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Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 2, 206–217

Persistent URL: `http://dml.cz/dmlcz/101738`

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

PRIME SELECTORS IN LATTICE-ORDERED GROUPS

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(Received May 22, 1979)

I. Introduction. If G is an l-group $\mathbf{P}(G)$ will denote the set of all its proper prime subgroups. (Recall that a convex l-subgroup N is *prime* if $a \wedge b = 1$ implies that $a \in N$ or $b \in N$. Fot several different characterizations of prime-ness the reader should look through Conrad [1].) A convex l-subgroup M of G is called a *value* of G if there is an element $g \in G$ such that M is maximal relative to *not* containing g. We also say that M is a value of g. It is well known that every value is a prime, and that every prime is the intersection of a chain of values; see Conrad [1]. $\mathbf{M}(G)$ will denote the set of values of G.

Before proceeding we point out that our groups shall be written multiplicatively, unless we expressly signal the contrary.

Let's review two basic facts about l-groups:

1. If $\phi : G \to H$ is an l-epimorphism, then the map $N \to N\phi^{-1}$ is a one-to-one correspondence between $\mathbf{P}(H)$ and the proper primes of G that contain Ker (ϕ).

2. Next, suppose that C is a convex l-subgroup of G. The map $N \to N \cap C$ is a one-to-one correspondence between the primes of G that do not contain C and $\mathbf{P}(C)$.

(See Conrad [1] or Martinez [13] for details.)

Both of the above correspondences can be restricted to the appropriate sets of values, yielding 'value' analogues of (1) and (2). In addition, when $N \in \mathbf{P}(G)$ and doesn't contain the subgroup C, then $N \cap C$ is a minimal prime of C if and only if N is a minimal prime of G.

Let us also review the definition of a torsion class as given in [13] or [14]. A class of l-groups $\mathcal{T}$ is a *torsion class* if it is closed under taking (a) l-homomorphic images and (b) joins of convex l-subgroups of an l-group, which happen to belong to $\mathcal{T}$.

Notice that we do not require (c): $\mathcal{T}$ is closed under taking convex l-subgroups. If $\mathcal{T}$ is a torsion class satisfying (c) we'll say its *hereditary*. Hereditary torsion classes were studied extensively by this author in [10], [11] and [14], and jointly with Charles Holland in [5]. [13] should be seen as a survey article with several new results, including the main theorem of [14] and most of the results in this paper.

As per Lemma 1 in [14] we have a choice of studying torsion classes as classes or through their radicals. For an l-group G, $\mathcal{T}(G)$ stands for the *$\mathcal{T}$-radical* of G: the join of all the convex l-subgroups of G belonging to $\mathcal{T}$. By definition $\mathcal{T}(G)$ is in $\mathcal{T}$ and a characteristic subgroup of G. In Lemma 1 of [14] torsion radicals are characterized as functions taking on values in the lattice of convex l-subgroups such that

(i) $\mathcal{T}(C) \leqq \mathcal{T}(G)$ for each convex l-subgroup C of G.

(ii) If $\phi : G \to H$ is an l-epimorphism, then $\mathcal{T}(G)\,\phi \leqq \mathcal{T}(H)$.

(iii) $\mathcal{T}(\mathcal{T}(G)) = \mathcal{T}(G)$.

II. Prime selectors. A *prime selector* is a function which assigns to each l-group G a subset $\boldsymbol{H}(G)$ of $\boldsymbol{P}(G)$ subject to the following conditions:

(1) If $\phi : G \to H$ is an l-epimorphism, $N \geqq \operatorname{Ker}(\phi)$ and $N \in \boldsymbol{H}(G)$, then $N\phi \in \boldsymbol{H}(H)$.

(2) If C is a convex l-subgroup of G and N is a prime subgroup such that $N \ngeqq C$, then $N \cap C \in \boldsymbol{H}(C)$ implies that $N \in \boldsymbol{H}(G)$.

If $\boldsymbol{H}$ is a prime selector we define $\mathsf{TOR}(\boldsymbol{H})$ to be $\{G \mid \boldsymbol{H}(G) = \boldsymbol{P}(G)\}$. We then have:

Proposition 1. $\mathsf{TOR}(\boldsymbol{H})$ *is a torsion class.*

Proof. Condition (1) evidently implies that $\mathsf{TOR}(\boldsymbol{H})$ is closed under taking l-homomorphic images. Suppose then that $A = \bigvee A_i$ $(i \in I)$ in the lattice of convex l-subgroups of G, and that each $A_i \in \mathsf{TOR}(\boldsymbol{H})$. If $N \in \boldsymbol{P}(A)$ there is an $i \in I$ such that $N \ngeqq A_i$, and so $N \cap A_i \in \boldsymbol{P}(A_i) = \boldsymbol{H}(A_i)$. Hence $N \in \boldsymbol{H}(A)$ by condition (2), $A \in \in \mathsf{TOR}(\boldsymbol{H})$. This proves $\mathsf{TOR}(\boldsymbol{H})$ is a torsion class.

If $\mathcal{T}$ is a torsion class and $\boldsymbol{H}$ is a prime selector so that $\mathcal{T} = \mathsf{TOR}(\boldsymbol{H})$, we say that $\boldsymbol{H}$ *presents* $\mathcal{T}$, or is a *presentation* of $\mathcal{T}$. Our next result shows that every torsion class has a presentation, and indeed a least one. To understand that, let us decide on how to compare prime selectors: $\boldsymbol{H}_1 \leqq \boldsymbol{H}_2$ if $\boldsymbol{H}_1(G) \leqq \boldsymbol{H}_2(G)$, for each l-group G.

