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INVERSE SEMIGROUPS DETERMINED BY THEIR LATTICES OF INVERSE SUBSEMIGROUPS

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This paper is concerned with the determinability of inverse semigroups, or of classes of inverse semigroups, by their lattices of inverse subsemigroups. In a previous paper [14] the author showed after proving some general results that every lattice isomorphism of the free inverse semigroup $\mathscr{F}\mathscr{I}_X$ upon an inverse semigroup T is induced by a unique isomorphism of $\mathscr{F}\mathscr{I}_X$ upon T. In this paper the general results are extended to arbitrary inverse semigroups and applied to various classes of inverse semigroups. In particular it is shown that the free product of two groups in the category of inverse semigroups is determined by its lattice of inverse subsemigroups.

In Section 1 we show that if Φ is a lattice isomorphism between inverse semigroups S and T, there is a "weak isomorphism" of E_S upon E_T, an $\mathscr{R}$-and $\mathscr{L}$-preserving bijection of the "non-subgroup" elements of S upon those of T and a comparability-preserving bijection of the poset $\mathscr{J}(S)$ of $\mathscr{J}$-classes of S upon the poset $\mathscr{J}(T)$ of $\mathscr{J}$-classes of T.

Using these results we show, for example (in Section 2) that if E is a semilattice, *the class of simple inverse semigroups with semilattice of idempotents isomorphic with E is closed under lattice isomorphisms.*

In Section 3, the general results are specialized a little to *E-unitary* inverse semigroups. Under some natural weak assumptions, it is shown (Theorems 3.4, 3.10) that *if S is E-unitary and $\mathscr{L}(S) \cong \mathscr{L}(T)$, then T is also E-unitary, $E_S \cong E_T$, and moreover $\mathscr{L}(S/\sigma) \cong \mathscr{L}(T/\sigma)$*, the lattice isomorphisms commuting in a certain sense. In particular this is true if S has a set of non-idempotent generators. As a consequence we prove (Theorem 3.12) that *the class of quasi-free inverse semigroups* [29] *of given rank is closed under lattice isomorphisms.*

The foregoing results are applied in Section 4 to prove the result stated earlier: *the free product of two groups in the category of inverse semigroups* [18, 21] *is determined by its lattice of inverse subsemigroups*, (thus answering in this special case a question posed by Preston in [27, Problem 7]). If the groups are torsion-free the lattice isomorphism is induced by a (semigroup) isomorphism but whether the same is true in general the author does not know. This theorem is the analogue to that of Sadovskiĭ for the free product of two non-trivial groups in the category of groups. It is of interest to note that either of the groups may be trivial in our case.

Recall from [14] that by a *lattice isomorphism* [L-isomorphism, structural isomorphism or projectivity] Φ of an inverse semigroup S upon another T we mean an isomorphism of $\mathscr{L}(S)$ upon $\mathscr{L}(T)$ (where $\mathscr{L}(S)$ denotes the lattice of inverse subsemigroups of S, including the empty inverse subsemigroup). A mapping ϕ is said to *induce* Φ if $A\Phi = A\phi$ for all A in $\mathscr{L}(S)$.

To say that a class $\mathscr{C}$ of inverse semigroups is *closed* under lattice isomorphisms means that if $S \in \mathscr{C}$ and $\mathscr{L}(S) \cong \mathscr{L}(T)$ then $T \in \mathscr{C}$. A stronger property is to say that an inverse semigroup is *determined* by its lattice of inverse subsemigroups – that is if $\mathscr{L}(S) \cong \mathscr{L}(T)$ then $S \cong T$. (Equivalently the class $\{S\}$ is closed under lattice isomorphisms.) We say S is *strongly determined* by $\mathscr{L}(S)$ if every lattice isomorphism of S upon T is *induced* by an isomorphism of S upon T. We prove results of each type, (for example the theorems quoted above, on quasi-free inverse semigroups, free products of groups and free inverse semigroups respectively).

1. LATTICE ISOMORPHISMS

Throughout this section S and T are inverse semigroups and Φ is a lattice isomorphism of S upon T. If $A \subseteq S$, then E_A or $E(A)$ will denote the set of idempotents of A. For all other terminology the reader is referred to [14] (or to [3, 12] for basic properties of inverse semigroups and [2] for lattice-theoretic results).

Result 1.1. ([14, Lemma 1.1]). *There is a unique bijection ϕ_E of E_S upon E_T such that $\langle e \rangle \Phi = \langle e\phi_E \rangle$ for all $e \in E_S$. In addition ϕ_E satisfies*

(i) *$e \nparallel f$ if and only if $e\phi_E \nparallel f\phi_E$,*

(ii) *$e \parallel f$ implies $(ef)\phi_E = (e\phi_E)(f\phi_E)$.*

(Recall that if $(P, \leqq)$ is a poset then $p \nparallel q$ if p and q are comparable, and $p \parallel q$ otherwise). Where no confusion arises we will write ϕ for ϕ_E. Such bijections between semigroups were called *weak isomorphisms* by Severin [32]. It is easily seen that a weak isomorphism is an *isomorphism* if and only if it is order-preserving, a fact which will be used without comment.

As an immediate consequence of Result 1.1, Φ maps a maximal subgroup H_e of S, $e \in E_S$, to the maximal subgroup $H_{e\phi}$ of T. Further,

Result 1.2. ([14, Proposition 1.4]). *If S is completely semisimple, so is T.*

Most of the results in [14] were proved for completely semisimple inverse semigroups (those for which no two distinct $\mathscr{D}$-related idempotents are comparable). Before going on to generalize those results we state first a result from [6] and [17], whose fundamental importance to this paper stems from the well known fact that every inverse semigroup which is *not* completely semisimple contains a copy of bicyclic semigroup as an inverse subsemigroup.

Result 1.3. ([6, 17]). *The bicyclic semigroup B is strongly determined by its lattice of inverse subsemigroups – in fact every lattice isomorphism of B upon T is induced by a unique isomorphism.*

By Result 1.1, the restriction of this isomorphism to the idempotents of B is just ϕ_E.

Ersova [6] proved the weaker version of this result but also proved more generally that any infinite elementary inverse semigroup is determined, weakly, by its lattice of inverse subsemigroups.

Lemma 1.4. *If $S = \langle x \rangle$, an elementary inverse semigroup, then T is also elementary.*

Proof. If S is a group, this follows from results of Ore [26]; see [33, Chapter 1].

If $xx^{-1} \parallel x^{-1}x$, the proof is identical with that of [14, Lemma 1.5] noting again that $M = S \setminus \{x, x^{-1}\}$ is the maximum proper inverse subsemigroup of S.

