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EXTENSIONS OF LEBESGUE SETS AND OF REAL-VALUED FUNCTIONS*)

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0. Definitions. For a topological space X, $C(X)$ (resp. $C^*(X)$) denotes the algebra of all real-valued (resp. bounded real-valued) continuous functions on X. For $f \in C(X)$, $A \subset C(X)$, and $a \in \boldsymbol{R}$ (the reals), set $Z(f) = \{x \in X : f(x) = 0\}$, $\operatorname{coz} f = X - Z(f)$, $\mathscr{Z}(X) = \{Z(f) : f \in C(X)\}$, $\operatorname{coz} A = \{\operatorname{coz} f : f \in A\}$, $L_a(f) = \{x \in X : f(x) \leqq a\}$, and $L^a(f) = \{x \in X : f(x) \geqq a\}$. Sets of the form $Z(f)$ (resp. $\operatorname{coz} f$) with $f \in C(X)$ are *zero-sets* (resp. *cozero-sets*) in X, and sets of the form $L_a(f)$ or $L^a(f)$ are *Lebesgue sets of* f (see [M_2] and [M_3]). (Lebesgue sets of functions in $C(X)$ are obviously zero-sets in X.) If $S \subset X$, S is *C-embedded* (resp. *C^*-embedded*) in X if every $f \in C(S)$ (resp. $f \in C^*(S)$) extends to a function in $C(X)$, and S is *z-embedded* in X if every zero-set in S extends to a zero-set in X. If $S \subset X$ and $f \in C(S)$, f is z-embedded in X if every Lebesgue set of f extends to a zero-set in X (see [B_1]).

For Tychonoff S and $f \in C(S)$: S is *absolutely* C- (resp. C^*-, resp. z-) *embedded* if S is C- (resp. C^*-, resp. z-) embedded in every Tychonoff superspace of S, and f is *absolutely extendible* (resp. *absolutely z-embedded*) if f extends continuously over (resp. is z-embedded in) every Tychonoff superspace of S.

1. Introduction. The point of this paper can best be described by first citing the following four known results:

(1) A subset S of X is C^*-embedded in X if and only if any two disjoint zero-sets in S are completely separated in X. (This is the Gillman-Jerison version of Urysohn's Extension Theorem [GJ, 1.17].)

(2) A subset S of X is C-embedded in X if and only if S is C^*-embedded in X and completely separated from every zero-set in X disjoint from S (Gillman and Jerison [GJ, 1.18]).

(3) A Tychonoff space S is absolutely C-embedded if and only if, of any two disjoint zero-sets in S, at least one is compact (Hewitt [He] and Smirnov [S]).

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(4) A Tychonoff space S is absolutely z-embedded if and only if S is Lindelöf or absolutely C-embedded (Hager and Johnson [HaJ, Theorem 3] and Blair and Hager [BH, 4.1]).

Each of (1)–(4) is global in the sense that it characterizes simultaneous extendibility of every function in $C^*(S)$ or in $C(S)$, or of every zero-set in S. Here we localize (and generalize) these results in such a way as to obtain theorems concerning extendibility of a *single* function f in $C^*(S)$ or in $C(S)$, or of every Lebesgue set of f. These localized versions of (1)–(4) appear as parts of 3.2, 3.8, 4.1, and 4.7, respectively. (Other studies in the same vein are in [M_2] and [M_4]. In particular, 3.2, for which we provide a self-contained proof, is due to Mrówka [M_2].) The present paper generalizes much of [BH].

Separation properties are assumed only in 2.5, 3.10, 3.15, and § 4. For notation and terminology not defined here, and for general background, see [GJ].

2. z-embedded functions. For S a subset of a topological space X, set $C_z(S, X) = \{f \in C(S): f \text{ is } z\text{-embedded in } X\}$. This section collects the relevant facts about approximation and partial extension of z-embedded functions, and about $C_z(S, X)$ (2.2, 2.3, and 2.5); for the most part, these are implicit in [BH], [Ha_2], and [M_3].

We first note the following:

2.1. Proposition. *Let $S \subset X$. S is z-embedded in X if and only if every (bounded) $f \in C(S)$ is z-embedded in X.*

Proof. Assume every function in $C^*(S)$ is z-embedded in X and let $f \in C(S)$. Then there is $Z \in \mathscr{Z}(X)$ with $S \cap Z = L_0(|f| \wedge 1) = Z(f)$. The converse is clear.

2.2. Theorem. *Let $S \subset X$ and let $f \in C(S)$. f is z-embedded in X if and only if f can be uniformly approximated on S by continuous functions on cozero-sets in X which contain S.*

Proof. Assume first that $f \in C(S)$ can be so approximated, and let $a \in \mathbf{R}$. For each integer $n > 0$, choose $P_n \in \operatorname{coz} C(X)$ with $S \subset P_n$ and $f_n \in C(P_n)$ with $|f_n(x) - f(x)| \leqq 1/n$ for every $x \in S$. Let $Z'_n = \{x \in P_n : f_n(x) \leqq a + (1/n)\}$. Since cozero-sets are z-embedded [BH, 1.1], there is $Z_n \in \mathscr{Z}(X)$ with $Z_n \cap P_n = Z'_n$. Then $\bigcap_n Z_n \in \mathscr{Z}(X)$ and $(\bigcap_n Z_n) \cap S = L_a(f)$; and since $L^a(f) = L_{-a}(-f)$, it follows also that there is $Z'' \in \mathscr{Z}(X)$ with $L^a(f) = S \cap Z''$. The (nontrivial) converse follows immediately from the proof of [BH, 2.2].