Proposition 2. *Suppose $\mathcal{T}$ is a torsion class. For each l-group G let $\boldsymbol{h}(\mathcal{T})(G) = \{N \in \boldsymbol{P}(G) \mid N \ngeqq \mathcal{T}(G)\}$. Then $\boldsymbol{h}(\mathcal{T})$ is a prime selector and $\mathsf{TOR}(\boldsymbol{h}(\mathcal{T})) = \mathcal{T}$. Moreover, $\boldsymbol{h}(\mathcal{T})$ is the smallest prime selector presenting $\mathcal{T}$.*

Proof. Let us suppose for the moment that $\boldsymbol{h}(\mathcal{T})$ has been shown to be a prime selector. Observe then that $G \in \mathsf{TOR}(\boldsymbol{h}(\mathcal{T}))$ if and only if $\boldsymbol{h}(\mathcal{T})(G) = \boldsymbol{P}(G)$. Thus, $G \in \mathsf{TOR}(\boldsymbol{h}(\mathcal{T}))$ if and only if every proper prime fails to contain $\mathcal{T}(G)$. This happens precisely when $G = \mathcal{T}(G)$. Conclusion $\mathsf{TOR}(\boldsymbol{h}(\mathcal{T})) = \mathcal{T}$.

Now to prove that $\boldsymbol{h}(\mathcal{T})$ is a prime selector: suppose $\phi : G \to H$ is an l-epimorphism and $N \in \boldsymbol{P}(G)$ with $N \geqq \operatorname{Ker}(\phi)$ and $N \ngeqq \mathcal{T}(G)$. If $N\phi \geqq \mathcal{T}(H)$ then

$$N = (N\phi)\,\phi^{-1} \geqq \mathcal{T}(H)\,\phi^{-1} \geqq \mathcal{T}(G)\,,$$

since $\mathcal{T}$ is a torsion class. This is a contradiction and therefore $N\phi \ngeqq \mathcal{T}(H)$. This shows that $N \in \boldsymbol{h}(\mathcal{T})(H)$.

Next, suppose C is a convex l-subgroup of G and N is a prime of G that does not contain C. Evidently then $\mathscr{T}(C) \nleqq N \cap C$ implies that $\mathscr{T}(G) \nleqq N$. Thus, $\mathbf{h}(\mathscr{T})$ is a prime selector.

If $\mathbf{H}$ is any presentation of $\mathscr{T}$ and N is a prime subgroup which fails to contain $\mathscr{T}(G)$, then $N \cap \mathscr{T}(G) \in \mathbf{P}(\mathscr{T}(G)) = \mathbf{H}(\mathscr{T}(G))$. Since $\mathbf{H}$ is a prime selector $N \in \mathbf{H}(G)$, and we conclude that $\mathbf{h}(\mathscr{T}) \leqq \mathbf{H}$. $\mathbf{h}(\mathscr{T})$ is then indeed the smallest presentation of $\mathscr{T}$.

If a prime selector $\mathbf{H}$ satisfies the stronger condition: (2′): for each convex l-subgroup C of G and each $N \in \mathbf{P}(G)$,

$$N \cap C \in \mathbf{H}(C) \quad \text{if and only if} \quad N \ngeqq C \quad \text{and} \quad N \in \mathbf{H}(G),$$

we say that $\mathbf{H}$ is a *hereditary* selector. The following result is easy to verify.

Proposition 3. *If $\mathbf{H}$ is a hereditary prime selector, then* $\mathsf{TOR}(\mathbf{H})$ *is a hereditary torsion class. Conversely, every hereditary torsion class $\mathscr{T}$ has a hereditary presentation, namely $\mathbf{h}(\mathscr{T})$.*

In proving that $\mathbf{h}(\mathscr{T})$ is the smallest prime selector presenting $\mathscr{T}$ we proved that $\mathscr{T}(G) \leqq \bigcap\{N \in \mathbf{P}(G) \mid N \notin \mathbf{H}(G)\}$, where $\mathsf{TOR}(\mathbf{H}) = \mathscr{T}$. If equality holds we speak of $\mathbf{H}$ being an *exact* presentation. It is evident that $\mathbf{h}(\mathscr{T})$ is an exact presentation for $\mathscr{T}$. Moreover, if $\mathbf{H}$ is hereditary and $\mathscr{T} = \mathsf{TOR}(\mathbf{H})$, then $\mathbf{H}$ is exact. The proof can be found in [13].

Let us now consider some examples:

(a) Suppose $\mathbf{M}_0$ is the prime selector of minimal primes. Then $\mathsf{TOR}(\mathbf{M}_0)$ is the hereditary class $\mathscr{A}\imath$ of hyper-archimedean l-groups. However $\mathbf{M}_0 \neq \mathbf{h}(\mathscr{A}\imath)$: it is known that in the free abelian l-group F on two generators – see Conrad [2] – each prime is either minimal or maximal, and thus $\mathbf{M}_0(F) \neq \emptyset$. Yet $\mathscr{A}\imath(F) = 1$, which says that $\mathbf{h}(\mathscr{A}\imath)(F) = \emptyset$.

Incidentaly, notice that for any torsion class $\mathscr{T}$, $\mathbf{h}(\mathscr{T})$ is an ideal of primes; meaning that if $N \in \mathbf{h}(\mathscr{T})(G)$ and $M \leqq N$, $(M \in \mathbf{P}(G))$, then $M \in \mathbf{h}(\mathscr{T})(G)$. This property does not characterize the minimality of a presenting selector; just look at $\mathbf{M}_0$.

(b) Suppose $\mathbf{F}$ is the selector that picks all non-values and every special value. Then $\mathsf{TOR}(\mathbf{F}) = \mathscr{F}\!v$, the class of finite valued l-groups. Once again, there are examples to show $\mathbf{F} \neq \mathbf{h}(\mathscr{F}\!v)$.

(c) Consider $\mathsf{TOR}(\mathbf{M}) = \mathscr{D}$. $G \in \mathscr{D}$ if and only if every prime is a value. This is equivalent to saying that $\mathbf{M}(G)$ satisfies the DCC. This class is complete. $\mathscr{D}$ has been studied by Conrad [3].