If $xx^{-1} \nparallel x^{-1}x$ then S is bicyclic, by [3, Lemma 1.31], so T is also, by Result 1.3.

In order to strengthen this lemma and for use in the sequel we require a description of the elementary (or monogenic) inverse semigroups. These were classified by Gluskin [7] and more succinctly by Ersova [5]; for a lucid account see [4, p. 27–28]. An elementary inverse semigroup $S = \langle x \rangle$ is of one of the following types:

(I) a cyclic group;

(II) the free elementary inverse semigroup $\mathscr{F}\mathscr{I}_1$;

(III) defined by the relations $x^n x^{-n} = x^{-n} x^n$, $x^{n+m} = x^n$, $n \geqq 2$, $m \geqq 0$, whence S is completely semisimple, with a group kernel generated by $x^{n+1}x^{-n}$ (as an inverse subsemigroup);

(IV) bicyclic;

or (V) defined by the relation $(x^n x^{-n})(x^{-n}x^n) = x^{-n}x^n$, $n \geqq 2$, (or its dual) whence S has a bicyclic kernel, generated by $x^{n+1}x^{-n}$, (or, dually, by $x^{-(n+1)}x^n$).

When S has a kernel K, S is an ideal extension of K by $\mathscr{F}\mathscr{I}_1/I_n$, where I_n is the ideal of $\mathscr{F}\mathscr{I}_1$ generated by x^n.

Lemma 1.5. *If $S = \langle x \rangle$, not a group, then there exists a unique generator y of T such that $(xx^{-1})\phi = yy^{-1}$ and $(x^{-1}x)\phi = y^{-1}y$.*

Proof. Clearly S has one of the types (II) to (V) above. Types (II) and (III) were covered in parts (i) and (ii) of the proof of [14, Lemma 1.6].

If S is of type (IV), that is bicyclic, let y be the image of x under the unique isomorphism upon T which induces Φ (Result 1.3).

If S is of type (V), then by the comments following Result 1.3, $T \cong S$, for S is certainly infinite. Moreover if K_S, K_T denote the kernels of S and T respectively, $K_S \Phi = K_T$, and the restriction of Φ_E to $E(K_S)$ is an isomorphism upon $E(K_T)$, by Result 1.3. Then an argument almost identical with that of part (ii) of the proof of [14, Lemma 1.6], (replacing the identity 0_x of the *subgroup* kernel there by the semilattice $E(K_S)$ of idempotents of the bicyclic kernel and noting the final sentence in the description of the elementary inverse semigroups above) yields the required result.

(Note that even if S is a (cyclic) group there is still of course such a generator y, but the properties above do not then define y uniquely).

Combining this lemma with Result 1.1 we have

Proposition 1.6. *For each x in S, either idempotent or in no subgroup of S, there is a unique element y of T such that $\langle x \rangle \Phi = \langle y \rangle$, $(xx^{-1}) \phi = yy^{-1}$ and $(x^{-1}x) \phi = y^{-1}y$.*

Combining this proposition with the comments preceding it and with those preceding Result 1.2 yields a technical point which will be useful in the sequel: for any x in S there exists y in T such that $y \in R_{(xx^{-1})\phi} \cap \langle x \rangle \Phi$, (and y generates $\langle x \rangle \Phi$).

Corollary 1.7. *The mapping $x \mapsto y$ defines a partial one-to-one map ϕ of S into T, extending ϕ_E and taking $S \setminus (\bigcup_{e \in E_S} H_e)$ onto $T \setminus (\bigcup_{f \in E_T} H_f)$. Further ϕ and ϕ^{-1} $\mathscr{R}$- and $\mathscr{L}$-preserving. If θ is any homomorphism of S upon T which induces Φ, then $x\theta = x\phi$, wherever $x\phi$ is defined.*

Proof. The bijectivity of ϕ between the non-subgroup elements of S and those of T comes from applying the above arguments to the lattice isomorphism Φ^{-1} of T upon S.

Now suppose $s, t \in S$ and $s\phi$, $t\phi$ are defined. Then $s \mathscr{R} t$ iff $ss^{-1} = tt^{-1}$ iff $(s\phi) \cdot (s\phi)^{-1} = (ss^{-1}) \phi = (tt^{-1}) \phi = (t\phi)(t\phi)^{-1}$ iff $s\phi \mathscr{R} t\phi$.

The final statement follows from the uniqueness of y in Proposition 1.6.

In particular if S is combinatorial (whence by [14, Corollary 1.3] so is T), ϕ is therefore a bijection of S upon T and if θ is a homomorphism of S upon T inducing Φ, then $\theta = \phi$. For instance, in Result 1.3 the isomorphism of B inducing a lattice isomorphism Φ of $\mathscr{L}(B)$ must in fact be ϕ.

Denote by $\mathscr{D}(U)$ the set of $\mathscr{D}$-classes of a semigroup U.

Proposition 1.8. *If $e, f \in E_S$, then $e \mathscr{D} f$ if and only if $e\phi \mathscr{D} f\phi$. Hence there is a bijection $\phi_{\mathscr{D}}$ of $\mathscr{D}(S)$ upon $\mathscr{D}(T)$. Further for each $\mathscr{D}$-class D of S*

$$|E_T \cap D\phi_{\mathscr{D}}| = |E_S \cap D| .$$

Proof. If $e, f \in E_S$, $e \neq f$, then $e \mathscr{D} f$ implies $e = xx^{-1}$, $x^{-1}x = f$ for some $x \in S$, in no subgroup of S. Then by Corollary 1.7, $e\phi = (x\phi)(x\phi)^{-1}$ and $(x\phi)^{-1} \cdot$ $\cdot (x\phi) = f\phi$, that is $e\phi \mathscr{D} f\phi$. The converse is similar.

Define $\phi_{\mathscr{D}}$ by $D\phi_{\mathscr{D}} = D_{e\phi}$ for some (any) $e \in E_D$. This is clearly well-defined and bijective. The final statement is then immediate from the bijectivity of ϕ_E.

An application is the following. An idempotent e (and similarly any subgroup of H_e) is called *isolated* [13] if $E_S \cap D_e = \{e\}$ (that is, if $D_e = H_e$), and otherwise non-isolated.

Corollary 1.9. *If S has no isolated non-trivial subgroups then there is a one-to-one correspondence between the cardinalities of the $\mathscr{H}$-classes of S and the cardinalities of those of T. Hence $|S| = |T|$.*

Proof. Since the $\mathscr{H}$-classes within any given $\mathscr{D}$-class of S have the same number of elements it is sufficient to show, by the proposition, that for each $\mathscr{D}$-class of S, an $\mathscr{H}$-class of D has the same number of elements as some $\mathscr{H}$-class of $D\phi_{\mathscr{D}}$.