By a *C_δ-set* in X we mean the intersection of a countable family of cozero-sets in X.

2.3. Corollary. *Let $S \subset X$ and $f \in C(S)$. If f is z-embedded in X, then f extends continuously over some C_δ-set in X.*

Proof. The proof is like that of [BH, 2.4].

2.4. Remarks. (a) The converse of 2.3 fails. (As noted in [BH, 2.5(a)], the x-axis S of the tangent disk space X is a non-z-embedded C_δ in X; and by 2.1 there is a non-z-embedded $f \in C(S)$.)

(b) The converse of 2.3 holds if each C_δ in X containing S is z-embedded, e.g. if X is Tychonoff and each C_δ in X is Lindelöf (see [HeJ, 5.3] or [BH, 4.1]). (As noted in [BH, 2.5(b)], each C_δ in a compact space is Lindelöf.)

A subset A of $C(S)$ is *inversion-closed* if $1/f \in A$ whenever $f \in A$ and $Z(f) = \emptyset$; and A *separates points and closed sets in* S if for every closed set F in S and $x \in S - F$ there is $f \in A$ with $F \subset Z(f)$ and $x \in \operatorname{coz} f$.

2.5. Proposition. *If X is Tychonoff and $S \subset X$, then $C_z(S, X)$ is a uniformly closed, inversion-closed subalgebra of $C(S)$ that contains the constant functions on S and that separates points and closed sets in S.*

Proof. By [M_2, 4.10], the vector lattice $V = \{f \mid S : f \in C^*(X)\}$ is uniformly closed in $C(S)$, so $\langle S, \operatorname{coz} V\rangle$ is an Alexandroff space [Ha_2, 2.2] and the set $A(\langle S, \operatorname{coz} V\rangle)$ of all A-maps of $\langle S, \operatorname{coz} V\rangle$ is a uniformly closed, inversion-closed subalgebra of $C(S)$ ([Ha_2, 2.3] or [M_3, 3.5]). But (as one easily verifies) $f \in C_z(S, X)$ if and only if for every $a < b$ in $\mathbf{R}$, $f^{-1}(a, b) = S \cap P$ for some $P \in \operatorname{coz} C(X)$, and from this it follows that $C_z(S, X)$ is precisely $A(\langle S, \operatorname{coz} V\rangle)$. Moreover, since V contains the constant functions on S and separates points and closed sets in S, so does $C_z(S, X)$.

2.6. Remark. An alternative proof of 2.5 (that avoids the theory of Alexandroff spaces) can be based on 2.2: Let $Q = \{(f \mid S)/(g \mid S) : f, g \in C(X) \text{ and } Z(g \mid S) = \emptyset\}$. By 2.2 and [BH, 3.1], it is easily seen that $C_z(S, X)$ is the uniform closure of Q in $C(S)$; and $C_z(S, X)$ is inversion-closed because of formulas like $L_a(1/f) = L^{1/a}(f) \cup L_0(f)$ $(a > 0)$. To see that $C_z(S, X)$ is actually a ring, let $f, g \in C_z(S, X)$ and note that f and g are z-embedded in the Stone-Čech compactification βX of X. By 2.3, f and g extend continuously over some C_δ-set T in βX, and hence $f + g$ and fg also extend continuously over T. In view of 2.4(b), $f + g$ and fg belong to $C_z(S, X)$.

3. Extension theorems. Let S be a subset of a topological space X. The following three conditions on the embedding $S \subset X$ are studied in [BH, §3] and [B_1, §4]:

(α) Disjoint zero-sets of S are completely separated in X.

(β) If $Z_1, Z_2 \in \mathscr{Z}(X)$ with $Z_1 \cap Z_2 \cap S = \emptyset$, then $Z_1 \cap S$ and $Z_2 \cap S$ are completely separated in X.

(γ) S is completely separated from every zero-set in X that is disjoint from S.

These are important because:

(I) S is C^*-embedded in X if and only if (α) holds [GJ, 1.17].

(II) S is C-embedded in X if and only if (α) and (γ) hold [GJ, 1.18].

(III) S is C-embedded in X if and only if S is z-embedded in X and (γ) holds ([BH, 3.6B] or [B_1, 4.1B]).

(IV) S is C^*-embedded in X if and only if S is z-embedded in X and (β) holds ([BH, 3.6A] or [B_1, 4.1A]).

(For characterizations of (β) and (γ), see [BH, 3.4] and [B_1, 4.2].)

If $S \subset X$, $f \in C(S)$, and $A \subset X$, then A and f are *completely separated* in case A and $f^{-1}[a, b]$ are completely separated for every $a, b \in \boldsymbol{R}$.

Here we consider the following conditions on the embedding $S \subset X$, relative to a given function $f \in C(S)$:

(α_f) If $a < b$ in $\boldsymbol{R}$, then $L_a(f)$ and $L^b(f)$ are completely separated in X.

(γ_f) S is completely separated from every zero-set in X that is completely separated from f.

