$ denote the interval of H with left end Q and note that

$$[4.8] \qquad g_N(l) \leqq g_N(s)$$

by (4.6) and [4.7] since by C.5.c of the construction $g_h(l) = g_N(l)$ for each h, $h = 1, 2, \ldots, N$.

Also $g_N(Q) \geqq g_N(r)$. (If $r = w$, then $Q = r$.) Thus either $[Q, l] = [r, s]$ which implies statement 2 or $[Q, l]$ is an interval $[\gamma, \delta]$ with the necessary properties to insure statement 1.

Finally, if $g_N(j) > g_N(i)$ a similar proof suffices.

If $n + 1 = N$, then since $[r, s]$ is a subinterval of $[i, j]$, either $[r, s]$ is a proper subset of $[i, j]$ or $[r, s] = [i, j]$. If $[r, s]$ is a proper subset of $[i, j]$ then either $Wf_N[r, s] = 0$ or $g_N(s) - g_N(r) = 0$ by Definitions 4 and 5 and the construction of (f_N, g_N). If $[r, s] = [i, j]$ then $[r, s]$ is in H by the definition of $[r, s]$ and of G. So if $n + 1 = N$ then either statement 2 or statement 3 follows.

Thus if $[r, s]$ is an interval of A and [4.1] holds, then one of statements 1, 2 and 3 holds.

Finally if $g_N(j) > g_N(i)$ a similar proof suffices.

Definition. Let H_n denote the subdivision such that x is an end of H_n if and only if x is an end of $E(g_n)$ which is an end of H, $n = 1, 2, \ldots, N$. Recall that $E(g_n)$ is a refinement of $E(g_i)$ if $i < n$.

Definition. Recall that d denotes a positive number which is less than half the shortest length of an interval in $E(g_N)$.

Definition. For each positive integer n, $n = 1, 2, \ldots, N$, let H'_n denote the collection of subdivisions of $[a, b]$ such that a subdivision D is in H'_n if and only if

1. each end of D is either an end x of H_n or is $x - d$ with x and end of H_n and
2. the set of ends of D contains one and only one term from each number pair in the set $\{(x, x - d) : x \text{ an end of } H_n\}$.

Note that G is in H'_N.

Lemma 5. *Suppose n is a positive integer, $n = 1, 2, \ldots, N - 1$, and D is a member of H'_{n+1}. Then there is a member D_n of H'_n such that $[D_n, f_N, g_N] > O[D, f_N, g_N] - \varepsilon/2N$.*

Proof of Lemma 5. Suppose D is in H'_{n+1}. Let $[p, q]$ denote an interval of H_n. Let p_1 denote the end of D closest to p. Thus p_1 is either p or $p - d$. Let q_1 denote the end of D closest to q. Thus q_1 is either q or $q - d$. Now let p_2 denote p_1 if p is an end of $E(g_{n-1})$ and let p_2 denote p otherwise. And similarly, let q_2 denote q_1 if q is an end of $E(g_{n-1})$ and let q_2 denote q otherwise.

Let w denote the first end of D in $(p, q]$.

The following is proved and then used to prove 5.5, which is then used to prove Lemma 5.

[5.1] $$Wf_N[p_2, q]\,|g_N(q) - g_N(w)| \geqq O[D\,.\,[w, q_1], f_N, g_N] - \frac{3\varepsilon}{8Nt_n}\,.$$

In the following proof of this inequality 5.1 refer to Figure 5.

If there is only one interval in $D\,.\,[p_1, q_1]$, then 5.1 is true since in this case, $w = q_1$ and thus

$$Wf_N[p_2, q]\,|g_N(q) - g_N(w)| = 0 \quad \text{and} \quad O[D\,.\,[w, q_1], f_N, g_N] \equiv 0\,.$$

Suppose there is more than one member in $D\,.\,[p_1, q_1]$. Let $[w, x]$ denote the interval of $D\,.\,[p_1, q_1]$ with left end w. And let $[y, q_1]$ denote the interval of $D\,.\,.\,[p_1, q_1]$ with right end q_1.

Let k denote the number of intervals in $E(S_n)\,.\,[p, q]$ and let $\{G_i\}_{i=1}^k = \{a, b_i\}_{i=1}^k$ denote the set of intervals of $E(S_n)\,.\,[p, q]$. Let j_1 denote the positive integer such that G_j contains w. Let j_2 denote the positive integer such that G_{j_2} contains y. (Usual ordering is assumed.)

Now note that

$$O[E(P_n)\,.\,[p, q], P_{nR}, P_n] = Wf_N[p, q]\,|g_N(q) - g_N(p)|$$

by C.4.b of the construction of f_N, g_N. Also if h is a positive integer, $h \leqq k$, then

[5.2] $$O[E(P_n)\,.\,G_h, P_{nR}, P_n] \leqq \frac{Wf_n[p, q]\,|g_n(q) - g_n(p)|}{k} + \frac{\varepsilon}{8w_nNt_n}$$

by C.4.c of the construction and Definition 7. Also $O[E(g_n + S_n + P_n)\,.\,G_h, f_N, g_N] \leqq \leqq O[E(P_n)\,.\,G_h, P_{nR}, P_n] + \varepsilon/8w_nNt_n$ since if $[s, t]$ is in $E(g_{n+1}) = E(g_n + S_n + P_n)$ then $Wf_{n+1}[s, t]\,|g_{n+1}(t) - g_{n+1}(s)| < \varepsilon/16w_nNt_n$ by C.4.a and the difference between $O[E(g_n + S_n + P_n)\,.\,G_h, f_N, g_N]$ and $O[E(P_n)\,.\,G_h, P_{nR}, P_n]$ depends only on the intervals of $E(g_n + S_n + P_n)$ at each end of G_h. So if h is a positive integer, $h \leqq k$, then

