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GRAPH REPRESENTATIONS OF FINITE ABELIAN GROUPS

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To every graph G we can assign the group $\operatorname{Aut} G$ of all automorphisms of G. Making use of the results of A. Cayley, R. Frucht [1] proved that for every finite group there exists a graph whose automorphism group is isomorphic to this group.

V. G. Vizing [2] suggested the investigation of special types of graphs assigned to groups which will be called here *graph representations of groups.*

A graph representation of a group $\mathfrak{G}$ is a graph with the property that its automorphism group is isomorphic to $\mathfrak{G}$ and to any two vertices x, y of this graph there exists exactly one automorphism φ of this graph such that $\varphi(x) = y$.

A graph representation of a group may be an undirected graph or a directed one. Therefore for a group $\mathfrak{G}$ we shall distinguish its undirected graph representation $UR(\mathfrak{G})$ and its directed graph representation $DR(\mathfrak{G})$.

Proposition. *Let $\mathfrak{G}$ be a group for which an undirected graph representation exists. Then there exists also a directed graph representation of $\mathfrak{G}$.*

Proof. If in the undirected graph representation of $\mathfrak{G}$ each undirected edge xy is substituted by the pair of directed edges $\overrightarrow{xy}$, $\overrightarrow{yx}$, we obtain a directed graph representation of $\mathfrak{G}$.

We shall study graph representations of finite Abelian groups. It is well-known that each non-trivial finite Abelian group can be expressed as a direct product of primary cyclic groups, i.e., cyclic groups whose orders are powers of prime numbers. (By a non-trivial group we mean a group with more than one element.) We shall prove some theorems on directed graph representations of these groups.

Theorem 1. *Let $\mathfrak{G}$ be a non-trivial finite Abelian group which can be expressed as a direct product of cyclic groups of pairwise different orders. Then the group $\mathfrak{G}$ has a directed graph representation.*

Proof. Let $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$ be the factors in the expression of $\mathfrak{G}$ as a direct product of cyclic groups and let them have pairwise different orders. Let c_i be the order of $\mathfrak{C}_i$ for $i = 1, \ldots, n$. Without loss of generality we may suppose that $c_1, \ldots, c_n$ is an increasing sequence. Now we shall construct a graph G. The vertex set of G will be the support of $\mathfrak{G}$. In each $\mathfrak{C}_i$ for $i = 1, \ldots, n$ we choose its generator a_i; denote $A = \{a_1, \ldots, a_n\}$. If x and y are two vertices of G (elements of $\mathfrak{G}$), then the edge $\overrightarrow{xy}$ exists in G if and only if $x^{-1}y \in A$.

Now for each $b \in \mathfrak{G}$ let φ_b be a mapping of $\mathfrak{G}$ onto $\mathfrak{G}$ such that $\varphi_b(x) = bx$. The mapping φ_b for each $b \in \mathfrak{G}$ is a bijection of $\mathfrak{G}$ onto $\mathfrak{G}$; this follows from the fact that $\mathfrak{G}$ is a group. Let x, y be two vertices of G. We have $(\varphi_b(x))^{-1} (\varphi_b(y) = (bx)^{-1} by = x^{-1}b^{-1}by = x^{-1}y$, which implies that the edge $\overrightarrow{\varphi_b(x)\,\varphi_b(y)}$ exists in G if and only if the edge $\overrightarrow{xy}$ exists in G. Hence φ_b is an automorphism of G for each $b \in \mathfrak{G}$. Further, if $b \in \mathfrak{G}$, $c \in \mathfrak{G}$, then $\varphi_b\varphi_c(x) = \varphi_b(cx) = bcx = \varphi_{bc}(x)$ for each $x \in \mathfrak{G}$, therefore the mappings φ_b for all $b \in \mathfrak{G}$ form a group (with respect to the superposition) isomorphic to $\mathfrak{G}$.