The main extension theorems of this section (3.2 and 3.8) provide necessary and sufficient conditions in order that a given function in $C^*(S)$, or in $C(S)$, be continuously extendible over X. These theorems are analogues, for (α_f) and (γ_f), of (I) and (II) above (and, together with 3.1, quickly yield (I) and (II)). The corresponding analogue for (III) fails; see 3.11. We have no such analogue for (IV).

3.1. Proposition. *Let* $S \subset X$.

(a) (α) *holds if and only if* (α_f) *holds for every* (*bounded*) $f \in C(S)$.

(b) (γ) *holds if and only if* (γ_f) *holds for every* $f \in C(S)$.

Proof. (a) Assume (α_f) holds for every $f \in C^*(S)$ and let Z, Z' be disjoint zero-sets in S. Then there is $f \in C^*(S)$ with $f(Z) \subset \{0\}$, $f(Z') \subset \{1\}$. By (α_f), $L_0(f)$ and $L^1(f)$ (*a fortiori*, Z and Z') are completely separated in X. The converse is obvious.

(b) Assume (γ_f) holds for every $f \in C(S)$ and let $g \in C(X)$ with $S \cap Z(g) = \emptyset$. Then $f = 1/|g \mid S| \in C(S)$; and for every $a, b \in \boldsymbol{R}$, $|g| \geqq 1/b$ on $f^{-1}[a, b]$, so $Z(g)$ and $f^{-1}[a, b]$ are completely separated. By (γ_f), S and $Z(g)$ are completely separated. Conversely, assume (γ), let $f \in C(S)$, and let $Z \in \mathscr{Z}(X)$ with Z completely separated from f. Clearly $S \cap Z = \emptyset$, so S and Z are completely separated; hence (γ_f) holds.

Theorem 3.2 below is due to Mrówka [M_2, 4.11], who deduces it from a very general approximation theorem [M_2, 2.7]. Since the proof of the latter is omitted in [M_2], we present a self-contained proof of 3.2 here. (Our proof is based on an elegant proof of [M_2, 2.7] communicated to the author by H. E. White, Jr.)

3.2. Theorem (Mrówka). *Let* $S \subset X$ *and* $f \in C^*(S)$. f *has a continuous extension over* X *if and only if* (α_f) *holds.*

Proof. Assume (α_f) and choose a positive integer m with $|f| \leqq m$. For each integer $n \geqq 0$ and each integer j with $0 \leqq j \leqq p(n) = m\,2^{n+2} - 1$, there is $f_{nj} \in C^*(X)$ with $f_{nj} = 0$ on $\{x \in S : f(x) \leqq -m + j\,2^{-n-1}\}$, $f_{nj} = 1$ on $\{x \in S : f(x) \geqq -m + (j+1)\,2^{-n-1}\}$, and $0 \leqq f_{nj} \leqq 1$. Set $f_n = -m + 2^{-n-1}\sum_{j=0}^{p(n)} f_{nj}$, $g_n = ((f_{n+1} - f_n) \wedge 2^{-n}) \vee -2^{-n}$, and $g = \sum_{n=0}^{\infty} g_n$, and note that $g \in C^*(X)$. Consider any n and any $x \in S$. Since $0 \leqq (f + m)/2m \leqq 1$, there is an integer k with $0 \leqq k \leqq p(n)$ and $k/m\, 2^{n+2} \leqq (f(x) + m)/2m \leqq (k + 1)/m\, 2^{n+2}$. Then $f_{nj} = 1$ (resp. 0) if $j \leqq k - 1$ (resp. $j \geqq k + 1$), so $f_n(x) = -m + 2^{-n-1}(k + f_{nk}(x))$. Thus $f(x)$ and $f_n(x)$ both lie in the interval $[-m + k\, 2^{-n-1}, \ -m + (k + 1)\, 2^{-n-1}]$, and hence $|f(x) - f_n(x)| \leqq 2^{-n-1}$. It follows that $|f_{n-1} - f_n| \leqq 2^{-n}$ on S, and hence $g_n \mid S = f_{n+1} - f_n$. Then $\sum_{i=0}^{n} g_i \mid S = (f_{n+1} - f_0) \mid S$, and we conclude that $(g + f_0) \mid S = f$. The converse is clear.

By a *truncation* of a function $f \in C(S)$ we mean a function of the form $(f \wedge a) \vee \vee -a$, where $a \in \boldsymbol{R}$ and $a \geqq 0$.

3.3. Corollary. *Let $S \subset X$ and $f \in C(S)$. (α_f) holds if and only if every truncation of f has a continuous extenston over X.*

3.4. Remarks. (I) above (Urysohn's Extension Theorem) is an immediate consequence of 3.2 and 3.1(a). It is worth remarking that the above proof of 3.2 is quite different from the usual proof of (I) (as, for example, in [GJ, 1.17]). Another proof of 3.2 is implicit in the proof of (1) $\Rightarrow$ (2) of [BH, 3.4A], and still another can be based on 3.5, a result of independent interest. (The proof of the nontrival implication (b) $\Rightarrow$ (a) of 3.5 is implicit in the proof of [T, Theorem 2].)

3.5. Theorem. *If f and g are real-valued functtons on X such that $f \leqq g$, then the following are equivalent:*

(a) *There exists $h \in C(X)$ such that $f \leqq h \leqq g$.*

(b) *For every $a < b$ in $\boldsymbol{R}$, $L_a(g)$ and $L^b(f)$ are completely separated in X.*

To deduce 3.2 from 3.5, assume (α_f) holds for $S \subset X$ and $f \in C^*(S)$, and let $|f| \leqq m$. Define functions u and v on X by $u = f = v$ on S and $u = -m$ and $v = m$ on $X - S$. By 3.5 there is $h \in C(X)$ with $u \leqq h \leqq v$, and thus $f = h \mid S$.