The mapping $\mathscr{T} \to \mathbf{h}(\mathscr{T})$ embeds the proper class of torsion classes into the 'lattice' of prime selectors in view of the identity $\mathsf{TOR}(\mathbf{h}(\mathscr{T})) = \mathscr{T}$. If $\mathscr{T} = \bigwedge_{i \in I} \mathscr{T}_i$ is a join of torsion classes and $N \in \mathbf{P}(G)$, then $N \geqq \mathscr{T}(G)$ if and only if N contains each $\mathscr{T}_i(G)$. Conclusion: $\mathbf{h}(\mathscr{T}) = \bigwedge_{i \in I} \mathbf{h}(\mathscr{T}_i)$. The embedding also assigns a hereditary selector to a hereditary class. It preserves the meet of $\mathscr{T}_1$ and $\mathscr{T}_2$ if and only if

$$(\mathscr{T}_1 \cap \mathscr{T}_2)(G) = \mathscr{T}_1(G) \cap \mathscr{T}_2(G),$$

for each l-group G. This is valid if $\mathscr{T}_1$ and $\mathscr{T}_2$ are hereditary. In general though, it is unknown whether $\boldsymbol{h}$ preserves finite meets. Furthermore, the previous argument breaks down for infinite meets, even in the hereditary case, thus foiling the proof of Theorem 1.5 in [12].

The map TOR, on the other hand, preserves arbitrary meets; the proof is easy. This fact does not seem to help in resolving the issue of what radical an intersection of torsion classes has. It would be useful to know whether TOR preserves joins. If so then every torsion class has a largest presentation.

Some intriguing questions:

1. Is $\boldsymbol{M}_0$ the largest presentation of $\mathscr{A}\imath$?
2. Is $\boldsymbol{F}$ the largest presentation of $\mathscr{F}v$?

III. Selectors and completeness. In this section we turn to the question: can the completeness of a torsion class be reflected by a suitable choice of presentation? Recall from [13] that a torsion class $\mathscr{T}$ is *complete* if for each l-group G and each l-ideal A of G belonging to $\mathscr{T}$ so that $G/A \in \mathscr{T}$, it follows that $G \in \mathscr{T}$. It is easy to show that a complete torsion class is closed under *finite* subdirect products. (See [13].)

Now we strengthen the first defining condition for prime selectors. If $\boldsymbol{H}$ is a prime selector and in addition the following holds:

(1′) for each $N \in \mathbf{P}(G)$ and each l-epimorphism $\phi : G \to H$ for which $N \geqq \operatorname{Ker}(\phi)$, $N \in \boldsymbol{H}(G)$ if and only if $N\phi \in \boldsymbol{H}(H)$,

we say that $\boldsymbol{H}$ is a *strong* prime selector. If $\mathscr{T} = \mathrm{TOR}(\boldsymbol{H})$ we'll say that $\mathscr{T}$ is *strongly presented*. If $\mathscr{T}$ is such a class it is easy to show that $\mathscr{T}$ is complete. On the other hand, $\mathscr{A}\imath^*$, the completion of the class of hyper-archimedean l-groups, is complete but cannot be strongly presented. We shall indicate the proof at the end of this section.

By way of example, notice that $\boldsymbol{M}$ is a strong selector, and hence that $\mathscr{D}$ is strongly presented. Neither $\boldsymbol{M}_0$ nor $\boldsymbol{F}$ are strong selectors.

So far we've only exhibited hereditary selectors. But consider the selector $\boldsymbol{H}$ that picks a prime if it's not a value, or when it is, selects it if it is not both maximal and normal. $\boldsymbol{H}$ is strong; $\mathscr{T} = \mathrm{TOR}(\boldsymbol{H})$ is the class of l-groups having no convex l-subgroups which are both maximal and normal. Neither $\mathscr{T}$ nor $\boldsymbol{H}$ are hereditary.

We need to recall some notions from [10]. The 'lattice' structure of the class of hereditary torsion classes is Brouwerian. In particular, for each hereditary torsion class $\mathscr{T}$ there is a largest hereditary torsion class $\mathscr{T}'$ such that $\mathscr{T} \cap \mathscr{T}' = 0$. $\mathscr{T}'$ is called the *polar* of $\mathscr{T}$. $\mathscr{T}$ is a *polar class* if $\mathscr{T} = \mathscr{T}''$. We have the following theorems from [10]; all apply to hereditary torsion classes only.

(Note: if A and B are convex l-subgroups of G and $A \leqq B$ with A normal in B, we call B/A a *subquotient* of G. B/A is a *non-trivial* subquotient if $A \neq B$.)

Theorem (2.3, [10]) $\mathscr{T}' = \{G \mid \mathscr{T}(B/A) = 1$ *for each subquotient of G.*$\}$

Theorem (2.5, [10]) *$\mathcal{T}''$ consists of all l-groups G for which each non-trivial subquotient B/A possesses a non-trivial subquotient $D/C(A \leqq C \leqq D \leqq B)$ in $\mathcal{T}$. Thus, $\mathcal{T}$ is a polar class if and only if $G \in \mathcal{T}$ whenever each non-trivial subquotient of G has a non-trivial subquotient in $\mathcal{T}$.*

Theorem (2.4, [10]) $\mathcal{T}^* \leqq \mathcal{T}''$. *($\mathcal{T}^*$ is the completion of $\mathcal{T}$.)*

We also mention one result from Holland-Martinez [5] whose proof involves wreath products:

Theorem 4. *If $\mathcal{T}$ is a polar class and $\mathcal{U}^* = \mathcal{T}$ ($\mathcal{U}$ hereditary) then $\mathcal{U} = \mathcal{T}$.*

It is evident from this that $\mathcal{T}^* < \mathcal{T}''$ for every non-complete hereditary torsion class. Any hereditary class which distinguishes itself among all the hereditary classes according to the condition of Theorem 4 will be called *inaccessible*. In view of Theorem 4 and the fact that polar classes are strongly presented – which we shall prove in the sequel – we conjecture that every strongly presented hereditary torsion class is inaccessible. Equivalently, we are betting that $\mathcal{T}^*$ is never strongly presented if $\mathcal{T}$ is non-complete.

To show that polar classes are strongly presented, we actually produce a bit more: suppose $\boldsymbol{H}$ is an arbitrary prime selector and define $\boldsymbol{H}^s$ as follows: $N \in \boldsymbol{H}^s(G)$ if there is a non-trivial subquotient B/A of G such that $N \geqq A$, $N \ngeqq B$ and $N \cap B/A \in \boldsymbol{H}(B/A)$.