Let x, y be two vertices of G such that $x^{-1}y = a_1$. There exists the edge $\overrightarrow{xy}$ in G and it is contained in a cycle $F_1(x)$ of the length c_1; this cycle has the vertices xa_1^k for $k = 1, \ldots, c_1$ and the edges going from xa_1^k into xa_1^{k+1} for each such k. We shall prove that each cycle of the length c_1 in G is $F_1(x)$ for some $x \in \mathfrak{G}$. Let D be a cycle of the length c_1, let $d_1, \ldots, d_{c_1}$ be its vertices, let its edges be $\overrightarrow{d_i d_{i+1}}$ for $i = 1, \ldots, c_1 - 1$ and $\overrightarrow{d_{c_1} d_1}$. For each $i = 1, \ldots, c_1 - 1$ let $z_i = d_i^{-1}d_{i+1}$ and let $z_{c_1} = d_{c_1}^{-1}d_1$. We have $z_i \in A$ for $i = 1, \ldots, c_1$. Further, evidently $d_1 = d_1 z_1 \ldots z_{c_1}$, which implies $z_1 \ldots z_{c_1} = e$, where e is the unit element of $\mathfrak{G}$. The set A is a set of independent generators of $\mathfrak{G}$ and $\mathfrak{G}$ is Abelian, therefore the number of occurrences of each a_i for $i = 1, \ldots, c_1$ among the elements $z_1, \ldots, z_{c_1}$ must be an integral multiple of c_i. As c_1 is less than c_i for $i > 1$, we have $z_j = a_1$ for $j = 1, \ldots, c_1$ and $D = F_1(d_1)$. If we denote $E_i = \{\overrightarrow{xy} \mid x \in \mathfrak{G},\ y \in \mathfrak{G},\ x^{-1}y = a_i\}$ for $i = 1, \ldots, n$, then we may assert that an edge of G belongs to E_1 if and only if it belongs to a cycle of G of the length c_1. Hence each automorphism ψ of G maps each edge of E_1 again onto an edge of E_1 and we have $(\psi(x))^{-1}\,\psi(y) = a_1$ if and only if $x^{-1}y = a_1$. Inductively we can prove that for each $i = 1, \ldots, n$ an edge of G belongs to E_i if and only if it belongs to a cycle of G of the length c_i and does not belong to a cycle of G of the length c_j for $j < i$. This implies that for each $i = 1, \ldots, n$ we have $(\psi(x))^{-1}\,\psi(y) = a_i$ if and only if $x^{-1}y = a_i$, where ψ is an arbitrary automorphism of G. In other words, $(\psi(x))^{-1}\,\psi(y) = x^{-1}y$ for each x and y such that $x^{-1}y \in A$. But then $\psi(x)\,x^{-1} = \psi(y)\,y^{-1}$ for each x, y such that $\overrightarrow{xy}$ is an edge of G. The graph G is strongly connected; this follows from the fact that A is a set of generators of $\mathfrak{G}$ and hence if x and y are arbitrary elements of $\mathfrak{G}$, the element y can be obtained from x by successive multiplication by elements of A. Thus by induction according to the length of the shortest path from x into y we can prove that $\psi(x)\,x^{-1} = \psi(y)\,y^{-1}$ for any two elements of $\mathfrak{G}$. If we denote $b = \psi(x)\,x^{-1}$ for $x \in \mathfrak{G}$, we see that $\psi = \varphi_b$. As ψ was an arbitrary automorphism of G, we see that the automorphism group of G is the group consisting of all φ_b for $b \in \mathfrak{G}$; as was proved above, this group is isomorphic to $\mathfrak{G}$ and G is a directed graph representation of $\mathfrak{G}$.

Theorem 2. *Let $\mathfrak{G}$ be a group with the property that for each set A of generators of $\mathfrak{G}$ there exists a non-identical automorphism α of $\mathfrak{G}$ such that $\alpha(A) = A$. Then there exists no graph representation of $\mathfrak{G}$.*

Proof. Suppose that $\mathfrak{G}$ has the mentioned property and that there exists a directed graph representation $DR(\mathfrak{G})$ of $\mathfrak{G}$. We shall study its structure. If we choose a vertex v_0 in $DR(\mathfrak{G})$, then each vertex of $DR(\mathfrak{G})$ can be uniquely expressed as $\gamma(v_0)$, where γ is an automorphism of $\mathfrak{G}$. We choose an isomorphism of Aut $DR(\mathfrak{G})$ onto $\mathfrak{G}$ and then we assign the vertex $\gamma(v_0)$ that element of $\mathfrak{G}$ which is the image of γ in this isomorphism. In this way we can identify the elements of $\mathfrak{G}$ with vertices of $DR(\mathfrak{G})$ (the vertex v_0 is then identified with the unit element e of $\mathfrak{G}$). Thus, in the sequel, we shall treat vertices of $DR(\mathfrak{G})$ as elements of $\mathfrak{G}$.

Let x, y be two elements of $\mathfrak{G}$. Then $x = \gamma_x(e)$, $y = \gamma_y(e)$, where γ_x, γ_y are automorphisms of $DR(\mathfrak{G})$ which correspond to x, y in the mentioned isomorphism of Aut $DR(\mathfrak{G})$ onto $\mathfrak{G}$. There exists a unique automorphism of $DR(\mathfrak{G})$ which maps x onto y; this automorphism is $\gamma_y\gamma_x^{-1}$ and its image in the mentioned isomorphism is yx^{-1}. In order to simplify the notation, we can say (not distinguishing between isomorphic groups) that x is mapped onto y by the automorphism yx^{-1}.