If $S \subset X$ and $f \in C(S)$, then the following result shows that (α_f) suffices for a certain kind of partial extendibility of f, and that (γ_f) then suffices for the extendibility over X of the already partially extended f.

3.6. Theorem. *Let $S \subset X$ and $f \in C(S)$.*

(a) *If (α_f) holds, then f extends continuously over some cozero-set P containing S such that $X - P$ and f are completely separated.*

(b) *If (γ_f) holds, and if $f = g \mid S$ with $g \in C(T)$, where $S \subset T$, g is z-embedded in X, and $X - T$ is completely separated from f, then f extends continuously over X.*

Proof. Let $\phi : \boldsymbol{R} \to (-1, 1)$ be an order-preserving homeomorphism from $\boldsymbol{R}$ onto the open interval $(-1, 1)$.

(a) Assume (α_f). Since for every $a \in \boldsymbol{R}$,

$$(*) \qquad L_a(f) = L_{\phi(a)}(\phi \circ f), \quad L^a(f) = L^{\phi(a)}(\phi \circ f),$$

Added in proof, February 9, 1981: Theorem 3.5 has been obtained independently (with a proof based on techniques of Katětov) by E. P. Lane, Insertion of a continuous function, Topology Proceedings *4* (1979), 463—478, Theorem 2.1.

$(\alpha_{\phi\circ f})$ holds. By 3.2, $\phi \circ f = k \mid S$ for some $k \in C(X)$. Let $P = X - (L_{-1}(k) \cup \cup L^1(k))$, and note that P is cozero in X and contains S. Now if $a, b \in \boldsymbol{R}$, then $L_{-1}(k) \cup L^1(k)$ and $L^{\phi(a)}(k) \cap L_{\phi(b)}(k)$ are completely separated in X. Hence, by $(*)$, $X - P$ and $L^a(f) \cap L_b(f)$ are completely separated, and thus $X - P$ and f are completely separated. To complete the proof of (a), note that $\phi^{-1} \circ (k \mid P)$ is a continuous extension of f over P.

(b) Assume (γ_f), and let g and T be as described in 3.6(b). By $(*)$ (with f replaced by g), $\phi \circ g$ is z-embedded in X. By 2.2, for each integer $n > 0$ there is a cozero-set P_n containing T and $g_n \in C(P_n)$ with $|(\phi \circ g)(x) - g_n(x)| < 1/n$ for all $x \in T$; and since $\phi \circ g$ is bounded, we may assume that g_n is bounded. By (γ_f), there is $h_n \in C(X)$ with $h_n = 1$ on S and $h_n = 0$ on $X - P_n$. Define f_n by $f_n = g_n h_n$ on coz h_n and $f_n = 0$ on $Z(h_n)$. Then $f_n \in C(X)$ and $f_n = g_n$ on S, so $f_n \to \phi \circ g$ uniformly on S. By [BH, 2.3], $\phi \circ g$ has an extension $h \in C(X)$. Let $Z = \{x \in X : |h(x)| \geqq 1\}$. Then $Z \subset \subset X - T$ so, by (γ_f), there is $u \in C(X)$ with $u = 1$ on S, $u = 0$ on Z, and $0 \leqq u \leqq 1$. Now $|u(x)\, h(x)| < 1$ for all $x \in X$, so $\phi^{-1} \circ (uh)$ is well-defined. But $uh \mid S = = (\phi \circ g) \mid S$, and thus $\phi^{-1} \circ (uh)$ is a continuous extension of f over X.

3.7. Corollary. *Let $S \subset X$ and $f \in C(S)$. If (α_f) holds, then f is z-embedded in X.*

Proof. This is immediate from 3.6(a) and 2.2. (A direct proof can, of course, also easily be given.)

3.8. Theorem. *Let $S \subset X$ and $f \in C(S)$. f has a continuous extension over X if and only if (α_f) and (γ_f) hold.*

Proof. Assume first that $f = g \mid S$ with $g \in C(X)$. Clearly (α_f) holds. Suppose that $A \subset X$ and that A and f are completely separated. For each integer n there is $h_n \in C(X)$ such that

$$h_n(L_{n-2}(g) \cup L^{n+2}(g)) \subset \{0\}\,, \quad h_n(L^{n-1}(g) \cap L_{n+1}(g)) \subset \{1\}\,,$$

and $h_n \geqq 0$, and there is $k_n \in C(X)$ such that $k_n(A) \subset \{0\}$,

$$k_n(L^{n-1}(f) \cap L_{n+1}(f)) \subset \{1\}\,,$$

and $k_n \geqq 0$. Let $u_n = h_n k_n$. Since

$$\operatorname{coz} u_n \subset \operatorname{coz} h_n \subset \{x \in X : n - 2 < g(x) < n + 2\}\,,$$

the family $(\operatorname{coz} u_n)_n$ is locally finite (in fact, star-finite) in X, and hence $u = \sum_n u_n$ is in $C(X)$. If $x \in S$, let m be the largest integer with $m \leqq f(x)$. Then $x \in L^{m-1}(f) \cap \cap L_{m+1}(f) \subset L^{m-1}(g) \cap L_{m+1}(g)$, so $u_m(x) = 1$. Thus $u(x) \geqq 1$ for every $x \in S$ and $u = 0$ on A, and hence S and A are completely separated. Therefore (γ_f) holds. In view of 2.1 and the fact that cozero-sets are always z-embedded [BH, 1.1], the converse follows immediately from 3.6.