Let $e \in E_D$. If $D = \{e\}$, then $D\phi_{\mathscr{D}} = \{e\phi\}$. Otherwise by assumption, $e \mathscr{D} f$ for some $f \in E_S$, $f \neq e$. Since the $\mathscr{H}$-class $H = R_e \cap L_f$ of S intersects no subgroup of S, ϕ is defined on H and moreover maps ϕ bijectively upon the $\mathscr{H}$-class $R_{e\phi} \cap L_{f\phi}$ of T (since ϕ is $\mathscr{R}$- and $\mathscr{L}$-preserving). Thus $|H| = |H\phi|$.

Since $|E_T \cap D\phi_{\mathscr{D}}| = |E_S \cap D|$, we have therefore $|D\phi_{\mathscr{D}}| = |D|$ for each $D \in \mathscr{D}(S)$. But $|\mathscr{D}(S)| = |\mathscr{D}(T)|$, so $|S| = |T|$.

Before proving the analogue for $\mathscr{J}$-classes of Proposition 1.8, we prove a result of interest in its own right. Some applications are considered in Section 2.

Lemma 1.10. *The weak isomorphism ϕ_E is order-preserving on the idempotents of each $\mathscr{J}$-class of S.*

Proof. Let J be a $\mathscr{J}$-class of S and let $e, f \in E_J$, $e < f$. Then $f \mathscr{D} g$ for some $g \in E_S$, $g \leqq e$ (see, for example [3, Section 8.4, Exercise 3]). Let $x \in R_f \cap L_g$: clearly $xx^{-1} = f > g = x^{-1}x$, that is x is *strictly right regular*, and so $\langle x \rangle$ is bicyclic. By the comments following Corollary 1.7, ϕ_E is therefore an isomorphism on the idempotents of $\langle x \rangle$. Thus $f\phi > g\phi$, (whence $x\phi$ is also strictly right regular).

Now $ex \mathscr{R} e$ and $ex \neq e$ (for if $ex = e$, then $gx = g$, that is $x^{-1}x^2 = x^{-1}x$, impossible in a bicyclic semigroup). Further by the comments following Proposition 1.6, applied to ex, there exists y in T such that $y \in R_{e\phi} \cap \langle ex \rangle \Phi$. Thus $y \in \langle E_S, x \rangle \Phi =$ $= \langle E_T, x\phi \rangle$ (since $x\phi$ is not in a subgroup), and by Lemma 2.2 of [16], $yy^{-1} \leqq$ $\leqq (x\phi)(x\phi)^{-1}$, that is $e\phi \leqq f\phi$.

Denote by $\mathscr{J}(U)$ the poset of $\mathscr{J}$-classes of an inverse semigroup U (where $J_u \leqq J_v$ if $UuU \subseteq UvU$, $u, v \in U$).

Corollary 1.11. *If $e, f \in E_S$, then $e \mathscr{J} f$ if and only if $e\phi \mathscr{J} f\phi$. Hence there is bijection $\phi_{\mathscr{J}}$ of $\mathscr{J}(S)$ upon $\mathscr{J}(T)$ such that*

(i) $|E_T \cap J\phi_{\mathscr{J}}| = |E_S \cap J|$,

(ii) $J_1 \nparallel J_2$ *if and only if* $J_1\phi_{\mathscr{J}} \nparallel J_2\phi_{\mathscr{J}}$,

and

(iii) J *and* $J\phi_{\mathscr{J}}$ *have the same number of* $\mathscr{D}$*-classes, for all* $J, J_1, J_2 \in \mathscr{J}(S)$.

Proof. Let $e, f \in E_S$, $e \mathscr{J} f$. Again $f \mathscr{D} g$ for some $g \in E_S$, $g \leqq e$. By Proposition 1.8, $f\phi \mathscr{D} g\phi$ and by the lemma $g\phi \leqq e\phi$. Hence $J_{f\phi} = J_{g\phi} \leqq J_{e\phi}$. Similarly $J_{e\phi} \leqq$ $\leqq J_{f\phi}$, whence $e\phi \mathscr{J} f\phi$. The converse is similar.

If $J \in \mathscr{J}(S)$, define $J\phi_{\mathscr{J}} = J_{e\phi}$ for some (any) $e \in E_J$. Then $\phi_{\mathscr{J}}$ is clearly well-defined and bijective, and (iii) (and therefore (i)) follow from the corresponding result for $\mathscr{D}$-classes.

To prove (ii) suppose $J_1 \nparallel J_2$, $J_1, J_2 \in \mathscr{J}(S)$. Then by [3, Section 8.4, Exercise 3] again, $e \nparallel f$ for some $e \in E(J_1), f \in E(J_2)$. By Result 1.1, $e\phi \nparallel f\phi$, whence $J_1\phi_{\mathscr{J}} =$ $= J_{e\phi} \nparallel J_{f\phi} = J_2\phi_{\mathscr{J}}$. The converse is similar.

From part (ii) of this corollary we can deduce (as in Result 1.1) that $\phi_{\mathscr{J}}$ is an *order isomorphism* if and only if it is order preserving.

Using Lemma 1.10 and its corollary the conditions under which the weak isomorphism ϕ_E is an isomorphism can be substantially weakened. The results of Sections 3 and 4 depend strongly on this weakening.

Proposition 1.12. *The following are equivalent:*

a) ϕ_E *is order-preserving (that is, an isomorphism),*

b) *for all* $J_1, J_2 \in \mathscr{J}(S)$ *such that* $J_1 < J_2$, *there exist* $e \in E(J_1), f \in E(J_2)$ *such that* $e\phi < f\phi$,

c) $\phi_{\mathscr{J}}$ *is order-preserving (that is, an order-isomorphism).*

Proof. We prove a) $\Rightarrow$ *b*) $\Rightarrow$ c) $\Rightarrow$ a).

a) $\Rightarrow$ b): Let $J_1 < J_2 (J_1, J_2 \in \mathscr{J}(S))$. As above, there exist $e \in E(J_1)$, $f \in E(J_2)$, $e < f$. By a) $e\phi < f\phi$.

b) $\Rightarrow$ c): Let $J_1, J_2 \in \mathscr{J}(S)$, $J_1 < J_2$. By b) $e\phi < f\phi$ for some $e \in E(J_1), f \in E(J_2)$, that is $J_1\phi_{\mathscr{J}} = J_{e\phi} < J_{f\phi} = J_2\phi_{\mathscr{J}}$.

c) $\Rightarrow$ a): Let $e, f \in E_S$, $e < f$. If $e \mathscr{J} f$ then by Lemma 1.10, $e\phi < f\phi$. Otherwise $J_e < J_f$, so that by c), $J_{e\phi} < J_{f\phi}$. By Result 1.1, $e\phi \nparallel f\phi$, possible only if $e\phi < f\phi$.