We then have:

Proposition 5. *$\boldsymbol{H}^s$ is a strong selector, $\boldsymbol{H} \leqq \boldsymbol{H}^s$ and $\boldsymbol{H}^s$ is the smallest strong selector containing $\boldsymbol{H}$. If $\boldsymbol{H}$ is hereditary so is $\boldsymbol{H}^s$.*

The proof of this proposition involves routine verifications; it appears in [13], and we leave it to the interested reader to look it up.

Theorem 6. *If $\mathcal{T}$ is any torsion class then $\mathrm{TOR}(\boldsymbol{h}(\mathcal{T})^s)$ is the smallest strongly presented torsion class that contains $\mathcal{T}$,*

Proof. $\mathrm{TOR}(\boldsymbol{h}(\mathcal{T})^s)$ is a strongly presented class by Proposition 5. Next, suppose that $\mathcal{U} = \mathrm{TOR}(\boldsymbol{H})$ and $\boldsymbol{H}$ is a strong selector. Suppose also that $\mathcal{T} \leqq \mathcal{U}$. Since TOR preserves meets, $\boldsymbol{h}(\mathcal{T}) \leqq \boldsymbol{H}$; ($\boldsymbol{H} \cap \boldsymbol{h}(\mathcal{T})$ is a presentation for $\mathcal{T}$.) Hence $\boldsymbol{h}(\mathcal{T})^s \leqq \boldsymbol{H}$, by Proposition 5, and so $\mathrm{TOR}(\boldsymbol{h}(\mathcal{T})^s) \leqq \mathcal{U}$.

Theorem 7. *If $\mathcal{T}$ is a hereditary torsion class, then $\mathrm{TOR}(\boldsymbol{h}(\mathcal{T})^s) \leqq \mathcal{T}''$. In particular, if $\mathcal{T} = \mathcal{T}''$, then $\mathcal{T}$ is strongly presented.*

Proof. We use Theorem 2.5 of [10]. Suppose $G \in \mathrm{TOR}(\boldsymbol{h}(\mathcal{T})^s)$ and D/C is a non-trivial subquotient of G. Select a prime N of G such that $N \geqq C$ but $N \ngeqq D$. Then we have a subquotient B/A for which $A \leqq N$, $B \nleqq N$ and $B/A \in \mathscr{T}$. It is easy to verify that $(B \cap C) \vee (A \cap D)$ is normal in $B \cap D$, and that $Q = [(B \cap D)/(B \cap C) \vee \vee (A \cap D)]$ is an l-homomorphic image of $B \cap D/A \cap D$. The latter is in $\mathscr{T}$ since $\mathscr{T}$ is hereditary. But Q is also an l-homomorphic image of $B \cap D/B \cap C$ which is l-isomorphic to a convex l-subgroup of D/C. Hence D/C has a subquotient in $\mathscr{T}$.

Q is non-trivial: $B \nleqq N$ and $D \nleqq N$ imply that $B \cap D \nleqq N$. On the other hand, $A \cap D \leqq N$ and $B \cap C \leqq N$, so that $(A \cap D) \vee (B \cap C) \leqq N$. By Theorem 2.5 in [10] $G \in \mathscr{T}''$.

With regard to Theorem 6 we make the following definition: if $\mathscr{T}$ is a torsion class, we call $\mathscr{T}^s = \mathsf{TOR}(\mathbf{h}(\mathscr{T})^s)$ the *strong completion* of $\mathscr{T}$.

The class $\mathscr{D} = \mathsf{TOR}(\mathbf{M})$ is strongly presented, but is not a polar class: $\mathscr{D}''$ contains all the normal-valued l-groups. On the other hand, we know that $\mathscr{N}$, the class of normal-valued l-groups is inaccessible (and strongly presented), but not whether it is a polar class. Also unknown are whether $\mathscr{D} \leqq \mathscr{N}$ and whether $\mathscr{D}$ is inaccessible. (Note: to prove that $\mathscr{N}$ is inaccessible one uses the approach of Holland-Martinez in [5], via wreath products. This construction produces descending chains of values, and is therefore inapplicable to $\mathscr{D}$.)

IV. Automorphically defined torsion classes. Suppose $\mathbf{H}$ is a strong prime selector, and let $\mathscr{N}(\mathbf{H})$ stand for the class of all l-groups G in which each $M \in \mathbf{H}(G)$ is a normal subgroup. Straightaway we have the following proposition:

Proposition 8. *If* $\mathbf{H}$ *is a strong, hereditary selector then* $\mathscr{N}(\mathbf{H})$ *is a hereditary torsion class.*

Proof. Suppose $G \in \mathscr{N}(\mathbf{H})$ with K an l-ideal of G. A prime in $\mathbf{H}(G/K)$ is of the form N/K with $N \in \mathbf{H}(G)$ since $\mathbf{H}$ is a strong selector. Thus, N is normal in G, which means that N/K is normal in G/K. We conclude that $G/K \in \mathscr{N}(\mathbf{H})$.

Suppose C is a convex l-subgroup of $G \in \mathscr{N}(\mathbf{H})$. If $M \in \mathbf{H}(C)$ then $M = C \cap M_0$, where $M_0 \in \mathbf{H}(G)$. M_0 is normal in G, whence M is normal in C. Thus: $\mathscr{N}(\mathbf{H})$ is closed under taking convex l-subgroups.

Finally, suppose G is the join of convex l-subgroups G_λ $(\lambda \in \Lambda)$, and that each G_λ belongs to $\mathscr{N}(\mathbf{H})$. If $M \in \mathbf{H}(G)$ then since $M \ngeqq G_\mu$ for some $\mu \in \Lambda$, $M \cap G_\mu \in \mathbf{H}(G_\mu)$ since $\mathbf{H}$ is hereditary. For each such μ, G_μ normalizes M because $G_\mu \in \mathscr{N}(\mathbf{H})$. If on the other hand $M \geqq G_\mu$ it is trivial that G_μ normalizes M. Thus, M is an l-ideal, and we've proved that $G \in \mathscr{N}(\mathbf{H})$.

For example, if $\mathbf{H} = \mathbf{M}$, then $\mathscr{N}(\mathbf{H})$ is the class of l-groups in which each value, and hence each convex l-subgroup is normal. These are the so-called *c-archimedean* l-groups. They were introduced in [8]. They are characterized by the condition: $1 \leqq a \in G$, $g \in G$ imply that $a^g = g^{-1}ag \leqq a^n$ for some natural number n. (Note: the choice of n depends on $g \in G$.)