3.9. Remarks. (a) (II) above follows immediately from 3.8 and 3.1.

(b) 3.2 and 3.8 provide conditions for the extendibility of a function $f \in C(S)$ in terms of the Lebesgue sets of f. When S is dense in X, techniques different from (and more elementary than) those of this paper can be used to provide similar, but more tractable, conditions for the extendibility of f; see [B_2, Theorem 2]. We note also that for a dense embedding $S \subset X$ and $f \in C^*(S)$, (α_f) holds if and only if $\mathrm{cl}_X L_a(f) \cap \mathrm{cl}_X L^b(f) = \emptyset$ for every $a < b$ in $\boldsymbol{R}$. (This follows from (b) $\Rightarrow$ (a) of [B_2, Theorem 2] and the easy half of 3.2.)

The remainder of this section is devoted to specialized results connecting extendibility of a function $f \in C(S)$ with z-embedding of f and with the conditions (β), (γ), and (γ_f).

We show first that (γ_f) holds for *every* embedding $S \subset X$ if and only if f is bounded:

3.10. Proposition. *If S is a topological space (resp. Tychonoff space) and if $f \in C(S)$, then the following are equivalent:*

(a) *f is bounded.*

(b) *Whenever S is embedded in a space (resp. Tychonoff space), (γ_f) holds.*

Proof. (a) $\Rightarrow$ (b): Let $|f| \leqq a$ and let $S \subset X$. If $A \subset X$ and if A is completely separated from f, then A is completely separated from $f^{-1}[-a, a] = S$.

(b) $\Rightarrow$ (a): Suppose f is unbounded and let $g = |f| \vee 1$. Choose $p \notin S$ and let $X = S \cup \{p\}$, where basic neighborhoods of p in X are sets of the form $\{p\} \cup \{x \in S : g(x) \geqq n\}$ $(n = 1, 2, \ldots)$. Since f is unbounded, S is dense in X. Define $h : X \to \boldsymbol{R}$ by $h = 1/g$ on S and $h(p) = 0$. Then $h \in C(X)$ (which implies that X is Tychonoff if S is Tychonoff). If $a, b \in \boldsymbol{R}$, choose an integer n with $n \geqq |a| \vee |b| \vee 1$. Then $h(x) \geqq 1/n$ for every $x \in f^{-1}[a, b]$, so $\{p\}$ is completely separated from f. But $\{p\}$ is not completely separated from S, and thus (γ_f) fails for the embedding $S \subset X$.

From 3.10 and 3.1(b) we have:

3.11. Corollary [BH, 4.3]. *The following conditions on S are equivalent:*

(a) *S is pseudocompact.*

(b) *Whenever $S \subset X$, (γ) holds.*

3.12. Remarks. As noted in (III) above, C-embedding is equivalent to the conjunction of z-embedding and (γ). The single function analogue of (III) fails: Let S be the subspace $\{1/n\colon n = 1, 2, \ldots\}$ of $\boldsymbol{R}$ and define $f \in C(S)$ by $f(1/n) = 1$ (resp. 0) if n is odd (resp. even). Then f is z-embedded in $\boldsymbol{R}$ and the embedding $S \subset \boldsymbol{R}$ satisfies (γ_f) (because f is bounded (3.10)), but f has no continuous extension over $\boldsymbol{R}$. We have been unable to isolate a condition on f (necessarily stronger than (γ_f)) which, when coupled with z-embedding of f, yields extendibility of f. But (III) and (IV) are partially recovered, for a single function, by 3.13:

3.13. Proposition. *Let $S \subset X$ and $f \in C^*(S)$ (resp. $f \in C(S)$). If f is z-embedded in X and if (β) (resp. (γ)) holds, then f has a continuous extension over X.*

Proof. Let $f \in C(S)$, let $a < b$ in $\boldsymbol{R}$, and choose $Z_1, Z_2 \in \mathscr{Z}(X)$ with $L_a(f) = S \cap Z_1$ and $L^b(f) = S \cap Z_2$. Then $Z_1 \cap Z_2 \cap S = \emptyset$, and therefore, if (β) holds, $L_a(f)$ and $L^b(f)$ are completely separated in X; i.e. (α_f) holds. And if (γ) holds, then both (β) and (γ_f) hold ([BH, 3.3] and 3.1(b)). The result therefore follows from 3.2 and 3.8.

3.14. Corollary. *Let $S \subset X$ and $f \in C(S)$, and assume that either S is pseudocompact, S is a zero-set in X, or S is G_δ-dense in X. Then f extends continuously over X if and only if f is z-embedded in X.*

Proof. If S is pseudocompact, (γ) holds by 3.11, if S is a zero-set, (γ) holds by [GJ, 1.15], and if S is G_δ-dense, (γ) holds vacuously. Now apply 3.13.

Recall that a Tychonoff space X is an F-space (resp. P-space) if and only if every cozero-set in X is C^*-embedded (resp. C-embedded) in X [GJ, 14.25 and 14.29].