[5.3] $$O[E(g_n + S_n + P_n)\,.\,G_h, f_N, g_N] \leqq \frac{Wf_n[p, q]\,|g_n(q) - g_n(p)|}{k} + \frac{\varepsilon}{4w_nNt_n}\,.$$

Let z denote the absolute value of the jumps of S_n in $[p, q]$. See Figure 5. Note that $Wf_N[p_2, q] \geqq Wf_N[p, q]$. Thus

$$Wf_N[p_2, q]\,|g_N(q) - g_N(w)| \geqq Wf_N[p, q]\,(k - j_1)\,z =$$
$$= Wf_N[p, q]\,(k - j_2)\,z + Wf_N[p, q]\,(j_2 - j_1)\,z\,.$$

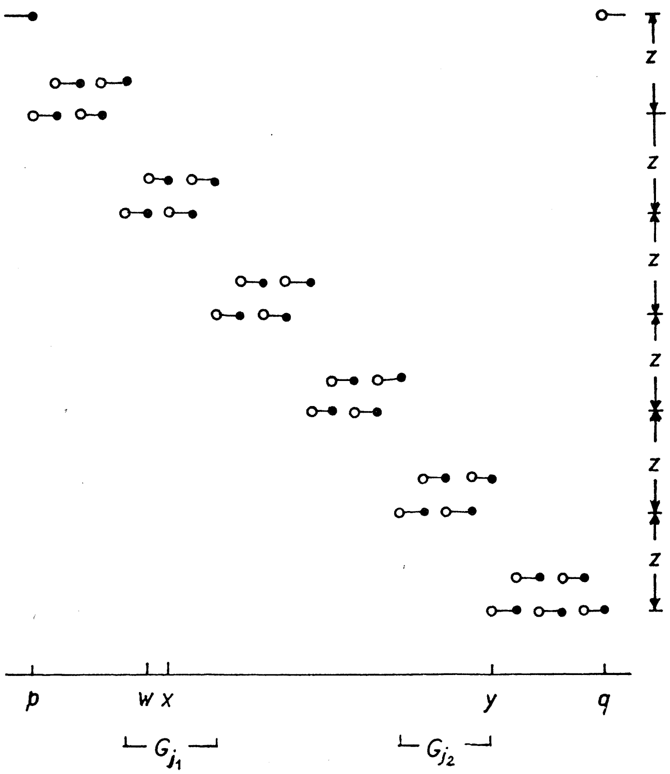


Fig. 5.

Also

[5.4]
$$O[D\,.\,[w, q_1], f_N, g_N] \leqq$$
$$\leqq \sum_{i=j_1}^{j_2} \{O[E(P_n)\,.\,G_i, f_N, g_N]\} + Wf_N[p, q]\,(k - j_2)\,z$$

since $O[D\,.\,[w, y], f_N, g_N] \leqq O[E(P_n)\,.\,[a_{j_1}, b_{j_2}], f_N, g_N]$, and $O[D\,.\,[y, q_1], f_N, g_N] \leqq \leqq Wf_N[p, q]\,|g_N(q_1) - g_N(y)|$, with $D\,.\,[y, q_1]$ containing only the interval $[y, q_1]$ by the definitions of H, H'_{n+1} and D. Thus

$$O[D\,.\,[w, q_1], f_N, g_N] \leqq$$
$$\leqq \left[\frac{Wf_N[p, q]\,|g_N(q) - g_N(p)|}{k} + \frac{\varepsilon}{4w_n N t_n}\right](1 + j_2 - j_1) + Wf_N[p, q]\,(k - j_2)\,z$$

by [5.3] and [5.4]. And since $|g_N(q) - g_N(p)|/k = z$ by Definition 4 and

$$\frac{Wf_N[p, q]\,|g_N(q) - g_N(p)|}{k} = (Wf_N[p, q])\,z < \frac{\varepsilon}{8Nt_n}$$

by C.2 of the construction, then

$$O[D\,.\,[w, q_1], f_N, g_N] \leqq (Wf_N[p, q])\,(k - j_1)\,z + $$
$$+ \frac{\varepsilon}{8Nt_n} + \frac{(1 + j_2 - j_1)\,\varepsilon}{4w_nNt_n} \leqq Wf_N[p, q]\,(k - j_1)\,z + \frac{3\varepsilon}{8Nt_n}$$

since $1 + j_2 - j_1 \leqq w_n$.

This ends the proof of 5.1 since $Wf_N[p_2, q] \geqq Wf_N[p, q]$ and thus $Wf_N[p_2, q]\,.\,.\,|g_N(q) - g_N(w)| \geqq Wf_N[p, q]\,(k - j_j)\,z$.

Since 5.1 is true and $O[D\,.\,[p_1, q_1], f_N, g_N] = Wf_N[p_1, w]\,|g_N(w) - g_N(p_1)| + + O[D\,.\,[w, q_1], f_N, g_N]$ and $Wf_N[p_2, q]\,|g_N(q) - g_N(p_2)| = Wf_N[p_2, q]\,|g_N(w) - - g_N(p_2)| + Wf_N[p_2, q]\,|g_N(q) - g_N(w)|$ and since $Wf_N[p_1, w]\,|g_N(w) - g_N(p_1)| \leqq \leqq Wf_N[p_2, q]\,|g_N(w) - g_N(p_2)|$ then by 5.1,

$$[5.5] \qquad Wf_N[p_2, q]\,|g_N(q) - g_N(p_2)| \geqq O[D\,.\,[p_1, q_1], f_N, g_N] - \frac{3\varepsilon}{8Nt_n}\,.$$

Note that $\{[p_2, q_2] : [p, q]$ in $H_n\} \equiv D_n$ is a subdivision of $[a, b]$ and that $[p_2, q] = [p_2, q_2]$ unless q is an end of $E(g_{n-1})$ and thus

$$[5.6] \qquad Wf_N[p_2, q_2]\,|g_N(q_2) - g_N(p_2)| \geqq O[D\,.\,[p_1, q_1], f_N, g_N] - \frac{3\varepsilon}{8Nt_n}$$

by [5.5] for each interval $[p, q]$ of H_n with q not an end of $E(g_{n-1})$.