Corollary 1.13. *If S is an elementary inverse semigroup then* ϕ_E *is an isomorphism.*

Proof. Recall the five types of elementary inverse semigroup listed earlier. Let $S = \langle x \rangle$. If S is a group, the result is obvious, if S is free, then it follows from [14, Theorem 2.1] and if S is bicyclic, it follows from Result 1.3. Otherwise S is of type (III) or (V). In either case, we have, for some $n \geqq 2$

$$\mathscr{J}(S) = \{J_x, J_{x^2}, \ldots, J_{x^{n-1}}, K\}$$

where $J_x > J_{x^2} > \ldots > J_{x^{n-1}} > K$ and the kernel $K = J_{x^n} = J_{x^{n+1}} = \ldots$ (Since $\mathscr{J}(\mathscr{F}\mathscr{I}_1)$ is a chain ,so is $\mathscr{J}(S)$. Then use the final comment before Lemma 1.5). Now $x^{n-1}x^{-(n-1)} \in J_{x^{n-1}}$ and $x^{n-1}x^{-(n-1)} \| x^{-1}x$ (since the same is true in $\mathscr{F}\mathscr{I}_1$) so if $e = (x^{n-1}x^{-(n-1)})(x^{-1}x)$, then $J_e < J_{x^{n-1}}$, that is $e \in K$. By Lemma 1.1, $e\phi = (x^{n-1}x^{-(n-1)})\,\phi(x^{-1}x)\,\phi$, that is $e\phi < (x^{n-1}x^{-(n-1)})\,\phi$, so $K\phi_{\mathscr{J}} < J_{x^{n-1}}\phi_{\mathscr{J}}$.

Again using the comments before Lemma 1.5, $J_{x^r}\phi_{\mathscr{J}} < J_{x^{r-1}}\phi_{\mathscr{J}}$ for $2 \leqq r \leqq n-1$, since $x^{r-1}x^{-(r-1)} \in J_{x^{r-1}}$, $x^{r-1}x^{-(r-1)} \| x^{-1}x$ and $x[(x^{r-1}x^{-(r-1)})(x^{-1}x)]\,x^{-1} = x^r x^{-r}$, that is $e = (x^{r-1}x^{-(r-1)})(x^{-1}x) \in J_{x^r}$, whence $e\phi < (x^{r-1}x^{-(r-1)})\,\phi$. Hence $\phi_{\mathscr{J}}$ is an isomorphism and by the proposition, ϕ_E is an isomorphism.

To complete this section we consider for a moment fundamental inverse semigroups. An inverse semigroup is *fundamental* [22, 24] if it admits no non-trivial idempotent-separating congruences.

Alternatively S is fundamental if $\mu = \iota$, where μ is the largest idempotent-separating congruence on S, defined [11] by $x\,\mu\,y$ if $xex^{-1} = yey^{-1}$ for all $e \in E_S$. The congruence μ is the largest congruence contained in $\mathscr{H}$.

We now characterise fundamental inverse semigroups by properties of their lattices of inverse subsemigroups – from this it easily follows that the class of fundamental inverse semigroups is closed under lattice isomorphisms. An application will be considered in Section 2.

Proposition 1.14. *An inverse semigroup S is fundamental if and only if for every non-idempotent x of S, $\langle E_S, x\rangle$ is not a union of subgroups of S.*

Proof. Suppose S is fundamental and x is non-idempotent. Since $\mu = \iota$, $(x, x^{-1}x) \notin \mu$ and so $xfx^{-1} \neq (x^{-1}x)f(x^{-1}x) = f(x^{-1}x)$, for some $f \in E_S$. But then $(xf)\,.\,(xf)^{-1} \neq (xf)^{-1}(xf)$, that is xf is not a subgroup of S.

Conversely, suppose for all non-idempotents x of S, $\langle E_S, x\rangle$ is not a union of subgroups of S; let $a\,\mu\,b$, $a, b \in S$. Clearly $ab^{-1}\,\mu\,bb^{-1}$. Put $c = ab^{-1}$ and $e = bb^{-1}$. Since $\mu \subseteq \mathscr{H}$, $c \in H_e$.

If $c \notin E_S$, then $\langle E_S, c\rangle \nsubseteq E_S$. Let s be a non-idempotent of $\langle E_S, c\rangle$. Then (by commuting idempotents, if necessay), $s = hc^n$ for some $h \in E_S$, $n \neq 0$. Since $c^n\,\mu\,e$, $s\,\mu\,he$, that is $s \in H_{he}$. Therefore $\langle E, c\rangle$ is a union of subgroups, contradicting the hypothesis. So $c \in E_S$, that is $c \in E$. That $a = b$ then follows, whence $\mu = \iota$.

Theorem 1.15. *The class of fundamental inverse semigroups is closed under lattice isomorphisms.*

Proof. Let S be fundamental and Φ an isomorphism of $\mathscr{L}(S)$ upon $\mathscr{L}(T)$. If T is not fundamental, then for some non-idempotent t of T, $\langle E_T, t\rangle$ is a union of subgroups, and so $E_S \vee \langle t \rangle \Phi^{-1}$ is also. (If $s \in E_S \vee \langle t \rangle \Phi^{-1} = \langle E_T, t \rangle \Phi^{-1}$, then $\langle s \rangle \Phi \subseteq \langle E_T, t \rangle$ since $\langle s \rangle \Phi = \langle b \rangle$ for some $b \in T$, and $\langle s \rangle \Phi$ is a subgroup of T. Hence $\langle s \rangle$ is a subgroup of S). Let $s \in \langle t \rangle \Phi^{-1}$, non-idempotent. Then $\langle E_S, s \rangle$ is a union of subgroups, a contradiction, by the proposition. Hence T is fundamental.

2. SIMPLE INVERSE SEMIGROUPS

Applying 1.8 – 1.11 to simple inverse semigroups, we have the following.