If $\mathbf{H}$ selects values which are not normal in their covers, then an oblique, vacuous argument gives that $\mathscr{N}(\mathbf{H}) = \mathscr{N}$, the class of normal valued l-groups.

There are other classes which we would like to consider in this manner. However, as in the case of the representable l-groups, one must depart from looking at the primes specified by a selector. One has to look at selections of l-automorphisms instead.

$A(G)$ stands for the group of l-automorphisms of an l-group G. $I(G)$ will be used for the subgroup of inner automorphisms. For each l-group G let $\theta(G)$ be a subgroup of $A(G)$ chosen to conform to the following conditions:

(A1) If $\phi : G \to H$ is an l-epimorphism and $\sigma \in \theta(G)$ fixing $\operatorname{Ker}(\phi)$ (ie. $[\operatorname{Ker}(\phi)]\,\sigma = \operatorname{Ker}(\phi)$), then $\sigma^H \in \theta(H)$, where σ^H denotes the canonical l-automorphism of H induced by σ, defined by $(x\phi)\,\sigma^H = (x\sigma)\,\phi$.

(A2) If K is an l-ideal of G, $\sigma \in A(G)$ and fixes K, and the restriction σ_K of σ to K is in $\theta(K)$, while $\sigma^K \in \theta(G/K)$, it follows that $\sigma \in \theta(G)$.

(A3) Suppose $\{G_i \mid i \in I\}$ is a family of convex l-subgroups of G, and $G = \bigvee_{i\in I} G_I$. If $\sigma \in A(G)$ and fixes each G_i, while the restriction σ_i to G_i belongs to $\theta(G_i)$, then $\sigma \in \theta(G)$.

We call $\theta(G)$ an *exact* subgroup of $A(G)$, and will indeed refer to the function θ as an exact subgroup of A, since there can be no ambiguity in doing so. If in addition θ satisfies:

(A4) For each G and each convex l-subgroup C of G, and each l-automorphism σ fixing C, then $\sigma \in \theta(G)$ implies that the restriction σ_C to C is in $\theta(C)$, we say that θ is a *hereditary* subgroup of A.

For example, suppose $\boldsymbol{H}$ is a strong prime selector. Let $\theta_{\boldsymbol{H}}(G) = \{\sigma \in A(G) \mid N\sigma = N, \text{ for all } N \in \boldsymbol{H}(G)\}$. We then have the following result; the proof is straightforward but tedious. We shall therefore omit it and refer the reader to [13].

Proposition 9. *If $\boldsymbol{H}$ is a strong, hereditary selector then $\theta_{\boldsymbol{H}}$ is an exact, hereditary subgroup of A.*

For any exact subgroup θ of A we define $\mathcal{N}(\theta)$ to be the class of all l-groups G for which $I(G) \leqq \theta(G)$. For a strong selector $\boldsymbol{H}$ it is evident that $\mathcal{N}(\boldsymbol{H}) = \mathcal{N}(\theta_{\boldsymbol{H}})$. In general, $\mathcal{N}(\theta)$ comes very close to being a torsion class, and may very well be one; we have no counter-examples. We do have the following:

Proposition 10. *For each exact subgroup θ of A, $\mathcal{N}(\theta)$ is closed under taking:*

(a) *l-homomorphic images;*
(b) *convex l-subgroups, if θ is hereditary;*
(c) *unions of chains of convex l-subgroups;*
(d) *cardinal sums.*

(Note: G is the *cardinal sum* of its convex l-subgroups G_i $(i \in I)$ if the G_i generate G and $G_i \cap G_j = 1$ for all $i \neq j$.)

Proof. Checking (a) and (b) is routine, and we shall omit them. Let us exhibit (c) and (d) to see just how close $\mathcal{N}(\theta)$ comes to being a torsion class. (In [11], Proposition 1.5, it is shown that if a class $\mathcal{T}$ consists of *finite-valued* l-groups, then these four conditions are enough to make $\mathcal{T}$ a torsion class.)

(c) Suppose I is a chain and $\{C_i \mid i \in I\}$ is a chain of convex l-subgroups of G so that $C_i \leqq C_j$ for $i < j$. Suppose $G = \bigcup C_i$, and suppose each $C_i \in \mathcal{N}(\theta)$. If $g \in G$ then $g \in C_{i_0}$ for a suitable $i_0 \in I$. Conjugation of C_j by g belongs to $\theta(C_j)$ provided $j \geqq i_0$. But $G = \bigcup_{j \geqq i_0} C_j$, and so by (A3), conjugation by g belongs to $\theta(G)$; that is: $G \in \mathcal{N}(\theta)$.

(d) Suppose G is the cardinal sum of the convex l-subgroups G_λ ($\lambda \in \Lambda$). If $g \in G$ and $G_\lambda^g = G_\lambda$, then $G_\lambda^{g_\lambda} = G_\lambda$, where g_λ is the λ-th component of g. (Note: a cardinal sum of these G_λ is in particular a *direct* sum. Thus each $x \in G$ can be written uniquely as a finite sum $x = \sum x_\lambda$ with $x_\lambda \in G_\lambda$.)

Conjugation by g_λ is in $\theta(G_\lambda)$ for each $\lambda \in \Lambda$, and so by (A3) once more, conjugation by g is in $\theta(G)$.

Two special examples should be mentioned at this point:

A. Let $\Pi(G)$ consist of all the polar-preserving l-automorphisms of G; that is, $\sigma \in \Pi(G)$ if and only if $a \wedge b = 1$ implies that $a \wedge b\sigma = 1$. Polar-preserving l-automorphisms fix every minimal prime of G. We leave it to the reader to verify that Π is an exact, hereditary subgroup of A. $\mathcal{N}(\Pi) = \{G \mid a \wedge b = 1$ implies that $a \wedge b^g = 1$ for each $g \in G\}$. This is the class of representable l-groups. (See Conrad [1] or Martinez [13] for a list of alternate characterizations of this class $\mathcal{R}$. In particular, $G \in \mathcal{R}$ if it can be written as a subdirect product of totally-ordered groups.)