Finally note that if q is an end of $E(g_{n-1})$ then $[p_2, q] = [p, q]$ and thus

$$[5.7] \qquad Wf_N[p, q]\,|g_N(q) - g_N(p)| \geqq O[D\,.\,[p_1, q_1], f_N, g_N] - \frac{3\varepsilon}{8Nt_n}\,,$$

for each interval $[p, q]$ of H_n, by [5.5].

Now let

$W = \sum\{Wf_N[p_2, q_2]\,|g_N(q_2) - g_N(p_2)| : [p, q]$ in H_n with q not an end of $E(g_{n-1})\}$,
$X = \sum\{Wf_N[p_2, q_2]\,|g_N(q_2) - g_N(p_2)| : [p, q]$ in H_n with q an end of $E(g_{n-1})\}$,
$U = O[E_n, f_n, g_n] = \sum\{Wf_N[p, q]\,|g_N(q) - g_N(p)| : [p, q]$ in H_n with q an end of $E(g_{n-1})\}$,
$Y = \sum\{O[D\,.\,[p_1, q_1], f_N, g_N] : [p, q]$ in H_n with q not an end of $E(g_{n-1})\}$, and
$Z = \sum\{O[D\,.\,[p_1, q_1], f_N, g_N] : [p, q]$ in H_n with q an end of $E(g_{n-1})\}$.

Note that $O[D_n, f_n, g_n] = W + X$ and $O[D, f_N, g_N] = Y + Z$. Now $W \geqq Y - (3\varepsilon/8Nt_n)\,j$ with j the number of intervals $[p, q]$ in H_n with q not an end of $E(g_{n-1})$, by 5.6.

And $U \geqq Z - (3\varepsilon/8Nt_n)\,i$ with i the number of intervals $[p, q]$ of H_n with q an end of $E(g_{n-1})$, by 5.7.

$W + U \geqq Y + Z - [(3/8Nt_n)\,(j + i)]$ and since $(j + i) \leqq t_n$, by C.1 of the construction, then

$$W \geqq Y + Z - \left[\left(\frac{3\varepsilon}{8Nt_n}\right) t_n + U\right].$$

$W > Y + Z - (3\varepsilon/8N + \varepsilon/8N) = Y + Z - \varepsilon/2N$, since $U < \varepsilon/8N$ by C.1 and C.4.a of the construction and the definition of U.

Thus Lemma 5 holds, since X is positive:

$$W + X > Y + Z - \frac{\varepsilon}{2N}.$$

To show that $O[A, f_N, g_N] < \varepsilon$, assume Lemma 4 and Lemma 5.

Lemma 5 implies the existence of a sequence of subdivisions $D_1, D_2, D_3, \ldots, D_{N-1}$ such that D_n is in H'_n for each n, $n = 1, 2, \ldots, N - 1$ and

$$\begin{aligned} O[G, f_N, g_N] &< O[D_{N-1}, f_N, g_N] + \frac{\varepsilon}{2N} \\ &< O[D_{N-2}, f_N, g_N] + \frac{\varepsilon}{2N} + \frac{\varepsilon}{2N} \\ &\vdots \qquad\qquad\qquad (N - 1 \text{ times}) \\ &< O[D_1, f_N, g_N] + \frac{\varepsilon}{2N} + \frac{\varepsilon}{2N} + \frac{\varepsilon}{2N} + \ldots + \frac{\varepsilon}{2N} \\ &< O[D_1, f_N, g_N] + \tfrac{1}{2}\varepsilon . \end{aligned}$$

Since D_1 is in H'_1, then $O[D_1, f_N, g_N] = O[D_1, f_1, g_1] \leqq \frac{1}{2}\varepsilon$. Thus $O[G, f_N, g_N] < \varepsilon$ and thus $O[A, f_N, g_N] < \varepsilon$ by Lemma 4. Note that, if s is a positive integer less than N, then the previous proof uses

$$O[D_{N-s}, f_N, g_N] = O[D_{N-s}, f_{N-s}, g_{N-s}]$$

by C.5.c of the construction and Definitions 4 and 5.

The following Lemma 6 implies that if A is a subdivision, then there is a subdivision B such that $O[A, f, g] \leqq O[B, f_N, g_N]$ and thus $O[A, f, g] < \varepsilon$.

*__Lemma 6.__ *Suppose (F, G) is an ordered step function pair with coordinated alternating jumps on the subdivision $E(G)$ of the interval $[a, b]$.*

Let f, g denote the ordered function pair such that for each interval $[p, q]$ of $E(G)$ the ordinate $f(x) = F(x)$ if x is in $[p, \frac{1}{2}(p + q)]$ and $f(x) = F(q) + 2((q - x)/(q - p))(F(p) - F(q))$ if x is in $[\frac{1}{2}(p + q), q]$ and such that for each $[p, q]$ in $E(G)$ the ordinate $g(x) = G(p) + 2((x - p)/(q - p))(G(q) - G(p))$ if x is in $[p, \frac{1}{2}(p + q)]$ and $g(x) = G(x)$ if x is in $[\frac{1}{2}(p + q), q]$.

Suppose A is subdivision of $[a, b]$. Then there exists a subdivision B of $[a, b]$ such that $O[A, f, g] \leqq O[B, F, G]$.