Theorem 2.1. *Lst S be a simple inverse semigroup with α $\mathscr{D}$-classes. If $\mathscr{L}(T) \cong \mathscr{L}(S)$ then T is also simple with α $\mathscr{D}$-classes, and moreover $E_T \cong E_S$ and $|S| = |T|$.*

Proof. By Lemma 1.10, the induced weak isomorphism of E_S upon E_T is order-preserving, thus an isomorphism. The remainder of the theorem follows from Corollary 1.9 and 1.11.

In particular, if S is bisimple and $\mathscr{L}(T) \cong \mathscr{L}(S)$ then so is T.

In view of Corollary 1.9 we may be a little more precise than just $|S| = |T|$ in the theorem – there is a one-to-one correspondence between the cardinalities of the $\mathscr{H}$-classes of S and the cardinalities of those of T. If we apply the theorem to the semilattice C_ω of positive integers under the inverse of their usual order, we see that the class of *simple inverse ω-semigroups* with d $\mathscr{D}$-classes (d finite; see [19, 23] or [12, Chapter V]) is closed under lattice isomorphisms. More particularly if S is the *fundamental* simple inverse semigroup B_d with d $\mathscr{D}$-classes and $\mathscr{L}(T) \cong \mathscr{L}(S)$, then T is again fundamental, by Theorem 1.15, whence $T \cong S$. (It was shown by different methods in [17] that the inverse semigroups B_d are in fact *strongly* determined by their lattices of inverse subsemigroups).

Of course for some semilattices E there is no simple inverse semigroup having E as its semilattice of idempotents. Munn [25] characterized those semilattices which are the semilattice of idempotents of a simple inverse semigroup as being *subuniform* – E is subuniform if for all $e, f \in E$ there exists $g \in E$, $g \leqq f$, such that $Ee \cong Eg$. Given a subuniform semilattice E, every *fundamental* simple inverse semigroup having semilattice of idempotents E is isomorphic with a full inverse subsemigroup of T_E, the (fundamental and simple) inverse subsemigroup of $\mathscr{I}_E$ consisting of the isomorphisms between principal ideals of E.

Proposition 2.2. *Let E be a subuniform semilattice. If $\mathscr{L}(S) \cong \mathscr{L}(T_E)$ for some inverse semigroup S, then S is isomorphic with a full inverse subsemigroup of T_E.*

Further if each principal ideal of E has a finite group of automorphisms, then $S \cong T_E$.

Proof. Let $\phi : \mathscr{L}(S) \to \mathscr{L}(T_E)$ be the isomorphism. By Theorem 2.1 (applied to Φ^{-1}), S is simple, and by Theorem 1.15 S is fundamental. Further the weak isomorphism $\phi_E : E_S \to E(T_E)$ is an isomorphism. There is a monomorphism θ of S into T_E extending ϕ_E, and $S\theta$ is a full inverse subsemigroup of T_E.

In fact by the arguments of Corollary 1.9, we have $|H\theta| = |R_{e\phi} \cap L_{f\phi}|$ and $H\theta \subseteq R_{e\phi} \cap L_{f\phi}$, for each $\mathscr{H}$-class $H = R_e \cap L_f$ of S. If each principal ideal of E has finite automorphism group, then the $\mathscr{H}$-classes of T_E are finite. Thus $S\theta = T_E$ and $S \cong T_E$.

For example if E is an *ω-tree*, that is a semilattice in which each principal ideal is isomorphic with C_ω, then $E\omega$ is (sub)uniform and each principal ideal has trivial automorphism group. Thus T_E is determined by its lattice of inverse subsemigroups.

There is an analogue of Theorem 2.1 for 0-*simple* inverse semigroups. Note first, however, that if U and V are inverse semigroups such that $\mathscr{L}(U) \cong \mathscr{L}(V)$, then $\mathscr{L}(U^0) \cong \mathscr{L}(V^1)$. We show, rather surprisingly prehaps, that this is the "worst" that can happen if S is 0-simple: if $\mathscr{L}(T) \cong \mathscr{L}(S)$, either T is 0-simple or $T = V^1$ where $\mathscr{L}(V) \cong \mathscr{L}(S \setminus 0)$. More precisely (denoting $S \setminus 0$ by S^*).

Theorem 2.3. *Let S be a* 0*-simple inverse semigroup with α $\mathscr{D}$-classes, and suppose $\mathscr{L}(S) \cong \mathscr{L}(T)$. Either T is* 0*-simple with α $\mathscr{D}$-classes and $E_T \cong E_S$, or $S^* < S$ (that is $S = (S^*)^0$) and $T = (S^*\Phi)^1$, where $S^*\Phi$ is simple with $\alpha - 1$ $\mathscr{D}$-classes and $E(S^*\Phi) \cong E(S^*)$.*

Proof. Let $\Phi : \mathscr{L}(S) \to \mathscr{L}(T)$ be the isomorphism. Since S has two $\mathscr{J}$-classes, one of which is trivial, the same is true of T. Suppose $S^* \nless S$. Then $ef = 0$ for some $e, f \in E(S^*)$. By Result 1.1, $(e\phi)(f\phi) = 0\phi$ and so $0\phi < e\phi$. Thus $\phi_{\mathscr{J}}$ is order-preserving and by Proposition 1.12, ϕ_E is an isomorphism. Since $J_{0\phi}$ is trivial, 0ϕ is the zero of T, whence T is 0-simple.

Now suppose $S^* < S$ and let $f \in E(S^*)$. If $0\phi < f\phi$, then as above ϕ_E is an isomorphism and T is 0-simple. Otherwise $\phi_{\mathscr{J}}$ is order-inverting and so $0\phi > g$ for all $g \in E(S^*\Phi)$. Put $0\phi = e$. We must show $ey = ye = y$ for all $y \in S^*\Phi$.

Let y be a non-idempotent of $S^*\Phi$. Since $e > yy^{-1}$, $ey \mathscr{R} yy^{-1}$. Now $ey \in \langle e, y\rangle$, so

$$\langle ey\rangle \Phi^{-1} \subseteq \langle e\rangle \Phi^{-1} \vee \langle y\rangle \Phi^{-1} = \{0\} \vee \langle y\rangle \Phi^{-1} = \{0\} \cup \langle y\rangle \Phi^{-1}.$$

But since $e \notin \langle ey\rangle$, $0 \notin \langle ey\rangle \Phi^{-1}$, that is $\langle ey\rangle \Phi^{-1} \subseteq \langle y\rangle \Phi^{-1}$, or $ey \in \langle y\rangle$. Suppose $yy^{-1} \parallel y^{-1}y$; then $\langle y\rangle \cap R_{yy^{-1}} = \{yy^{-1}, y\}$, so that $ey = y$ (for if $ey = yy^{-1}$, then $y^{-1}ey = y^{-1}$, a contradiction, for y is non-idempotent.) Otherwise $yy^{-1} \not\parallel y^{-1}y$ and $\langle y\rangle$ is bicyclic. Then $ey \in \langle y\rangle$ implies $y^{-1}ey \in \langle y\rangle$ and $ey = y(y^{-1}ey)$ Thus $ey \,\sigma\, y$, and since $ey \mathscr{R} y$ and $\langle y\rangle$ is E-unitary (see Section 3), $ey = y$. Similarly $ye = y$.