B. Let $\Delta(G) = \{\sigma \in A(G) \mid N\sigma = N$ for each value N of G and $Nx\sigma = Nx$ for all $x \in \overline{N}$, the cover of $N\}$. Again, Δ is exact and hereditary. $\mathcal{N}(\Delta) = \mathcal{W}$, the class of weakly-abelian l-groups. (See Martinez [6]; G is *weakly-abelian* if $1 \leqq a \in G$ and $g \in G$ imply that $a^g \leqq a^2$. It is proved in [6] that $\mathcal{N}(\Delta) = \mathcal{W}$; ie. that $G \in \mathcal{W}$ if and only if each value M of G is normal in G and $\overline{M}/M$ is central in G/M.)

Thus, if $\mathcal{T}$ is a torsion class and $\mathcal{T} = \mathcal{N}(\theta)$ for a suitable exact subgroup θ of A, we say that $\mathcal{T}$ is *automorphically defined* (or *definable*). The next proposition identifies a curious subclass of an automorphically defined class; using it we can easily dismiss many classes as not being so definable.

Proposition 11. *Suppose $\mathcal{T} = \mathcal{N}(\theta)$ and $\gamma(\mathcal{T}) = \{G \mid \theta(G) = A(G)\}$. Then*

(a) $\gamma(\mathcal{T}) \leqq \mathcal{T}$

and

(b) *if $G \in \gamma(\mathcal{T})$ then each extension of G by an l-group in $\mathcal{T}$ once again belongs to $\mathcal{T}$. The converse is true if θ is hereditary.*

Proof. (a) is clear.

(b) Suppose $G \in \gamma(\mathscr{T})$ and an l-ideal of H, so that H/G is in $\mathscr{T}$. If $h \in H$ then $G^h = G$. As $\theta(G) = A(G)$, the restriction of conjugation by h belongs to $\theta(G)$. h induces a conjugation on H/G (by Gh) which belongs to $\theta(H/G)$. By (A2) conjugation by h belongs to $\theta(H)$, and so $H \in \mathscr{N}(\theta)$.

Conversely, suppose that θ is hereditary, and each extension of G by an l-group in $\mathscr{T}$ belongs to $\mathscr{T}$. $A(\boldsymbol{Z}) = 1$ ($\boldsymbol{Z} \equiv$ the additive integers) and so $\boldsymbol{Z} \in \mathscr{T}$. For each $\sigma \in A(G)$ we construct a lexicographic extension K of G by $\boldsymbol{Z}$ so that conjugation by the integer 1 is equivalent to the action by σ; that is, $g\sigma = g^1$, for all $g \in G$. By assumption $K \in \mathscr{T}$, which together with the hereditariness of θ implies that $\sigma \in \theta(G)$. This proves that $A(G) = \theta(G)$, and thus that $G \in \gamma(\mathscr{T})$.

Now a few observations: assume $\mathscr{T} = \mathscr{N}(\theta)$ and that θ is hereditary.

1. $\boldsymbol{Z}$, or for that matter, any l-group G for which $A(G) = 1$, belongs to $\gamma(\mathscr{T})$.
2. Every abelian l-group belongs to $\mathscr{T}$.
3. The intersection of a family of exact (hereditary) subgroups is exact (hereditary).
4. $\mathscr{N}(\bigcap_{\lambda\in\Lambda} \theta_\lambda) = \bigcap_{\lambda\in\Lambda} \mathscr{N}(\theta_\lambda)$. ($\bigcap_{\lambda\in\Lambda} \theta_\lambda$ is to the interpreted in the only reasonable way: $(\bigcap_{\lambda\in\Lambda} \theta_\lambda)(G) = \bigcap_{\lambda\in\Lambda} \theta_\lambda(G)$, for each l-group G.)
5. Thus, there is a *unique* minimal automorphically definable class. In the sequel we describe this class and prove it is an *equational* class. (By a theorem of Holland [4] it is also a torsion class.)

Theorem 12. *For each l-group G let $\theta_0(G)$ consist of all l-automorphisms σ in $A(G)$ for which there is an ascending sequence $1 = A_0 \leqq A_1 \leqq \ldots \leqq A_\beta = G$ (β an ordinal number) of convex l-subgroups, so that $A_\alpha\sigma = A_\alpha$ for all $\alpha < \beta$, $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ for each limit ordinal γ, and σ induces the identity on the right cosets of $A_{\alpha+1}/A_\alpha$; (ie. $A_\alpha x\sigma = A_\alpha x$, if $x \in A_{\alpha+1}$.)*

Then θ_0 is an exact, hereditary subgroup of A. Let $\mathscr{T}_{\text{aut}} = \mathscr{N}(\theta_0)$; then $G \in \mathscr{T}_{\text{aut}}$ if and only if to each $g \in G$ there exists an ascending sequence of convex l-subgroups $1 = A_0 \leqq A_1 \leqq \ldots \leqq A_\alpha \leqq \ldots \leqq A_\beta = G$ such that $A_\alpha^g = A_\alpha$ for all $\alpha < \beta$, $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ if γ is a limit ordinal, and $A_\alpha x^g = A_\alpha x$, if $x \in A_{\alpha+1}$.

$\mathscr{T}_{\text{aut}}$ is an equational class containing all l-nilpotent l-groups.

(Note: an l-group G is *l-nilpotent (of rank m)* if there is a sequence $1 = G_0 \leqq G_1 \leqq \ldots \leqq G_m = G$ of l-ideals such that each G_{i+1}/G_i is central in G/G_i. This notion is stronger than conventional nilpotency; see Martinez [9].)

Proof of Theorem 12. It is to our advantage to introduce, for this proof, the following terminology: if $\sigma \in \theta_0(G)$ and $1 = A_0 \leqq A_1 \leqq \ldots \leqq A_\beta = G$ is a sequence of convex l-subgroups as specified in the definition of θ_0, we shall refer to it as a *σ-central ascending sequence.*

Suppose $\phi : G \to H$ is an l-epimorphism and $K = \operatorname{Ker}(\phi)$. Suppose $\sigma \in \theta_0(G)$ and fixes K. Select a σ-central ascending sequence $1 = A_0 \leqq A_1 \leqq \ldots \leqq A_\beta = G$, and set $B_\alpha = A_\alpha\phi$, for each ordinal $\alpha < \beta$. Then $B_\alpha\sigma^K = A_\alpha\phi\sigma^K = A_\alpha\sigma\phi = B_\alpha$.