*Proof of Lemma 6. Assume the hypothesis and notation of the lemma.

In order to prove Lemma 6 it is shown that there exists a subdivision B of $[a, b]$ such that

1. if $[p, q]$ is in $E(G)$, then there is no end of B in the number set $(p, \frac{1}{2}(p + q)) + (\frac{1}{2}(p + q), q)$ and
2. $O[A, f, g] \leqq O[B, f, g]$.

To prove there is such a B, it is shown that if D is a subdivision of $[a, b]$ and $[p, q]$ is an interval of $E(G)$ then there is a subdivision D' of $[a, b]$ such that

1. there is no end of D' in $(p, \frac{1}{2}(p + q)) + (\frac{1}{2}(p + q), q)$ and
2. $O[D, f, g] \leqq O[D', f, g]$.

Suppose D is a subdivision of $[a, b]$ and $[p, q]$ is an interval of $E(G)$ such that

[6.1] $$g(q) > g(p).$$

(A similar argument holds if $g(q) < g(p)$.)

Suppose some end of D is in the segment $(p, \frac{1}{2}(p + q))$. Let $[r, s]$ denote the first interval of D which has a point in $[p, \frac{1}{2}(p + q)]$ and let $[t, u]$ denote the last interval of D which has a point in $[p, \frac{1}{2}(p + q)]$.

The following is a proof that if $[p, q]$ is an interval of $E(g)$ then there is a subdivision D' of $[a, b]$ such that no end of D' is in $(p, \frac{1}{2}(p + q))$ and $O[D', f, g] \geqq O[D, f, g]$. To facilitate this proof a subdivision of the interval $[r, u]$ is defined and a function V from ordered real number pairs into the reals is defined.

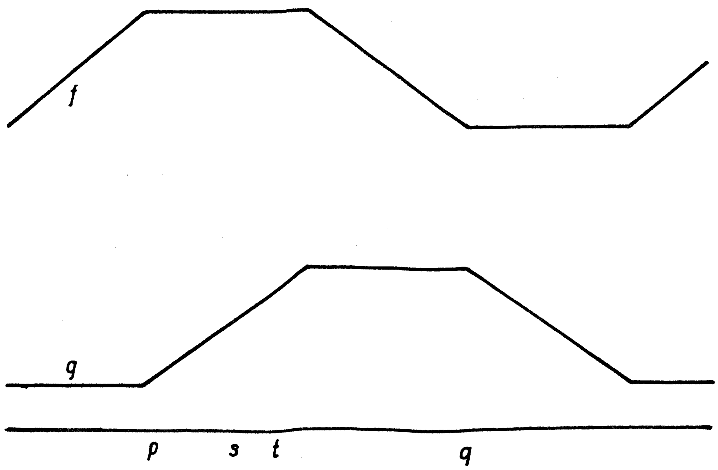


Fig. 6.

Let D_1 denote $D \,.\, [r, u]$. It is apparent that

$$O[D - D_1 + \{[r, s], [s, t], [t, u]\}, f, g] = O[D, f, g]$$

since f is constant and g is nondecreasing on the interval $[s, t]$ by hypothesis. See Figure 6. (If $s = t$, then $[s, t]$ is deleted and $D - D_1 + \{[r, s], [t, u]\} = D$.)

Let V denote the continuous function such that

$$[6.2] \qquad V(x, y) = O[D - D_1 + \{[r, x], [x, y], [y, u]\}, f, g]$$

with closed domain all ordered real number pairs (x, y) such that $x \in [p, \frac{1}{2}(p + q)]$, $y \in [p, \frac{1}{2}(p + q)]$ and $x \leqq y$. (In the expression $O[D - D_1 + \{[r, x], [x, y], [y, u]\}, f, g]$ if $x = y$ then $[x, y]$ is deleted, if $r = x = a$ then $[r, x]$ is deleted.)

The following is a proof that the inverse image I of max $V(x, y)$ contains one of the following three points: (p, p), $(p, \frac{1}{2}(p + q))$, $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$.

In this proof, for each of the inequalities concerning values of V the following two steps result in a proof of the inequality.

Step 1. Note that the inequality could be written in the form

$$\sum_{i=1}^{n} Wf[p_i, q_i]\, |g(q_i) - g(p_i)| \leqq \sum_{i=1}^{n} Wf[p_i', q_i']\, |g(q_i') - g(p_i')| \,.$$

Step 2. For each positive integer i such that $[p_i, q_i] \neq [p_i', q_i']$ compare the two numbers

$$Wf[p_i, q_i]\, |g(q_i) - g(p_i)| \quad \text{and} \quad Wf[p_i', q_i']\, |g(q_i') - g(p_i')| \,,$$

noting that

1. g is nondecreasing on $[p, q]$ by $[6.1]$ and the definition of g
2. f is constant on $[p, \frac{1}{2}(p + q)]$ by definition.

Suppose (x, y) is a point in I, with $x = y$ and with x in the segment (open interval) $(p, \frac{1}{2}(p + q))$. Either $Wf[r, x] \geqq Wf[x, u]$ or $Wf[r, x] < Wf[x, u]$.

If $Wf[r, x] \geqq Wf[x, u]$ then if $g(r) > g(x)$, the point (p, p) is in I since $V(p, p) \geqq V(x, x)$ and if $g(r) \leqq g(x)$, the point $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in I since $V(\frac{1}{2}(p + q), \frac{1}{2}(p + q)) \geqq V(x, x)$.

Similarly if $Wf[r, x] < Wf[x, u]$ then if $g(u) > g(x)$, the point (p, p) is in I since $V(p. p) \geqq V(x, x)$ and if $g(u) \leqq g(x)$, the point $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in I since $V(\frac{1}{2}(p + q), \frac{1}{2}(p + q)) \geqq V(x, x)$.