Hence $T = (S^*\Phi)^1$. The final statement is an application of Theorem 2.1.

Applications similar to those of Theorem 2.1 can be made for 0-subuniform semilattices [25].

In a slightly different direction, suppose S is a *Brandt semigroup*, that is a completely 0-simple inverse semigroup. If $S = G^0$, group with zero adjoined, and $\mathscr{L}(S) \cong \mathscr{L}(T)$, then $T = H^0$ or $T = H^1$ for some group H with $\mathscr{L}(H) \cong \mathscr{L}(G)$. If $S \neq G^0$, then T is again 0-simple, $|T| = |S|$ and $E_T \cong E_S$, whence T is completely 0-simple. If $S = \mathscr{M}^0(G; I, I; \Delta)$, say ([3, Chapter 3]), $|I| \geqq 2$, then $T \cong \mathscr{M}^0(H; I, I; \Delta)$ for some group H such that $\mathscr{L}(G) \cong \mathscr{L}(H)$ and $|G| = |H|$. (Note that by cardinality arguments similar to those used above, it can be seen that Φ is even a *strictly index-preserving isomorphism* of $\mathscr{L}(G)$ upon $\mathscr{L}(H)$, that is $|V : U| = |V\Phi : U\Phi|$ for all $U < V \leqq G$).

If for instance G is cyclic, then so is H, whence $G \cong H$ and $S \cong T$. However there exist non-isomorphic finite groups G and H with a strictly index-preserving lattice isomorphism between them (even preserving conjugacy), [33, p. 57]. The author does not know, however, whether for two such groups G and H $\mathscr{L}(\mathscr{M}^0(G; I, I; \Delta)) \cong \mathscr{L}(\mathscr{M}^0(H; I, I; \Delta))$.

3. E-UNITARY INVERSE SEMIGROUPS

An inverse semigroup S is called *E-unitary* (or proper, or reduced) if $ex = e$, $e^2 = e$ imply $x^2 = x$, $x, e \in S$. Equivalently [28] S is E-unitary if $\mathscr{R} \cap \sigma = \iota$, where σ is the least group congruence on S, defined by $a \,\sigma\, b$ if $ea = eb$ for some $e \in E_S$.

Before applying the results of Section 1 to E-unitary inverse semigroups, some observations are in order.

The existence of isolated subgroups in an inverse semigroup S can clearly lead to difficulties when considering lattice isomorphisms of S. There are, of course, many non-isomorphic groups having isomorphic lattices of subgroups. More importantly, from our point of view, are example like the semilattice $S_{G,H}$ of groups G and H, where $e_G > e_H$ and $gh = hg = h$ for all $g \in G$, $h \in H$. It is is easily seen that $\mathscr{L}(S_{G,H})$ is isomorphic with the lattice formed by adjoining a zero to $\mathscr{L}(G) \times \mathscr{L}(H)$. Then $\mathscr{L}(S_{G,H}) \cong \mathscr{L}(S_{H,G})$ though of course $S_{G,H}$ and $S_{H,G}$ need not be isomorphic. In particular if G is trivial, then $S_{G,H} = H^1$ and $S_{H,G} = H^0$. Here $S_{G,H}$ is E-unitary but $S_{H,G}$ is not.

Note however that in this example, the associated weak-isomorphism ϕ_E inverts the idempotents. We will show that if ϕ_E is an isomorphism then the property of being E-unitary is preserved. Moreover under some mild conditions, ϕ_E *must* be an isomorphism.

For the remainder of the section Φ will be a lattice isomorphism of S upon T and the various mappings ϕ_E, ϕ, $\phi_{\mathscr{D}}$ and $\phi_{\mathscr{J}}$ are those defined in Section 1.

Lemma 3.1. *If S is E-unitary and J is a non-trivial $\mathscr{J}$-class of S, then ϕ_E is an isomorphism on the ideal generated by J.*

Proof. By Proposition 1.12, applied to the ideal generated by J, it is sufficient to show that for any $J_1, J_2 \in \mathscr{J}(S)$ such that $J_1 < J_2 \leqq J$, then $e\phi < f\phi$ for some $e \in E(J_1), f \in E(J_2)$,

Now since J is non-trivial so is J_2. For, let $x \in J \setminus E_J$. Since $xx^{-1} \in E_J$, $g < xx^{-1}$ for some $g \in E(J_2)$ (as in the proof of Lemma 1.10), whence $gx \in R_g \subseteq J_2$ and $gx \neq g$ since S is E-unitary.

Let $a \in J_2 \setminus E(J_2)$; clearly aa^{-1} and $a^{-1}a$ are in $E(J_2)$. Again $e < aa^{-1}$ for some $e \in E(J_1)$ and again $ea \mathscr{R} e$, $ea \neq e$. By the comments following Lemma 1.5, there is an element y of T such that $\langle a \rangle \Phi = \langle y \rangle$, $yy^{-1} = (aa^{-1})\phi$ and $y^{-1}y = (a^{-1}a)\phi$. Similarly $\langle ea \rangle \Phi = \langle z \rangle$, $zz^{-1} = ((ea)(ea)^{-1})\phi = e\phi$ and $z^{-1}z = (a^{-1}ea)\phi$ for some $z \in T$, non-idempotent.

But $\langle z \rangle = \langle ea \rangle \Phi \subseteq E_S \vee \langle a \rangle \Phi = \langle E_S, y \rangle$, so z can be written in the form $z = hy^n$ for some $h \in E_T$, $n \neq 0$. Then $e\phi = zz^{-1} \leqq y^n y^{-n}$, so either $e\phi < yy^{-1} = (aa^{-1})\phi$ (if $n > 0$) or $e\phi < y^{-1}y = (a^{-1}a)\phi$ (if $n < 0$), as required.

Corollary 3.2. *If S is E-unitary and has no trivial $\mathscr{J}$-class, in particular if S has a set of non-idempotent generators then ϕ_E is an isomorphism of E_S upon E_T.*

Proof. Suppose no $\mathscr{J}$-class of S is trivial. Let $e, f \in E_S$, $e < f$. Then J_f is non-trivial and e, f lie in the ideal generated by J_f, so that $e\phi < f\phi$ by the lemma.