Furthermore, induces a canonical l-homomorphism ϕ_α from $A_{\alpha+1}/A_\alpha$ onto $B_{\alpha+1}/B_\alpha$. It is easy to verify that because σ induces the identity on $A_{\alpha+1}/A_\alpha$ σ^K induces the identity on $B_{\alpha+1}/B_\alpha$. It should be clear then that $1 = B_0 \leqq B_1 \leqq \dots \leqq B_\beta = H$ is a σ^K-central ascending sequence. Conclusion: $\sigma^K \in \theta_0(H)$.

That takes care of (A1). (A4) is proved analogously. (A2) is demonstrated by piecing together two ascending sequences which are central with respect to a restriction (to an l-ideal), and a quotient, respectively.

Suppose now that $\{G_i \mid i \in I\}$ is a family of convex l-subgroups of G and that $G = \bigvee G_i$. Pick $\sigma \in A(G)$ and suppose that σ fixes each G_i, while the restriction σ_i to G_i belongs to $\theta_0(G_i)$. Select a σ_i-central ascending sequence in G_i (for each $i \in I$), denoted: $A_{0,i} \leqq A_{1,i} \leqq \dots \leqq A_{\beta_i i}/= G_i$. Since the β_i are bounded we may without loss of generality assume that all the β_i are equal (to β). For each $\alpha < \beta$ let $A_\alpha = \bigvee_{i\in I} A_{\alpha,i}$; clearly $A_\alpha\sigma = A_\alpha$ for each $\alpha < \beta$. Also, $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ for limit ordinals γ. The inclusions of $A_{\alpha,i} \leqq A_\alpha$ induce canonical l- homomorphisms $\tau_{\alpha,\iota}$ from $A_{\alpha+1,i}/A_{\alpha,i}$ into $A_{\alpha+1}/A_\alpha$. The images generate $A_{\alpha+1}/A_\alpha$. Since each σ_i induces the identity on each $A_{\alpha+1,i}/A_{\alpha,i}$, σ does the same on $A_{\alpha+1}/A_\alpha$. This proves that $\sigma \in \theta_0(G)$ and settles (A3).

Conclusion: θ_0 is an exact hereditary subgroup of A.

It is clear that $G \in \mathscr{N}(\theta_0)$ if and only if each $g \in G$ determines a sequence $1 = A_0 \leqq \leqq A_1 \leqq \dots \leqq A_\beta = G$ of convex l-subgroups of G such that $A_\alpha^g = A_\alpha$ for each $\alpha < \beta$, $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ for limit ordinals γ, and $A_\alpha x^g = A_\alpha x$ for all $\alpha < \beta$ and $x \in A_{\alpha+1}$. We let $\mathscr{T}_{\text{aut}} = \mathscr{N}(\theta_0)$.

Now to establish the minimality of $\mathscr{T}_{\text{aut}}$ among the automorphically defined classes, (whether torsion or no). To do this we shall first establish that each $\sigma \in \theta_0(G)$ fixes every value of G. It implies that if $G \in \mathscr{T}_{\text{aut}}$, then each convex l-subgroup of G is normal. Accepting this, suppose $\mathscr{T} = \mathscr{N}(\theta)$, where θ is an exact subgroup of A. If $G \in \mathscr{T}_{\text{aut}}$, select for a given $g \in G$ a g-central ascending sequence (that is, relative to conjugation by g), $1 = A_0 \leqq A_1 \leqq \dots \leqq A_\beta = G$. On each factor $A_{\alpha+1}/A_\alpha$ conjugation by g induces the identity, which certainly belongs to $\theta(A_{\alpha+1}/A_\alpha)$. Now apply (A2) and (A3) repeatedly to conclude that the conjugation by g is in $\theta(G)$. Clearly $G \in \mathscr{T}$.

So let's prove that each $\sigma \in \theta_0(G)$ fixes every value N of G. Select a σ-central ascending sequence $1 = C_0 \leqq C_1 \leqq \dots \leqq C_\delta = G$. Find the least ordinal δ_0 for which $C_{\delta_0} \nleqq N$. It is clear that δ_0 is not a limit ordinal. Further, if $x \in N \cap C_{\delta_0}$ then $x\sigma \in C_{\delta_0 - 1}x \leqq N$, so that $x\sigma \in N \cap C_{\delta_0}$. All of this holds for σ^{-1} and therefore $N\sigma \cap C_0 = N \cap C_0$. This proves $N\sigma = N$.

It is evident that $\mathscr{T}_{\text{aut}}$ contains all the l-nilpotent l-groups.

To prove that $\mathscr{T}_{\text{aut}}$ is equational it is enough, by Birkhoff's Theorem on equational classes, to prove that $\mathscr{T}_{\text{aut}}$ is closed under taking l-subgroups and direct products. We shall only carry out the proof for l-subgroups. For direct products the argument is quite similar to that of (A3) above.

So suppose H is an l-subgroup of $G \in \mathcal{T}_{\text{aut}}$. If $h \in H$, let $1 = A_0 \leqq A_1 \leqq \ldots \leqq \leqq A_\beta = G$ be an h-central ascending sequence. Set $B_\alpha = A_\alpha \cap H$; each B_α is a convex l-subgroup of H, $B_\alpha^h = B_\alpha$ for all $\alpha < \beta$ and $B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha$ if γ is a limit ordinal. Finally, if $x \in B_{\alpha+1}$ then $x^h x^{-1} \in A_\alpha \cap H = B_\alpha$. Conclusion: $1 = B_0 \leqq B_1 \leqq \ldots \ldots \leqq B_\beta = H$ is h-central, and $H \in \mathcal{T}_{\text{aut}}$. Thus, $\mathcal{T}_{\text{aut}}$ is equational, and the theorem is at long last proved.