3.15. Corollary. *The following conditions on a Tychonoff space X are equivalent:*

(a) *X is an F-space (resp. P-space).*

(b) *For each $S \subset X$ and each $f \in C^*(S)$ (resp. $f \in C(S)$), if f is z-embedded in X, then f extends continuously over X.*

Proof. (a) $\Rightarrow$ (b): If X is an F-space (resp. P-space), then (β) (resp. (γ)) holds for each embedding $S \subset X$ [BH, 4.5]. Now apply 3.13.

(b) $\Rightarrow$ (a): Since cozero-sets are z-embedded, (b) and 2.1 imply that each cozero-set in X is C^*-embedded (resp. C-embedded) in X.

4. Absolute extendibility and absolute z-embedding. In this final section we restrict to Tychonoff spaces. The main results (4.1 and 4.7) provide numerous characterizations of absolutely extendible functions and of absolutely z-embedded functions.

If $f \in C(S)$, $f^\beta : \beta S \to \boldsymbol{R}^*$ will denote the Stone extension of f, where $\boldsymbol{R}^*$ is the one-point compactification of $\boldsymbol{R}$. If $A \subset S$, the oscillation of f on A is defined by $\operatorname{osc}(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$. As in [$M_4$], *$f$ has vanishing oscillation outside compact subsets of S* if for every $\varepsilon > 0$ there is a compact subset K of S with $\operatorname{osc}(f, S - K) < \varepsilon$.

The equivalence of (a), (b), and (f) of 4.1 is proved by Mrówka (in an entirely different way) in [M_4].

4.1. Theorem. *If $f \in C(S)$, then the following are equivalent:*

(a) *f is absolutely extendible.*

(b) *f extends continuously over every compactification of S.*

(c) *f is z-embedded in each compactification of S and f^β is constant and real-valued on $\beta S - \upsilon S$.*

(d) *f is constant and real-valued on $\beta S - S$.*

(e) *f is bounded and for every $a < b$ in $\boldsymbol{R}$, either $L_a(f)$ or $L^b(f)$ is compact.*

(f) *f has vanishing oscillation outside compact subsets of S.*

Proof. (a) $\Rightarrow$ (b): Trivial.

(b) $\Rightarrow$ (c): (b) obviously implies that f is z-embedded in each compactification of S. Moreover, by (b), $f = g \mid S$ for some $g \in C(\beta S)$. Since g and f^β agree on S, $f^\beta = g$, and hence f^β is real-valued.

Next let $p, q \in \beta S - \upsilon S$, let $X = \beta S/\{p, q\}$ be the compactification of S obtained from βS by identifying p and q, and let $\tau : \beta S \to X$ be the resulting canonical map. By (b), $f = h \mid S$ for some $h \in C(X)$. Then $h \circ \tau = f = f^\beta$ on S, so $h \circ \tau = f^\beta$. Hence $f^\beta(p) = f^\beta(q)$, and thus f^β is constant on $\beta S - \upsilon S$.

(c) $\Rightarrow$ (d): Let $f^\upsilon : \upsilon S \to \boldsymbol{R}$ be the continuous extension of f over υS [GJ, 8.7]. Since $f^\beta = f^\upsilon$ on $S, f^\beta \mid \upsilon S = f^\upsilon$. Thus (c) implies that f^β is real-valued on βS, and hence on $\beta S - S$. If $S = \upsilon S$, then (d) follows trivially from (c), so we may assume there exists $p \in \upsilon S - S$. If $q \in \beta S - S$, let $X = \beta S/\{p, q\}$ be the compactification of S obtained from βS by identifying p and q, and let $\tau : \beta S \to X$ be the resulting canonical map. By (c) and 2.3, there is a C_δ-set A in X containing S and $g \in C(A)$ with $f = g \mid S$. Write $A = \bigcap_n P_n$ with each P_n cozero in X. If $p \notin \tau^{-1}(A)$, then $p \notin \tau^{-1}(P_n)$ for some n. But then $\upsilon S - \tau^{-1}(P_n)$ is a nonempty zero-set in υS which misses S, a contradiction [GJ, 8.8(b)]. Thus $p, q \in \tau^{-1}(A)$. Let $h = g \circ (\tau \mid \tau^{-1}(A))$. Then $h = f^\beta$ on S, so $h = f^\beta \mid \tau^{-1}(A)$. Hence $f^\beta(p) = g(\tau(p)) = g(\tau(q)) = f^\beta(q)$, and we conclude that f^β is constant on $\beta S - S$.

(d) $\Rightarrow$ (e): By (d), there exists $r \in \boldsymbol{R}$ with $f^\beta(p) = r$ for all $p \in \beta S - S$. Then f^β is bounded, so f is bounded. Next let $a < b$. If $r \geqq b$, then $L_a(f) = L_a(f^\beta)$, and if $r < b$, then $L^b(f) = L^b(f^\beta)$; hence either $L_a(f)$ or $L^b(f)$ is compact.

(e) $\Rightarrow$ (a): Let $S \subset X$. By 3.2, it suffices to show that (α_f) holds for the embedding $S \subset X$. Let $a < b$ in $\boldsymbol{R}$ and assume, say, that $L_a(f)$ is compact. Since $L_a(f) \cap \cap \operatorname{cl}_X L^b(f) = \emptyset$, $L_a(f)$ and $L^b(f)$ are completely separated in X [GJ, 3.11(a)], and thus (α_f) holds.