If $S = \langle X \rangle$, where X consists of non-idempotents, then every $\mathscr{J}$-class of S is below J_x for some $x \in X$. Since each J_x is non-trivial, so is every $\mathscr{J}$-class by the second paragraph of the proof above.

The hypotheses of this corollary are by no means necessary – for example there are semilattices which are strongly determined by their lattices of subsemilattices [32]. (Note, however, that from the comments early in the proof of Lemma 3.1, in an E-unitary inverse semigroup any $\mathscr{J}$-class *above* a trivial $\mathscr{J}$-class (in the poset of $\mathscr{J}$-classes) is again trivial).

Lemma 3.3. *If S is any inverse semigroup and A, B and $A \cup B \in \mathscr{L}(S)$, then $A\Phi \cup B\Phi \in \mathscr{L}(T)$.*

Proof. Clearly $A\Phi \cup B\Phi \subseteq (A \cup B)\Phi$.

Conversely let $y \in (A \cup B)\Phi$, so that $\langle y \rangle \Phi^{-1} \subseteq A \cup B$. By Lemma 1.4, $\langle y \rangle \Phi^{-1} = \langle s \rangle$ for some $s \in S$. Either $s \in A$ or $s \in B$, so $\langle s \rangle \subseteq A$ or $\langle s \rangle \subseteq B$, that is $\langle y \rangle \subseteq A\Phi$ or $\langle y \rangle \subseteq B\Phi$. Hence $y \in A\Phi \cup B\Phi$ and $A\Phi \cup B\Phi = (A \cup B)\Phi \in \mathscr{L}(T)$.

Theorem 3.4. *If S is E-unitary and ϕ_E is an isomorphism then T is E-unitary.*

Proof. Suppose not. Then $ex = e = xe$ for some $x \in T \setminus E_T$, $e \in E_T$. (If $ex = e$, then $x \,\sigma\, e$, $xf = ef$ for some $f \in E_T$, whence $x(ef) = ef = (ef)x$). By induction $ex^n = e = x^n e$ for every integer $n \neq 0$. So $\langle e, x \rangle = \{e\} \cup \langle x \rangle$ and $e \leqq x^n x^{-n}$,

$e \leqq x^{-n}x^n$ for every integer $n \neq 0$. By the lemma, $\{e\}\,\Phi^{-1} \cup \langle x \rangle\,\Phi^{-1} \in \mathscr{L}(S)$. Let $f = e\phi^{-1}$, $g = (xx^{-1})\,\phi^{-1}$ and let s be a non-idempotent of S such that $\langle s \rangle = \langle x \rangle\,\Phi^{-1}$, $ss^{-1} = g$ and $s^{-1}s = (x^{-1}x)\,\phi^{-1}$, (see Lemma 1.5 and the comments following). Thus $\{f\} \cup \langle s \rangle \in \mathscr{L}(S)$. Now $fs \neq f$ since S is E-unitary, so $fs \in \langle s \rangle$. But since $e \leqq xx^{-1}$ and ϕ_E is an isomorphism, $f \leqq g = ss^{-1}$, so that $f = f(ss^{-1}) = (fs)\,s^{-1} \in \langle s \rangle$. Therefore $e \in \langle x \rangle$ and by the properties above e is a *zero* for $\langle x \rangle$. Thus $\langle x \rangle$ has a kernel $K = H_e \cap \langle x \rangle = \{e\}$. Since $\phi_{\mathscr{J}}^{-1}$ (where $\mathscr{J} = \mathscr{J}\langle s \rangle$) is an order – isomorphism of $\mathscr{J}(\langle x \rangle)$ upon $\mathscr{J}(\langle s \rangle)$ (by Proposition 1.12), $K\Phi^{-1}$ is the minimum $\mathscr{J}_{\langle s \rangle}$-class of $\langle s \rangle$, that is the kernel of $\langle s \rangle$. But $K\Phi^{-1} = \{f\}$, and so f is a zero for $\langle s \rangle$. This is a contradiction as since S is E-unitary so is $\langle s \rangle$.

Corollary 3.5. *The class of E-unitary inverse semigroups without trivial $\mathscr{J}$-classes is closed under lattice isomorphisms.*

We will now show that Theorem 3.4 can be strengthened: when ϕ_E is an isomorphism not only must T be E-unitary but there is a lattice isomorphism of S/σ upon T/σ which, in a certain sense, commutes with Φ. Two preliminary lemmas on elementary inverse semigroups are needed.

Lemma 3.6. *If $S = \langle a \rangle$ then $a^r\phi = (a\phi)^r$ for every integer $r(\neq 0)$ such that a^r is not in the kernel, if any, of S.*

Proof. If S is free, then $\phi : S \to T$ is an isomorphism [14, Theorem 2.1] so $a^r\phi = (a\phi)^r$ for all $r \neq 0$. (In this case S has no kernel).

Otherwise S has a kernel K, either a group or bicyclic. (See section 1.) Suppose $a^r \notin K$. Then a^r is not a subgroup of S, and therefore neither is a. Hence $a^r\phi$ and $a\phi$ are defined (Proposition 1.6).

From the description of the elementary inverse semigroups given in Section 1 (and in particular the comment immediately preceding Lemma 1.5) we see that, as in $\mathscr{F}\mathscr{I}_1$, the $\mathscr{J}$-class J_{a^r} of S contains a unique idempotent, $a^r a^{-r}$, below aa^{-1} and covered by at most one idempotent of S. Moreover since ϕ_E is an isomorphism by Corollary 1.13, $\phi_{\mathscr{J}}$ is an order-isomorphism of $\mathscr{J}(S)$ upon $\mathscr{J}(T)$, by Proposition 1.12. Thus $(J_{a^r})\,\phi_{\mathscr{J}}$ is not the kernel $K\Phi$ of T. In fact $J_{a^r}\phi_{\mathscr{J}} = J_{(a\phi)^r}$, since J_{a^r} and $J_{(a\phi)^r}$ have the same depth in their respective posets $\mathscr{J}(S)$, $\mathscr{J}(T)$, each chains. (Namely they have depth r, the depth of J_{x^r} in $\mathscr{J}(\mathscr{F}\mathscr{I}_1)$, $\mathscr{F}\mathscr{I}_1 = \langle x \rangle$). Since ϕ_E is an isomorphism, $J_{(a\phi)^r}$ thus has a unique idempotent $(a^r a^{-r})\,\phi$ below $(aa^{-1})\,\phi = (a\phi)\,(a\phi)^{-1}$ and covered by at most one idempotent of T. But $(a\phi)^r\,(a\phi)^{-r}$ also has this property in T, so

$$(a^r\phi)\,(a^{-r}\phi) = (a^r a^{-r})\,\phi = (a\phi)^r\,(a\phi)^{-r},$$

that is $a^r\phi \;\mathscr{R}\; (a\phi)^r$.

