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A CHARACTERIZATION OF POLARITIES WHOSE LATTICE OF POLARS IS BOOLEAN

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By a polarity in a set X we shall mean a symmetric binary relation δ in X. Sets closed under the polarity, the so called polars, are topics of particular interest. The set of polars $\Gamma_\delta(X)$ is a complete lattice in which infima are set meets [1] IV § 5. In some particular cases the lattice $\Gamma_\delta(X)$ is a Boolean algebra; let us recall – as an example for many others – a polarity (disjointness) in an l-group defined as follows: $x \delta y \equiv |x| \wedge |y| = 0$. In the paper [3] properties of a polarity δ are described, which are sufficient for $\Gamma_\delta(X)$ to be a Boolean algebra (see below properties (Da) to (Dd)).

In the present note we shall prove that the above mentioned conditions are necessary as well (cf. Theorem 4 below). An alternative proof of Theorem 4 could be established by using Theorem 2.3 of Bondarev's paper [2], which also deals with the problem of characterizing a polarity δ possessing the property that the lattice of its polars Γ_δ is a Boolean algebra. Note that Theorems 2.1 and 2.2 [2] are essentially known (see Theorem 3 below).

Throughout this paper X denotes a nonempty set.

Definition 1. A symmetric binary relation in a set X is called a *polarity in X*.

Definition 2. ([2] Definition 1.0.) Let δ be a polarity in a set X. Let $\prec$ be a binary relation in X defined as follows ($x, y, u \in X$):

$$x \prec y \Leftrightarrow (u \delta y \Rightarrow u \delta x).$$

We say that $\prec$ is *induced* (in X) by δ. Obviously, $\prec$ is a quasi-order in X, i.e. a reflexive and transitive binary relation.

Definition 3. (Cf. [2] Definition 1.1, [3] 1.3, [4] Sec. C, p. 85, [5] § 1.) Let δ be a polarity in a set X and let $\prec$ be induced by δ. Denote ($x, y, \in X$):
(Dα) antireflexivity of δ (i.e., $x \delta x \Rightarrow x \delta y$ for every $y \in X$),
(Dβ) $x \bar{\delta} y \Rightarrow$ there exists $z \in X$ such that $z \bar{\delta} z$, $z \prec x$, $z \prec y$.

A couple (X, δ) fulfilling $(D\alpha)$ is called a *D-set*. If it fulfils both $(D\alpha)$ and $(D\beta)$ it is called a *D*-set* (an n. v. D-set in [2]).

Lemma 1. *Let δ be a polarity in a set X. Then*

$$(X, \delta) \text{ is a D-set} \Leftrightarrow \{x \in X: x \, \delta \, x\} = \{x \in X: x \, \delta \, y \text{ for every } y \in X\}.$$

Proof is evident.

Remark 1. In Bondarev's definition of a D-set the following identity is supposed: $\{x \in X: x \, \delta \, x\} = \{x \in X: x \, \delta \, y$ for every $y \in X\} =$ a singleton or $= \emptyset$ ([2] p. 16).

Lemma 2. *Let (X, δ) be a D-set. If $\delta \neq \emptyset$ then the set N of all least elements of X with respect to the quasi-order $\prec$ induced by δ is equal to $\Lambda = \{x \in X: x \, \delta \, x\}$. If $\delta = \emptyset$ then $\Lambda = \emptyset$ and $N = X$.*

Proof. If $\delta = \emptyset$ then obviously $\Lambda = \emptyset$ and $N = X$. Let $\delta \neq \emptyset$. The inclusion $\Lambda \subseteq N$ holds by $(D\alpha)$. To prove $\Lambda \supseteq N$ fix $n \in N$. There exist $x, y \in X$ with $x \, \delta \, y$. Now, $x \, \delta \, y$, $n \prec y$ implies $x \, \delta \, n$ and this together with $n \prec x$ gives $n \, \delta \, n$, hence $n \in \Lambda$. Thus $\Lambda = N$ is proved.