(d) $\Rightarrow$ (f): By (d), there is $r \in \boldsymbol{R}$ with $f^\beta = r$ on $\beta S - S$. Let $\varepsilon > 0$ and set $K = = \{p \in \beta S : |f^\beta(p) - r| \geqq \varepsilon/4\}$. Then K is compact, $K \subset S$, and $|f(x) - f(y)| < < \varepsilon/2$ for all $x, y \in S - K$. Hence $\operatorname{osc}(f, S - K) < \varepsilon$.

(f) $\Rightarrow$ (d): We first show that f is bounded. By (f), there is a compact subset K of S with $\operatorname{osc}(f, S - K) < 1$. Since f is bounded on K, we may assume there is $p \in \in S - K$. Then for all $x \in S - K$ we have $|f(x) - f(p)| < 1$, so $|f| < |f(p)| + 1$ on $S - K$. It follows that f is bounded, and hence f^β is real-valued.

Next let $p_i \in \beta S - S$ $(i = 1, 2)$, let $\varepsilon = |f^\beta(p_1) - f^\beta(p_2)|$, and suppose that $\varepsilon > 0$. By (f), there is a compact subset K of S with $\operatorname{osc}(f, S - K) < \varepsilon/3$. Moreover, there is a neighborhood V_i of p_i in βS with $V_i \cap K = \emptyset$ and $|f^\beta(p_i) - f^\beta(q)| < \varepsilon/3$ for every $q \in V_i$, and there is a point $x_i \in V_i \cap (S - K)$. But then $|f^\beta(p_1) - f^\beta(p_2)| \leqq$ $\leqq |f^\beta(p_1) - f^\beta(x_1)| + |f^\beta(x_1) - f^\beta(x_2)| + |f^\beta(x_2) - f^\beta(p_2)| < \varepsilon$, a contradiction. The proof is now complete.

The following corollary is due in part to Hewitt [He] and in part to Smirnov [S]; see [GJ, 6J]. 4.3 generalizes [GJ, 3L.2].

4.2. Corollary. *The following conditions on S are equivalent*:

(a) *S is absolutely C-embedded.*

(b) *S is absolutely C^*-embedded.*

(c) $|\beta S - S| \leqq 1$.

(d) *Of any two disjoint zero-sets in S, at least one is compact.*

Proof. (a) $\Rightarrow$ (b) is trivial, and the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) follow readily from 4.1. If (d) holds, then S is pseudocompact [GJ, 1G.4], so the implication (e) $\Rightarrow$ (a) of 4.1 yields (a).

A space S that satisfies any of the equivalent conditions of 4.2 is called *almost compact.*

4.3. Corollary. *Let $\mathscr{F}$ be a filter on S such that* $\operatorname{cl}(S - A)$ *is compact for every $A \in \mathscr{F}$. If $f \in C(S)$ and if $f(\mathscr{F})$ is convergent in $\mathbf{R}$, then f is absolutely extendible.*

Proof. We verify 4.1(e): By hypothesis, $f(\mathscr{F}) \to r$ for some $r \in \mathbf{R}$, so there is $A \in \mathscr{F}$ with $f(A) \subset (r - 1, r + 1)$. Then f is bounded on A (as well as on $\operatorname{cl}(S - A)$), so f is bounded. Let $a < b$. If $a < r$, there is $B \in \mathscr{F}$ with $f(B) \subset (a, +\infty)$; then $L_a(f) \subset \operatorname{cl}(S - B)$, so $L_a(f)$ is compact. Similarly, $L^b(f)$ is compact if $a \geqq r$.

4.4. Corollary. *Let S be locally compact Hausdorff and let S^* be the one-point compactification of S. If $f \in C(S)$ and if f has a continuous extension over S^*, then f is absolutely extendible.*

We require the following two facts for our characterizations of absolutely z-embedded functions. 4.5 is due, independently, to Henriksen and Johnson [HeJ, 5.4] and to Mrówka [M_1]. (For other proofs, see [HaJ, Theorem 3] and [Ha_1, 3.10].) For the simple proof of 4.6, see $(*)$ of [BH, p. 50].

4.5. Proposition. *If S is Lindelöf, then $C(S)$ is the only uniformly closed, inversion-closed subalgebra of $C(S)$ that contains the constant functions on S and that separates points and closed sets in S.*

4.6. Proposition. *Let $S \subset X$. If $Z \in \mathscr{Z}(S)$ and $S - Z$ is Lindelöf, then there exists $Z' \in \mathscr{Z}(X)$ with $Z = S \cap Z'$.*

4.7. Theorem. *If $f \in C(S)$, then the following are equivalent*:

(a) *f is absolutely z-embedded.*

(b) *f is z-embedded in each compactification of S.*

(c) *Either f is absolutely extendible or S is Lindelöf.*

(d) *If $a < b$ in $\boldsymbol{R}$ and if one of the sets $L_a(f)$ or $L^b(f)$ is noncompact, then the other is Lindelöf.*

(e) *If L is any Lebesgue set of f, either L is compact or $S - L$ is Lindelöf.*

(f) *f belongs to every uniformly closed, inversion-closed subalgebra of $C(S)$ that contains the constant functions on S and that separates points and closed sets in S.*

Proof. $(a) \Rightarrow (b)$: Trivial.