Thus if there is a number x in the segment $(p, \frac{1}{2}(p + q))$ such that (x, x) is in I, then either the point (p, p) is in I or the point $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in I.

Now suppose (x, y) is a point in I with $x < y$. Either $g(u) < g(y)$, or $g(u) \geqq g(y)$.

If $g(u) \geqq g(y)$, then $V(x, x) \geqq V(x, y)$ and (x, x) is in I and thus either the point (p, p) is in I or the point $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in I.

Suppose $g(u) < g(y)$. Then $V(x, \frac{1}{2}(p + q)) \geqq V(x, y)$ and $(x, \frac{1}{2}(p + q))$ is in I. Either $g(r) \leqq g(x)$ or $g(r) > g(x)$.

If $g(r) \leqq g(x)$, then $V(\frac{1}{2}(p + q), \frac{1}{2}(p + q)) \geqq V(x, \frac{1}{2}(p + q))$ and $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in I. If $g(r) > g(x)$ then $V(p, \frac{1}{2}(p + q)) \geqq V(x, \frac{1}{2}(p + q))$ and the point $(p, \frac{1}{2}(p + q))$ is in I.

Thus I contains one of the following points: (p, p), $(p, \frac{1}{2}(p + q))$ and $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ and since $O[D - D_1 + \{[r, s], [s, t], [t, u]\}, f, g] = O[D, f, g]$ then one of the following is true:

$$O[D, f, g] \leqq O[D - D_1 + \{[r, p], [p, u]\}, f, g],$$

$$O[D, f, g] \leqq O[D - D_1 + \{[r, p], [p, \tfrac{1}{2}(p + q)], [\tfrac{1}{2}(p + q), u]\}, f, g],$$

and

$$O[D, f, g] \leqq O[D - D_1 + \{[r, \tfrac{1}{2}(p + q)], [\tfrac{1}{2}(p + q), u]\}, f, g].$$

So there exists a subdivision D' of $[a, b]$ such that no end of D' is in the segment $(p, \frac{1}{2}(p + q))$ and $O[D', f, g] \geqq O[D, f, g]$.

Now suppose D is a subdivision of $[a, b]$ and $[p, q]$ is an interval of $E(G)$ such that some end of D is in the segment $(\frac{1}{2}(p + q), q)$. Let $[r, s]$ denote the first interval of D which has a point in $[\frac{1}{2}(p + q), q]$ and let $[t, u]$ denote the last interval of D which has a point in $[\frac{1}{2}(p + q), q]$.

The following is a proof that there is a subdivision D' of $[a, b]$ such that no end of D' is in $[\frac{1}{2}(p + q), q]$ and $O[D', f, g] \geqq O[D, f, g]$.

Let D_1 denote $D \,.\, [r, u]$. It is apparent that

$$O[D - D_1 + \{[r, s], [s, t], [t, u]\}, f, g] = O[D, f, g].$$

(If $s = t$ then $[s, t]$ is deleted.)

Let V denote the continuous function such that $V(x, y) = O[D - D_1 + \{[r, x], [x, y], [y, u]\}, f, g]$ with closed domain all (x, y) such that $x \in [\frac{1}{2}(p + q), q]$, $y \in [\frac{1}{2}(p + q), q]$ and $x \leqq y$. (If $x = y$ then $[x, y]$ is deleted and if $y = u = b$ then $[y, u]$ is deleted.)

The following is a proof that the inverse image P of max $V(x, y)$ contains one of the following two points: $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ and (q, q).

In this proof, which is similar to the preceeding, each time an inequality concerning two points of V arises note the following:

1. g is constant on $[\frac{1}{2}(p + q), q]$ by definition
2. f is non-increasing on $[p, q]$ by definition and [6.1].

Suppose (x, y) is a point in P, with $x = y$ and with x in the segment $(\frac{1}{2}(p + q), q)$. Either $|g(x) - g(r)| > |g(u) - g(x)|$ or $|g(x) - g(r)| \leqq |g(u) - g(x)|$.

If $|g(x) - g(r)| > |g(u) - g(x)|$ then if $Wf[r, \frac{1}{2}(p + q)] < Wf[r, x]$ the point (q, q) is in P since $V(q, q) \geqq V(x, x)$ and if $Wf[r, \frac{1}{2}(p + q)] = Wf[r, x]$ the point $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in P since $V(\frac{1}{2}(p + q), \frac{1}{2}(p + q)) \geqq V(x, x)$.

Similarly if $|g(x) - g(r)| \leqq |g(u) - g(x)|$ then if $Wf[x, u] > Wf[q, u]$ the point $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in P since $V(\frac{1}{2}(p + q), \frac{1}{2}(p + q)) \geqq V(x, x)$ and if $Wf[x, u] = Wf[q, u]$, the point (q, q) is in P since $V(q, q) \geqq V(x, x)$.

Thus if there is a number x in the segment $(\frac{1}{2}(p + q), q)$ such that (x, x) is in P, then either $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ is in P or (q, q) is in P.

Suppose (x, y) is a point in P with $x < y$. Then $V(x, x) \geqq V(x, y)$ and thus (x, x) is in P. Therefore either $(\frac{1}{2}(p + q), \frac{1}{2}(p + q))$ or (q, q) is in P. This ends the proof that there exists a subdivision D' such that no end of D' is in the segment $(\frac{1}{2}(p + q), q)$ and $O[D', f, g] \geqq O[D, f, g]$.

Therefore if D is a subdivision of $[a, b]$ and $[p, q]$ is in $E(G)$, there exists a subdivision D' of $[a, b]$ such that D' has no ends in the segment $(p, \frac{1}{2}(p + q))$ nor the segment $(\frac{1}{2}(p + q), q)$ and $O[D', f, g] \geqq O[D, f, g]$.