Definition 4. (Cf. [4] Sec. C, p. 85, [3] 1,3.) Let $\lhd$ be a quasi-order in a set X, N the set of all least elements of X with respect to $\lhd$ and let δ be a polarity in X such that the following implications are satisfied $(x, y \in X)$:

(Da) $x \, \delta \, x \Rightarrow x \, \delta \, y$ for every $y \in X$ (antireflexivity of δ),
(Db) $x \, \delta \, y$, $x \lhd y \Rightarrow x \in N$,
(Dc) $x \, \delta \, y$, $z \lhd y \Rightarrow x \, \delta \, z$,
(Dd) $x \, \delta \, y \Rightarrow$ there exists $z \in X$ such that $z \bar{\in} N$, $z \lhd x$, $z \lhd y$.

Then the triple $(X, \lhd, \delta)$ is called a *P-set*.

Remark 2. In [3] and [4], N is supposed to be non empty. Also, the name of a "P-set" is not used there.

We shall prove that, if $\delta \neq \emptyset$, the notions of a D^*-set and a P-set are equivalent in the sense that the structure of one type can be transferred in a uniquely defined way onto the structure of the other type. A more detailed account is given in the following Theorems 1 and 2.

Theorem 1. *Let (X, δ) be a D*set and $\delta \neq \emptyset$. Then the relation $\prec$ induced by δ is a quasiorder and $(X, \prec, \delta)$ is a P-set.*

Proof. Denote by N the set of all least elements in X with respect to $\prec$ and $\Lambda = \{x \in X: x \, \delta \, x\}$. By Lemma 2, $\Lambda = N$. We shall prove that (Da) to (Dd) hold.

(Da) $\equiv$ $(D\alpha)$.
(Db): Suppose $x \, \delta \, y$, $x \prec y$. The second relation means that $u \, \delta \, y \Rightarrow u \, \delta \, x$. Since $x \, \delta \, y$, then $x \, \delta \, x$, hence $x \in \Lambda = N$.

(Dc): Suppose $x \delta y$, $z \prec y$. The second relation means that $u \delta y \Rightarrow u \delta z$. Since $x \delta y$, then $x \delta z$.
(Dd) and (Dβ) are identical conditions because $A = N$.

Theorem 2. *Let $(X, \lhd, \delta)$ be a P-set. Then (X, δ) is a D^*-set.*

Proof. (Dα) ≡ (Da), hence (X, δ) is a D-set.
(Dβ): Denote by N the set of all least elements of X with respect to $\lhd$. Suppose $x \lhd y$, $u \delta y$. Then $u \delta x$ by (Dc). So we have $x \lhd y \Rightarrow x \prec y$, where $\prec$ means the relation induced by δ. Next, by (Db), $A \subseteq N$ because $x \delta x$, $x \lhd x \Rightarrow x \in N$. Now evidently (Dd) implies (Dβ).

Definition 5. Let δ be a polarity in a set X, $(\emptyset \subseteq) A \subseteq X$. If there exists $(\emptyset \subseteq) B \subseteq X$ such that $A = B^\delta$, where $B^\delta = \{x \in X: x \delta b \text{ for every } b \in B\}$, then A is called a *polar*. The set of all polars in (X, δ) will be denoted by $\Gamma_\delta(X)$ (or briefly by $\Gamma(X)$ or Γ).

Several names have been used for the notion of a polar: komponenta in Df. 1.2 [2], δ-Komponente in [4], p. 85, or Komponente in 1,4,1 [3]. Below, we shall use the term of a polar which is currently used at present, e.g. in the theory of l-groups.

The following Theorem 3 is known.