(The equational aspect of $\mathcal{T}_{\text{aut}}$ was observed by A.M.W. Glass during a seminar.)

In the sequel we suppose $\mathcal{T} = \mathcal{N}(\theta)$ is an automorphically defined torsion class; suppose $\mathcal{U}$ is an arbitrary torsion class. Set $\Gamma(G) = \{\sigma \in A(G) \mid \sigma^{\mathcal{U}(G)} \in \theta(G/\mathcal{U}(G))\}$. It should be evident that $\Gamma(G)$ is a subgroup of $A(G)$. We will show that it is an exact subgroup.

Suppose $\sigma \in \Gamma(G)$ and $\phi : G \to H$ is an l-epimorphism fixing the kernel. From properties of torsion radicals: $\mathcal{U}(G) \leqq \mathcal{U}(H)\phi^{-1}$, and so $G/\mathcal{U}(H)\phi^{-1}$ is an l-homomorphic image of $G/\mathcal{U}(G)$. $\sigma^{\mathcal{U}(G)} \in \theta(G/\mathcal{U}(G))$, and therefore induces an l-automorphism τ of $G/\mathcal{U}(H)\phi^{-1}$, since $\sigma^{\mathcal{U}(G)}$ fixes the kernel of the canonical map $G/\mathcal{U}(G) \to G/\mathcal{U}(H)\phi^{-1}$. Since θ is exact, $\tau \in \theta(G/\mathcal{U}(H)\phi^{-1})$. Furthermore, $G/\mathcal{U}(H)\phi^{-1}$ is naturally l-isomorphic to $H/\mathcal{U}(H)$, in a way that τ induces an l-automorphism τ' in $\theta(H/\mathcal{U}(H))$. This suffices to show that the l-automorphism σ^* induced by σ on H belongs to $\Gamma(H)$, as it, in turn, induces τ'. This proves (A1).

We leave (A2) to the reader. As for (A3), suppose that $\{G_i \mid i \in I\}$ is a family of convex l-subgroups of G and $G = \bigvee_{i\in I} G_i$. Suppose $\sigma \in A(G)$ and fixes each G_i; in addition, we assume that each restriction σ_i belongs to $\Gamma(G_i)$. Once again, we have by properties of radicals: $\mathcal{U}(G_i) \leqq \mathcal{U}(G) \cap G_i$, and therefore $G_i/\mathcal{U}(G) \cap G_i$ is an l-homomorphic image of $G_i/\mathcal{U}(G_i)$. Both convex l-subgroups in question are fixed by σ, and σ therefore induces an l-automorphism of $G_i/\mathcal{U}(G) \cap G_i$ as well as $G_i/\mathcal{U}(G_i)$. Denote these by ϱ^i and σ^i respectively. Since $\sigma^i \in \theta(G_i/\mathcal{U}(G_i))$ and θ is exact, $\varrho^i \in \theta(G_i/\mathcal{U}(G) \cap G_i))$. Also $G_i/\mathcal{U}(G) \cap G_i \simeq G_i \vee \mathcal{U}(G)/\mathcal{U}(G)$, and ϱ^i induces an an l-automorphism τ^i of $G_i \vee \mathcal{U}(G)/\mathcal{U}(G)$ belonging to $\theta(G_i \vee \mathcal{U}(G)/\mathcal{U}(G))$. These τ^i are all restrictions of $\sigma^{\mathcal{U}(G)}$ on $G/\mathcal{U}(G)$. Since $G/\mathcal{U}(G) = \bigvee_{i\in I} G_i \vee \mathcal{U}(G)/\mathcal{U}(G)$, it follows that $\sigma^{\mathcal{U}(G)} \in \theta(G/\mathcal{U}(G))$, and hence $\sigma \in \Gamma(G)$.

Thus, $\Gamma(G)$ is an exact subgroup of $A(G)$ containing $\theta(G)$. $\mathcal{N}(\theta) = \{G \mid$ for each $g \in G$ conjugation by $g\,\mathcal{U}(G)$ belongs to $\theta(G/\mathcal{U}(G))\} = \{G \mid G/\mathcal{U}(G) \in \mathcal{T}\}$.

Recall from [13] that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are any two classes of l-groups then $\mathcal{C}_1 . \mathcal{C}_2$ is the class of all l-groups G having an l-ideal $A \in \mathcal{C}_1$ so that $G/A \in \mathcal{C}_2$.

We've therefore established the following:

Proposition 13. *Suppose $\mathcal{T} = \mathcal{N}(\theta)$ is an automorphically defined torsion class and $\mathcal{U}$ is an arbitrary torsion class. Let $\Gamma(G) = \{\sigma \in A(G) \mid \sigma^{\mathcal{U}(G)} \in \theta(G/\mathcal{U}(G))\}$. Then Γ is an exact subgroup of A. $\mathcal{N}(\Gamma) = \mathcal{U} . \mathcal{T}$. If θ and $\mathcal{U}$ are hereditary so is Γ.*

Corollary. *If $\mathcal{T}$ and $\mathcal{U}$ are torsion classes and $\mathcal{T}$ is automorphically defined then so is $\mathcal{U} . \mathcal{T}$.*

With regard to the binary operation $\cdot$ between torsion classes, it was observed in [10] that one distributive law, $\mathscr{U} . (\bigvee_{i \in I} \mathscr{T}_i) = \bigvee_{i \in I} \mathscr{U} . \mathscr{T}_i$, always works for hereditary classes. Hereditariness hardly plays a role; it's true for all torsion classes. Ditto for the dual law, relative to meets. Distributivity from the right does not work in general, even for hereditary classes. It does, for meets, if the class to be distributed is closed under taking l-subgroups. For hereditary classes it's enough to assume closure of the class $\mathscr{T}$ under *finite* subdirect products to distribute $\mathscr{T}$ over *finite* meets. The proof is easy and will be left as an exercise.

Now, complete torsion classes distributive over finite meets (from the right) because they are closed under taking finite subdirect products. In [13], Proposition 4.4.2, we show that if $\mathscr{T}$ is automorphically defined by a hereditary subgroup then $\mathscr{T}$ is closed under finite subdirect products. Thus, we have yet another class of torsion classes which can be distributed from the right over finite meets.

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