It follows from this that if A is a subdivision of $[a, b]$ then there exists a subdivision B of $[a, b]$ such that if $[p, q]$ is in $E(G)$ then no end of B is in the segment $(p, \frac{1}{2}(p + q))$ nor in the segment $(\frac{1}{2}(p + q), q)$ and $O(A, f, g) \leqq O[B, f, g] = O[B, F, G]$.

III. PROOF OF THEOREM 2

The following construction is a proof that Theorem 2 follows from Theorem 1. See Figure 7.

For each positive integer n, let (F_n, G_n) denote an ordered pair of continuous functions on the closed interval $I_n = [1/2n, 1/(2n - 1)]$ such that $\int F_n \,\mathrm{d}G_n > 1$ on I_n, such that $O[D, F_n, G_n] < (\frac{1}{2})^n$ for each subdivision D of I_n and such that each of $F_n(x)$ and $G_n(x)$ is in $[1/(n + 1), 1/n]$ for each x in I_n.

Let (F, G) denote the ordered function pair with x-projection the interval $[0, 1]$ such that if $n > 0$, then (1) $F(x) = F_n(x)$ and $G(x) = G_n(x)$ for each x in I_n, and (2) if x is in $[1/(2n + 1), 1/2n]$ then the point $(x, F(x))$ is on the straight line connecting the point $(1/(2n + 1), F_{n+1}(1/(2n + 1)))$ and the point $(1/2n, F_n(1/2n))$, and (3) the point $(x, G(x))$ is on the straight line connecting the point $(1/(2n + 1), G_{n+1}(1/(2n + 1)))$ and the point $(1/2n, G_n(1/2n))$ for each x in $[1/(2n + 1), 1/2n]$ and (4) $(0, 0)$ is a point of each of F and G.

This pair of functions (F, G) is an ordered pair having the following properties on the interval $[0, 1]$:

(1) each of F and G is continuous on $[0, 1]$,

(2) $\int_0^1 F \,\mathrm{d}G$ does not exist, and

(3) the function pair (F, G) satisfies the weak-Cauchy condition.

To show that (3) is true, assume $0 < \varepsilon < 1$ and let N denote a positive integer such that $12 < N\varepsilon$ and such that $\sum_{i=N}^{\infty} (\frac{1}{2})^i < \frac{1}{3}\varepsilon$. Note that $\int_a^1 F \,\mathrm{d}G$ exists for each a in $(0, 1)$.

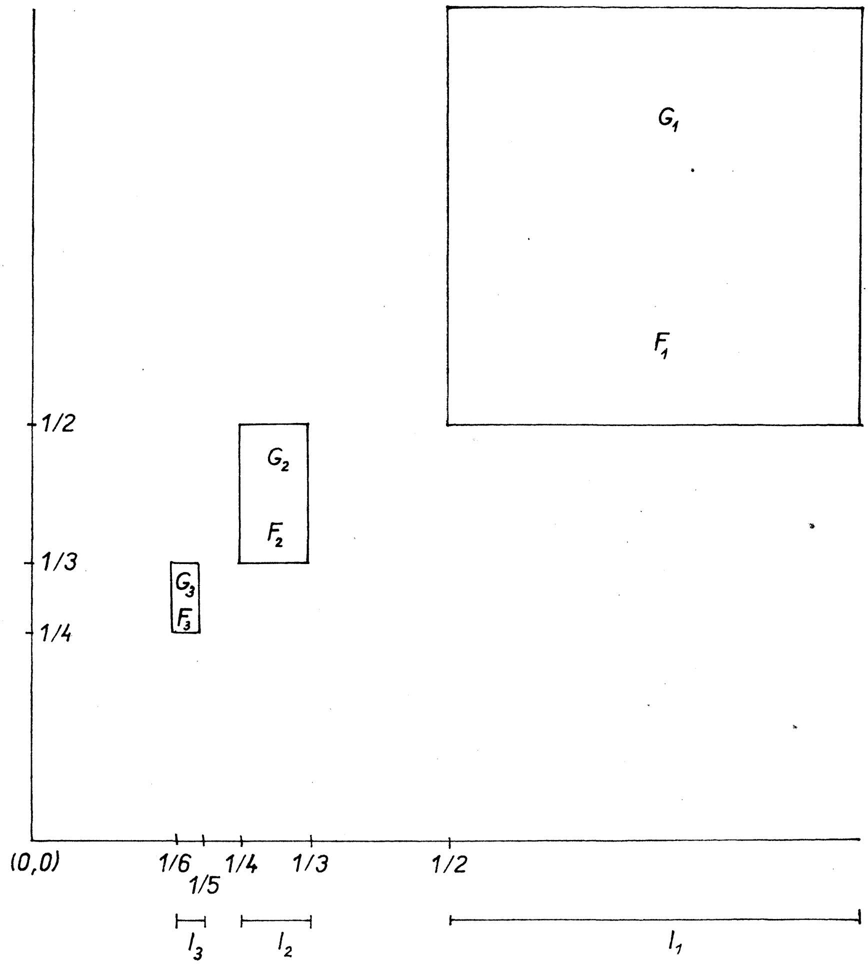


Fig. 7.

Let δ denote a positive number such that if E is a subdivision of $[1/(2N+1), 1]$ with norm less than δ, then $O[E, F, G] < \frac{1}{3}\varepsilon$ and such that

$$\delta < \frac{1}{2}\left(\frac{1}{2N} - \frac{1}{2N+1}\right).$$

Let D denote a subdivision of $[0, 1]$ with norm less than δ. Let t denote an end of an interval of D such that t is in $[1/(2N+1), 1/2N]$.