Theorem 3. A) *Let δ be a polarity in a set X. Then $\Gamma_\delta(X)$ is a complete lattice, infima in Γ are set meets, X and $\{x \in X: x \delta y \text{ for every } y \in X\}$ are the greatest and least elements of Γ, respectively, and the map $A \in \Gamma \to A^\delta$ is an involution, i.e. $A^{\delta\delta} = A$, $(\bigvee A_\alpha)^\delta = \bigwedge A_\alpha^\delta$, $(\bigwedge A_\alpha)^\delta = \bigvee A_\alpha^\delta$ for all $A, A_\alpha \in \Gamma$.*

B) *Let (X, δ) be a D-set. Then the lattice $\Gamma_\delta(X)$ is complemented and A^δ is a complement of $A \in \Gamma_\delta(X)$.*

C) *Let (X, δ) be a D^*-set. Then $\Gamma_\delta(X)$ is a complete Boolean algebra.*

For A) and B) see Corollary to Theorem 9 [1] IV § 5 (see also [4] Sec. A and B, p. 85 or 1,3,3 [3]). The statement C) is clear if $\delta = \emptyset$. If $\delta \neq \emptyset$, then C) is an immediate consequence of Theorem 1 and [3] Hauptsatz 1,4,4, which states that $\Gamma_\delta(X)$ is a complete Boolean algebra if $(X, \prec, \delta)$ is a P-set.

The converse of Theorem 3 is also true. We have the following result.

Theorem 4. A) *Let $\mathfrak{B}$ be a complete lattice of subsets of a set $Y (\neq \emptyset)$, let infima in $\mathfrak{B}$ be set meets and let $A \to A'$ be a map of $\mathfrak{B}$ into $\mathfrak{B}$ fulfilling $A'' = A$, $(\bigvee A_\alpha)' = \bigwedge A'_\alpha$ for all $A, A_\alpha \in \mathfrak{B}$. Denote by X the greatest element of $\mathfrak{B}$. Then there exists a polarity δ in X such that $\Gamma_\delta(X) = \mathfrak{B}$.*

B) *Let $\mathfrak{B}$ be as in A) and in addition, let A' be a complement of A for any A in $\mathfrak{B}$. Then (X, δ) is a D-set.*

C) *Let $\mathfrak{B}$ be a complete Boolean algebra of subsets of a set Y, let infima in $\mathfrak{B}$ be set meets. Denote by X the greatest element of $\mathfrak{B}$. Then there exists a polarity δ in X such that $\Gamma_\delta(X) = \mathfrak{B}$. Furthermore, (X, δ) is a D^*-set.*

Proof. A) First, $\mathfrak{B}$ is ordered by set inclusion, because of $A \geqq B \Leftrightarrow B = A \wedge \wedge B = A \cap B \Leftrightarrow A \supseteq B$. Next, for $x \in X$ put $\bar{x} = \bigcap\{A \in \mathfrak{B}: x \in A\}$. Obviously, $\bar{x} \in \mathfrak{B}$. Further, define for any $x, y \in X$

$$x \,\delta\, y \equiv y \in \bar{x}' .$$

δ is a polarity in X. In fact, $A, B \in \mathfrak{B}$, $A \supseteq B \Rightarrow B' = (A \cap B)' = A' \vee B' \Rightarrow \Rightarrow B' \supseteq A'$ and so $y \in \bar{x}' \Rightarrow \bar{y} \subseteq \bar{x}' \Rightarrow \bar{y}' \supseteq \bar{x}'' = \bar{x} \ni x$.

It follows that $\bar{x}' = x^\delta \in \Gamma_\delta(X)$ for every $x \in X$. Let $A \in \mathfrak{B}$ with $A' \neq \emptyset$. Then $A' = \bigvee\{\bar{x}: x \in A'\}$ and therefore $A = A'' = \bigcap\{\bar{x}': x \in A'\} = \bigcap\{x^\delta: x \in A'\} \in \Gamma_\delta(X)$, thus $A \in \Gamma_\delta(X)$. If $A' = \emptyset$ then $X = A$, since $X \supseteq A \Rightarrow X' \subseteq A' = \emptyset \Rightarrow X' = \emptyset \Rightarrow \Rightarrow X = X'' = \emptyset' = A'' = A$. Thus $X = \bigvee\{\bar{x}: x \in X\} \Rightarrow \emptyset = X' = \bigcap\{x^\delta: x \in X\} \in \in \Gamma_\delta(X)$, hence $A = X = \emptyset^\delta \in \Gamma_\delta(X)$. Conversely for $C \in \Gamma_\delta(X)$, $C^\delta \neq \emptyset$, we have $C^\delta = \bigvee_\Gamma\{x^{\delta\delta}: x \in C^\delta\}$, hence by Theorem 3(A) $C = C^{\delta\delta} = \bigcap\{x^\delta: x \in C^\delta\} = = \bigcap\{\bar{x}': x \in C^\delta\} \in \mathfrak{B}$ (since $x^{\delta\delta\delta} = x^\delta$). If $C^\delta = \emptyset$ then $X = C$ as above and $C = = X \in \mathfrak{B}$. We have proved that both $\mathfrak{B}$ and $\Gamma_\delta(X)$ are identical as sets and also as lattices, since their orders are the same.