(b) $\Rightarrow$ (c): Suppose that f is not absolutely extendible. By (d) $\Rightarrow$ (a) of 4.1, one of the following two cases must hold:

Case 1. There exists $t \in \beta S - S$ with $f^\beta(t) \notin \boldsymbol{R}$.

Case 2. There exist $p, q \in \beta S - S$ with $f^\beta(p) \neq f^\beta(q)$.

Let $\{G_\alpha\}_\alpha$ be a cover of S by open sets in βS and let $G = \bigcup_\alpha G_\alpha$. In Case 1 (resp. Case 2), let $F = (\beta S - G) \cup \{t\}$ (resp. $F = (\beta S - G) \cup \{p, q\}$), let $X = \beta S/F$ be the compactification of S obtained from βS by identifying the points of F, and let $\tau : \beta S \to X$ be the resulting canonical map. By hypothesis, f is z-embedded in X, so, by 2.3, there is a C_δ-set A in X containing S and $g \in C(A)$ such that $f = g \mid S$. Let $T = \tau^{-1}(A)$ and note that $S \subset T$. Since $f^\beta \mid T = f = g \circ (\tau|A)$ on S, $f^\beta \mid T = g \circ (\tau \mid A)$. Suppose there is $x \in T - G$. Then $\tau(x) \in A$ and $x \in F$. In Case 1 we have $\tau(t) = \tau(x)$, so $t \in T$ and $f^\beta(t) = g(\tau(t)) \in \boldsymbol{R}$, a contradiction; and in Case 2 we have $\tau(p) = \tau(x) = \tau(q)$ so $p, q \in T$ and $f^\beta(p) = g(\tau(p)) = g(\tau(q)) = f^\beta(q)$, again a contradiction. Thus $T \subset G$. Now T is a C_δ in βS and hence T is Lindelöf (see 2.4(b)). Therefore countably many G_α's cover T, and we conclude that S is Lindelöf.

(c) $\Rightarrow$ (d): Let $a < b$. If f is absolutely extendible, then, by (a) $\Rightarrow$ (e) of 4.1, either $L_a(f)$ or $L^b(f)$ is compact; and if S is Lindelöf, both $L_a(f)$ and $L^b(f)$ are Lindelöf.

(d) $\Rightarrow$ (e): Let $a \in \boldsymbol{R}$ and suppose that $L_a(f)$ is not compact. By (d), $L^{a+(1/n)}(f)$ is Lindelöf for every integer $n > 0$, so $S - L_a(f) = \bigcap_{n>0} L^{a+(1/n)}(f)$ is Lindelöf. The argument for $L^a(f)$ is similar.

(e) $\Rightarrow$ (a): If $S \subset X$ and $a \in \boldsymbol{R}$, we claim that $L_a(f)$ extends to a zero-set in X. By 4.6 we may assume that $S - L_a(f)$ is not Lindelöf. Then $A = L^{a+(1/m)}(f)$ is not compact for some positive integer m, so, by (e), $S - A$ is Lindelöf. By 4.6, there is $Z_1 \in \mathscr{Z}(X)$ with $A = S \cap Z_1$. By (e), $L_a(f)$ is compact and hence completely separated from Z_1 [GJ, 3.11(a)], so there is $Z_2 \in \mathscr{Z}(X)$ with $L_a(f) \subset Z_2$ and $Z_1 \cap Z_2 = \emptyset$. Next, $(S - A) - L_a(f)$ is Lindelöf because it is an F_σ in $S - A$, so, by 4.6, there is $Z_3 \in \mathscr{Z}(X)$ with $L_a(f) = (S - A) \cap Z_3$. Then $Z_2 \cap Z_3 \in \mathscr{Z}(X)$ and $L_a(f) = S \cap Z_2 \cap Z_3$. The argument for $L^a(f)$ is similar.

(f) $\Rightarrow$ (a): If $S \subset X$, then $f \in C_z(S, X)$ by (f) and 2.5; i.e. f is z-embedded in X.

(c) $\Rightarrow$ (f): Let A be a subalgebra of $C(S)$ of the kind described in (f). If S is Lindelöf, then $A = C(S)$ by 4.5, and thus $f \in A$. If f is absolutely extendible, let $A^* = \{g \in A : g \text{ is bounded}\}$. As shown in [I] (see also [HaJ, 2.3]), there is a compactification $H(A^*)$ of S with the property that $A^* = \{g \mid S : g \in C(H(A^*))\}$. Since f extends over $H(A^*)$, we have $f \in A^* \subset A$, and the proof is complete.

The preceding theorem (together with 2.1 and 4.2) quickly implies most of the known characterizations of absolutely z-embedded spaces (see [HaJ, Theorem 3] and [BH, 4.1]). We state these in 4.8, but suppress the easy proof.

4.8. Corollary. *The following conditions on S are equivalent:*

(a) *S is absolutely z-embedded.*

(b) *S is z-embedded in each compactification of S.*

(c) *S is either Lindelöf or almost compact.*

(d) *If Z_1 and Z_2 are disjoint zero-sets in S, either Z_1 is compact or Z_2 is Lindelöf.*

(e) *If Z is a zero-set in S, either Z is compact or $S - Z$ is Lindelöf.*

(f) *$C(S)$ is the only uniformly closed, inversion-closed subalgebra of $C(S)$ that contains the constant functions on S and that separates points and closed sets in S.*

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