The following subsets of D are used to show that $O[D, F, G] < \varepsilon$.

$D_1 = D \,.\, [t, 1]$.

$D_2 = \{[p, q] : [p, q] \in D,\ [p, q] \subset [0, t]$

$[p, q] \not\subset I_n$ for each n,

$q \in I_n$ for some n, $p \neq 0\}$.

$D_3 = \{[p, q] : [p, q] \in D,\ [p, q] \subset [0, t]$

$[p, q] \not\subset I_n$ for each n,

$p \in I_n$ for some $n\}$,

$D_4 = \{[p, q] : [p, q] \in D,\ [p, q] \subset [0, t]$,

$[p, q] \not\subset I_n$ for each n,

$[p, q] \notin D_2,\ [p, q] \notin D_3,\ p \neq 0\}$.

$D_6 = \{[p, q] : [p, q] \in D,\ [p, q] \subset I_n$ for some n,

$[p, q] \subset [0, t]\}$.

D_5 contains only the leftmost interval of D.

Note that $O[D_1, F, G] < \varepsilon/3$.

To show that $\sum\{Wf[p, q] : [p, q] \in D_2\} \leqq \varepsilon/3$ note the following. Each interval in the sequence $I_1, I_2, \ldots$ intersects no more than two members of D_2. If $[p, q]$ is an interval of D_2 then

$$WF[p, q] \leqq \sum \left\{\left(\frac{1}{n} - \frac{1}{n+2}\right) : I_n \text{ intersects } [p, q]\right\}.$$

Thus

$$\sum\{WF[p, q] : [p, q] \in D_2\} \leqq 2 \sum_{i=N}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2}\right) \leqq 4\varepsilon/12 = \varepsilon/3$$

since $1/N < \varepsilon/12$.

Likewise $\sum\{WF[p, q] : [p, q] \in D_3\} \leqq \varepsilon/3$.

To show that $\sum\{WF[p, q] : [p, q] \text{ in } D_4\} \leqq \varepsilon/6$ note that if n is a positive integer, then F is non-decreasing on $[1/(2n-1), 1/2n]$ and $(F(q) - F(p)) \leqq (1/n - 1/(n+2))$. Thus

$$\sum\left\{WF[p, q] : [p, q] \text{ in } D_4,\ [p, q] \subset \left[\frac{1}{2n-1}, \frac{1}{2n}\right]\right\} \leqq \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

and thus

$$\sum\{WF[p, q] : [p, q] \text{ in } D_4\} \leqq \sum_{i=N}^{\infty} \left(\frac{1}{i} - \frac{1}{i+2}\right) \leqq \varepsilon/6\,.$$

Also if $[p, q]$ is the interval in D_5 then $WF[p, q] \leqq \varepsilon/12$.

Therefore $\sum\{WF[p, q] : [p, q] \text{ in } \bigcup_{i=2}^{5} D_i\} \leqq \varepsilon$ since

$$\varepsilon/3 + \varepsilon/3 + \varepsilon/6 + \varepsilon/12 \leqq \varepsilon .$$

Since $0 \leqq G(x) \leqq \varepsilon/12$ for each x in $[0, t]$ then

$$O[\bigcup_{i=2}^{5} D_i, F, G] \leqq \varepsilon(\varepsilon/12) < \varepsilon/12 .$$

Finally since $O[D_6, F, G] < \sum_{n=N}^{\infty} O[D, F_n, G_n] < \sum_{i=N}^{\infty} (\frac{1}{2})^i < \varepsilon/3$ and since $O[D_1, F, G] < < \varepsilon/3$, then $O[D, F, G] < \varepsilon$.

IV. CONCLUSION

The work of this paper could be considered in the following context. Suppose f is a function on the interval $[0, 1]$, suppose $\mathcal{N}$ is a direction (filter base) in the set of all partitions of $[0, 1]$ and the following weak-Cauchy condition is true: for each positive number ε there exists a member N of $\mathcal{N}$ such that

$$\left|\sum_{i=1}^{k} f(x_i)\, mA_i - \sum_{i=1}^{k} f(x'_i)\, mA_i\right| < \varepsilon$$

where $(x_i, A_i)_{i=1}^k$ and $(x'_i, A_i)_{i=1}^k$ each is a partition in N and where partition is defined as in [4] and thus means the finite set of number-interval pairs used instead of just the subdivision consisting of the intervals used.

The weak Cauchy condition is equivalent to integrability or is not equivalent to integrability depending on how $\mathcal{N}$ and mA are defined. This paper has considered an instance where the weak Cauchy condition is not equivalent to integrability. If the following condition is added to the weak Cauchy condition, then, by the triangle inequality, the two conditions together are equivalent to f being integrable: for each positive number ε, there exists a member N of $\mathcal{N}$ such that if each of $(x_i, A_i)_{i=1}^k$ and $(y_i, B_i)_{i=1}^l$ is a partition in N then there exist sequences $(x'_i)_{i=1}^k$ and $(y'_i)_{i=1}^l$ such that each of $(x'_i, A_i)_{i=1}^k$ and $(y'_i, B_i)_{i=1}^l$ is in N and

$$\left|\sum_{i=1}^{k} f(x'_i)\, mA_i - \sum_{i=1}^{l} f(y'_i)\, mB_i\right| < \varepsilon$$

The following are a few example definitions for functions over an interval $[a, b]$ with mA meaning the length of the interval A. The closure of the A_i's is indicated by $\bar{A}_i$ so that one might consider the A_i's as right closed intervals or left closed intervals as well as intervals closed on each end.

Definition I. $N \in \mathcal{N}$ means there exists a subdivision $(C_i)_{i=1}^n$ of $[a, b]$ such that P is in N if and only if P is a partition $(x_i, A_i)_{i=1}^k$ with $(A_i)_{i=1}^k$ a refinement of $(C_i)_{i=1}^n$ and $x_i \in \bar{A}_i$, $i = 1, \ldots, k$.