B) $X^\delta = \{x \in X: x \,\delta\, y \text{ for } y \in X\}$ is the least element of $\Gamma_\delta(X)$, since by Theorem 3, δ is an involution and X the greatest element of $\mathfrak{B}$. Now evidently $\Lambda = \{x \in X: x \,\delta\, x\} \supseteq X^\delta$. To show $\subseteq$ suppose $x \,\delta\, x$. Then

(a) $$x \in x^{\delta\delta} \cap x^\delta = \bar{x} \cap \bar{x}' = X^\delta ,$$

so $\Lambda \subseteq X$. (The assertions of (a) can be proved as follows: 1. $x \in x^{\delta\delta}$ by Def. 5, 2. $x^\delta = \bar{x}'$ by (A), 3. $\bar{x} = \bigcap\{A \in \mathfrak{B}: x \in A\} = \bigcap\{A \in \mathfrak{B}: x^{\delta\delta} \subseteq A\} = x^{\delta\delta}$, since $\mathfrak{B} = = \Gamma_\delta(X)$ and for $A \in \mathfrak{B}$ we have $x \in A \equiv x^{\delta\delta} \subseteq (A^{\delta\delta} =) \subseteq A$, 4. $\bar{x} \cap \bar{x}' = X^\delta$, since $'$ is the symbol of a complement in $\mathfrak{B}$.) Hence $\{x \in X: x \,\delta\, x\} = \Lambda = X^\delta = \{x \in X: x \,\delta\, y \text{ for every } y \in X\}$. By Lemma 1, (X, δ) is a D-set.

C) Suppose (by way of contradiction) that $x, y \in X$ exist not fulfilling (Dβ), i.e. $x \,\bar{\delta}\, y$ and $(z \prec x, z \prec y \Rightarrow z \,\delta\, z)$, where $\prec$ is induced by δ. Since $z \in x^{\delta\delta} \Leftrightarrow (x \,\delta\, b \Rightarrow \Rightarrow z \,\delta\, b) \Leftrightarrow z \prec x$, we obtain $x^{\delta\delta} \cap y^{\delta\delta} \subseteq \{z \in X: z \,\delta\, z\} = \Lambda =$ the least element of $\Gamma_\delta(X)$ (by (B)), thus $x^{\delta\delta} \cap y^{\delta\delta} = \Lambda$. Because x^δ is a complement of $x^{\delta\delta}$ in $\Gamma_\delta(X) = \mathfrak{B}$ (by the proof of (B)), then $y^{\delta\delta} \subseteq x^\delta$, hence $x \,\delta\, y$, a contradiction. This comleptes the proof.

Remark 3. Theorem 4 implies Bondarev's Theorem 2.3 [2]. Theorem 2.3 [2]: If $\mathfrak{B}$ is a complete Boolean algebra and (X, δ) (defined in (C)) a D-set (in the stronger sense given in Remark 1), then (X, δ) is a D^*-set.

Corollary. *Let $\mathfrak{B}$ and δ be as in Theorem* 4(C), *let $\prec$ be induced by δ and $\delta \neq \emptyset$. Then $(X, \prec, \delta)$ is a P-set.*

Note that $\delta = \emptyset \Leftrightarrow \mathfrak{B} = \{X, \emptyset\}$.

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