If x_0y_0 is an edge of $DR(\mathfrak{G})$, then $\overrightarrow{xy}$ is an edge of $DR(\mathfrak{G})$ for any two vertices x, y such that $x^{-1}y = x_0^{-1}y_0$, because such elements x, y have the property that $x = ax_0$, $y = ay_0$ for some $a \in \mathfrak{G}$. Hence the graph $DR(\mathfrak{G})$ is uniquely determined by determining the set A of all elements a of $\mathfrak{G}$ such that $\overrightarrow{ea}$ is an edge of $DR(\mathfrak{G})$. If the order of $\mathfrak{G}$ is greater than two, then evidently $A \neq \emptyset$. Let $\mathfrak{G}(A)$ be the subgroup of $\mathfrak{G}$ generated by A. As we have seen, the vertices x and y are joined by an edge in $DR(\mathfrak{G})$ if and only if either $y = xa$ for $a \in A$ (then we have the edge $\overrightarrow{xy}$), or $y = xa^{-1}$ for $a \in A$ (then we have the edge $\overrightarrow{yx}$). By induction we can prove that x and y lie in the same connected component of $DR(\mathfrak{G})$ if and only if $y = xd$, where $d \in \mathfrak{G}(A)$. Therefore, the vertex set of each connected component of $DR(\mathfrak{G})$ is a left class of $\mathfrak{G}$ by $\mathfrak{G}(A)$. Evidently each of these connected components has a non-identical automorphism. If $\mathfrak{G}(A)$ is a proper subgroup of $\mathfrak{G}$, then there are at least two such components. If we choose a non-identical automorphism of one of them and extend it to the whole graph $DR(\mathfrak{G})$ by leaving all vertices of other components fixed, we obtain an automorphism β of $DR(\mathfrak{G})$. But then each vertex not belonging to the chosen component is mapped onto itself by both β and the identical automorphism of $DR(\mathfrak{G})$, which is a contradiction. Hence $\mathfrak{G}(A) = \mathfrak{G}$ and A is a system of generators of $\mathfrak{G}$. But then there exists an automorphism α of $\mathfrak{G}$ such that $\alpha(A) = A$ and α is not the identical automorphism of $\mathfrak{G}$. Let x and y be elements of $\mathfrak{G}$. As α is an automorphism of $\mathfrak{G}$, we have $\alpha(x^{-1}y) = (\alpha(x))^{-1}\,\alpha(y)$ and as $\alpha(a) = A$, we have $(\alpha(x))^{-1}\,\alpha(y) \in A$ if and only if $x^{-1}y \in A$. This implies that $\overrightarrow{\alpha(x)\,\alpha(y)}$ is an edge of $DR(\mathfrak{G})$ if and only if $\overrightarrow{xy}$ is an edge of $DR(\mathfrak{G})$ and α is an automorphism of $DR(\mathfrak{G})$. The vertex e is mapped onto itself by both α and the identical automorphism of $DR(\mathfrak{G})$, which is a contradiction. Therefore $DR(\mathfrak{G})$ does not exist. According to Proposition also no undirected graph representation of $\mathfrak{G}$ exists.

A simple example of a group fulfilling the condition of this theorem is the direct product of two cyclic groups of the order 2.

Conjecture. *Let $\mathfrak{G}$ be a finite group with the property that there exists a set A of generators of $\mathfrak{G}$ such that no non-identical automorphism of $\mathfrak{G}$ maps A onto itself. Then there exists a directed graph representation of $\mathfrak{G}$.*

Theorem 3. *Let $\mathfrak{G}$ be a finite Abelian group with at least one element of the order greater than 2. Then there exists no undirected graph representation of $\mathfrak{G}$.*

Proof. Let $\mathfrak{G}$ be a group with the mentioned property. Suppose that there exists an undirected graph representation of $\mathfrak{G}$. For this undirected graph representation of $\mathfrak{G}$ we can construct a directed graph representation of $\mathfrak{G}$ in the way described in the proof of Proposition. Define A in the same way as in the proof of Theorem 2. The graph $DR(\mathfrak{G})$ thus constructed has the property that for any two vertices x, y of $DR(\mathfrak{G})$ the existence of the edge $\overrightarrow{xy}$ is equivalent to the existence of the edge $\overrightarrow{yx}$. Hence for an arbitrary $a \in \mathfrak{G}$ we have $a \in A$ if and only if $a^{-1} \in A$. As $\mathfrak{G}$ is Abelian, the mapping α of $\mathfrak{G}$ onto $\mathfrak{G}$ such that $\alpha(x) = x^{-1}$ for each $x \in \mathfrak{G}$ is an automorphism of $\mathfrak{G}$. Evidently $\alpha(A) = A$. If x is an element of $\mathfrak{G}$ of an order greater than 2, then $\alpha(x) = x^{-1} \neq x$ and α is not the identical mapping of $\mathfrak{G}$. The mapping α is an automorphism of $DR(\mathfrak{G})$ (see the proof of Theorem 2). The unit element e is mapped onto itself by both α and the identical automorphism of $DR(\mathfrak{G})$, which is a contradiction. Hence there exists no undirected graph representation of $\mathfrak{G}$.

References

[1] *Frucht, R.:* Herstellung von Graphen mit vorgegebener abstrakter Gruppe. Compositio Math. *6* (1938), 239—250.

[2] *Визинг, В. Г.:* Некоторые нерешенные задачи в теории графов. Успехи мат. наук *23* (1968), 117—134.

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