Definition II. $N \in \mathcal{N}$ means there exists a number $\delta > 0$ such that P is in N if and only if P is a partition $(x_i, A_i)_{i=1}^k$ such that

$$A_i \subset (x_i - \delta,\ x_i + \delta) \quad \text{and} \quad x_i \in \bar{A}_i\,, \quad i = 1, \ldots, k\,.$$

Definition III. $N \in \mathcal{N}$ means there is a function δ with $\delta(x) > 0$ for each number x such that P is in N if and only if P is a partition $(x_i, A_i)_{i=1}^k$ with $A_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$ and $x_i \in \bar{A}_i$, $i = 1, \ldots, k$, [1, 2, 3].

Definition IV. $N \in \mathcal{N}$ means there is a function δ with $\delta(x) > 0$ for each number x such that P is in N if and only if P is a partition $(x_i, A_i)_{i=1}^k$ with $A_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, $i = 1, \ldots, k$.

For Definitions I and II the weak Cauchy condition implies integrability. An alteration by J. Kurzweil gives us Definition III where the weak Cauchy condition does not imply integrability since with Definition III the weak Cauchy condition holds for the function f where $f(x) = 1/x$ if $x \in (0, 1]$, $f(x) = 0$ if $x \in [-1, 0]$, but f is not integrable with respect to Definition III over the interval $[-1, 1]$.

To show the weak Cauchy condition holds for this example, suppose $\varepsilon > 0$. Let $p_1, p_2, p_3, \ldots$ denote an infinite decreasing sequence of numbers converging to 0 such that if n is a positive integer then $|f(p_n) - f(p_{n+1})| < \varepsilon$. For each positive integer n, denote by δ_n a positive valued function such that if $x \neq p_n$ then $\delta_n(x) = \frac{1}{2}|x - p_n|$ and $\delta_n(p_n) = 1$. Let δ' denote a positive valued function such that $\delta'(x) = \frac{1}{2}|x|$ if $x \neq 0$ and $\delta'(0) = 1$. Let δ denote $\min(\delta', \{\delta_n : n = 1, \ldots\})$. If $(x_i, A_i)_{i=1}^n$ and $(y_i, A_i)_{i=1}^n$ each is a partition of $[-1, 1]$ with A_i in $(x_i - \delta(x_i), x_i + \delta(x_i))$, $x_i \in \bar{A}_i$, $y_i \in \bar{A}_i$, then $\sum_{i=1}^n f(x_i)\, mA_i - \sum_{i=1}^k f(y_i)\, mA_i| < \varepsilon$.

Definition IV, a further alteration by E. J. Mc Shane, which merely delates the phrase "x_i is in $\bar{A}_i$" is enough to guarantee that if each of $(x_i, A_i)_{i=1}^k$ and $(y_i, B_i)_{i=1}^l$ is in N, then there exists a sequence $(C_j)_{j=1}^t$ of intervals each of which is the intersection of some member of $(A_i)_{i=1}^k$ and some member of $(B_i)_{i=1}^l$ such that there exist two members $(w_j, C_j)_{j=1}^t$ and $(v_j, C_j)_{j=1}^t$ of N with $w_j = x_i$ for some $i, j = 1, \ldots, t$ and $v_j = y_i$ for some $i, j = 1, \ldots, t$. Thus if some function is not integrable by Definition IV, then the weak Cauchy condition does not hold.

In a more general setting, this idea can be used with the definitions discussed in [4] which have property (4.4) of [4], to show that for each such integral definition, if the weak Cauchy condition has meaning then it implies integrability.

The question, "Why doesn't the example just given, $f(x) = 1/x$ if $x \in (0, 1]$, $f(x) = 0$ if $x \in [-1, 0]$, furnish a counter example with respect to Definition IV as well as with respect to Definition III?", is an enlightening puzzle concerning the two definitions, even though one could take a less challenging attitude that the preceeding proof is all the answer that is necessary.

The main results of this paper concern Definitions I and II with mA meaning g-length of A for a real valued function g, giving us the Riemann Stieltjes refinement integral and the Riemann Stieltjes norm integral respectively.

In fact, by the proof of Theorem 2, there exists an ordered function pair (F, G) such that F is not G integrable by the Riemann-Stieltjes refinement interior definition, but the weak Cauchy condition with respect to the Riemann-Stieltjes norm definition is satisfied by (F, G). Thus, with respect to each of the definitions, Riemann-Stieltjes refinement interior, Riemann-Stieltjes norm interior, Riemann-Stieltjes refinement, and Riemann-Stieltjes norm, F is not G integrable, but the weak Cuachy condition holds for F and G, with respect to each of the four definitions, where "interior" signifies that the numbers which determine the ordinates of F used in the Riemann sums are restricted to the corresponding segments (open intervals) of the Riemann sums, while without the word, interior, these numbers are in the corresponding closed intervals of the Riemann sums.

Another interesting example results from a sequence $(f_i, g_i)_{i=1}^{\infty}$ similar to the construction of Theorem 1 such that the uniform limits $F = \lim_{i \to \infty} f_i$ and $G = \lim_{i \to \infty} g_i$ exist with $\int F \, dG$ nonexistant and the weak Cauchy condition holding for F and G. There exists such an example which works for all four Riemann-Stieltjes definitions just mentioned.

The example constructed in the proof of Theorem 2 was chosen for presentation in this paper because the two functions F and G, are each continuous and because the number 0 is an isolated point of nonintegrability. That is, if t is a number in $(0, 1)$ then F is G integrable on the interval $[t, 1]$, and F is not G integrable on $[0, 1]$.

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