Dually $a^r\phi \;\mathscr{L}\; (a\phi)^r$, so $a^r\phi \;\mathscr{H}\; (a\phi)^r$. Since $J_{(a\phi)^r}$ does not meet the kernel of T, it has trivial subgroups. Hence $a^r\phi = (a\phi)^r$.

Lemma 3.7. *If S is E-unitary and $x \in S$, suppose $\langle x \rangle$ has kernel K. Then $K\sigma = \langle x \rangle \sigma$.*

Proof. (Note that strictly speaking we should perhaps be writing $K\sigma^{\#} \approx \langle x \rangle \sigma^{\#}$ where $\sigma^{\#}$ is the natural morphism of S upon S/σ.)

From the list of elementary inverse semigroups given in Section 1, the result is obvious if $\langle x \rangle$ is of type (I) or (IV) (when $K = \langle x \rangle$). Otherwise $K = \langle x^{n+1}x^{-n} \rangle$ for some $n > 2$, and since $x^{n+1}x^{-n}\sigma x$, we have $K\sigma = \langle x \rangle \sigma$.

The crucial technical lemma is the following.

Lemma 3.8. *If S is E-unitary and ϕ_E is an isomorphism, suppose $z = fx$, $x, z \in S \setminus E_S$, $f \in E_S$. Then*

$$(\langle z \rangle \Phi) \sigma = (\langle x \rangle \Phi) \sigma .$$

Proof. By Lemma 1.4, $\langle z \rangle = \langle c \rangle$ and $\langle x \rangle = \langle a \rangle$ for some $c, a \in T \setminus E_T$. Since $z = fx$, $\langle z \rangle \subseteq \langle E_S, x \rangle$, whence $\langle c \rangle \subseteq \langle E_T, a \rangle$. By commuting idempotents, if necessary, we may write $c = ku$ for some $k \in E_T$ and $u = a^s \in \langle a \rangle$, $s \neq 0$. Then $c \; \sigma \; u$, so $\langle c \rangle \sigma = \langle u \rangle \sigma \langle a \rangle \sigma$. We will show that $\langle a \rangle \sigma = \langle u \rangle \sigma$. Now $c \in \langle E_T, u \rangle$, so conversely, $\langle z \rangle \subseteq \langle E_S, y \rangle$, where $\langle y \rangle = \langle u \rangle \Phi^{-1} \subseteq \langle x \rangle$. Similarly $z = ly^m$ for some $l \in E_S$ and $m \neq 0$. Thus $x \; \sigma \; y^m$.

If (i) $\langle x \rangle$ is a group, then $y = x^r$ for some $r \neq 0$ and so $x \; \sigma \; x^{rm}$. Since σ is the identical relation on any subgroup of S, $x = x^{rm} \in \langle y \rangle = \langle u \rangle \Phi^{-1}$, that is $\langle a \rangle = \langle u \rangle$. Hence $\langle a \rangle \sigma = \langle u \rangle \sigma$.

If (ii) $\langle x \rangle$ is bicyclic, then $\phi : \langle x \rangle \to \langle a \rangle$ is an isomorphism (Result 1.3) and we may choose $y = u\phi^{-1} = a^s\phi^{-1} = (a\phi^{-1})^s = x^s$, so $x \; \sigma \; x^{sm}$. Since $\langle x\sigma \rangle$ is infinite cyclic, this is possible only if $sm = \pm 1$, so that $s = \pm 1$.

Thus $u = a^{\pm 1}$, so $\langle u \rangle = \langle a \rangle$ and $\langle a \rangle \sigma = \langle u \rangle \sigma$ again.

If (iii) y^m is not in the kernel (if any) of $\langle x \rangle$, (so that $\langle x \rangle$ is neither a group nor bicyclic), then neither is y, and hence u is not in the kernel of $\langle a \rangle$. We may therefore choose $y = u\phi^{-1}$. By Lemma 3.6 (applied to ϕ^{-1}), $y = u\phi^{-1} = a^s\phi^{-1} = (a\phi^{-1})^s = x^s$. Hence $x \; \sigma \; x^{sm}$ again. If $sm \neq 1$, then $x^{sm-1} \in E$ (since S is E-unitary), and so $\langle x \rangle$ has a kernel, which must contain $(x^{sm-1})\, x = x^{sm} = y$, contradicting the assumption. Hence $sm = 1$ and as in (ii), $\langle u \rangle \sigma = \langle a \rangle \sigma$.

If (iv) $\langle x \rangle$ is not a group but y^m is in a subgroup of $\langle x \rangle$, that is in the kernel K of $\langle x \rangle$, then $\langle x \rangle$ is of type (III) (see Section 1) and $K = \langle x^{n+1}x^{-n} \rangle$ for some $n \geqq 2$. But $x^{n+1}x^{-n} \; \sigma \; x \; \sigma \; y^m$, in the subgroup K, so $y^m = x^{n+1}x^{-n}$. Thus $z = l(x^{n+1}x^{-n})$ and using (i) $(\langle z \rangle \Phi) \sigma = (\langle x^{n+1}x^{-n} \rangle \Phi) \sigma = (K\Phi) \sigma$. As in the proof of Lemma 3.6, $\phi_{\mathscr{J}}$ is an order-isomorphism of $\mathscr{J}(\langle x \rangle)$ upon $\mathscr{J}(\langle x \rangle \Phi)$, so $K\Phi$ is the kernel of $\langle x \rangle \Phi = \langle a \rangle$. By Lemma 3.7, $(K\Phi) \sigma = \langle a \rangle \sigma$, so $\langle c \rangle \sigma = \langle a \rangle \sigma$.

If, finally, (v), $\langle x \rangle$ is not bicyclic but y^m is in a bicyclic kernel K of $\langle x \rangle$ (so $\langle x \rangle$ is of type (V)) then since $z = ly^m$ we have, applying (ii), $\langle c \rangle \sigma = (\langle z \rangle \Phi) \sigma = (\langle y^m \rangle \Phi) \sigma$. Now as in (iv) $K\Phi$ is the kernel (again bicyclic) of $\langle a \rangle$. Further

$\Phi \mid \mathscr{L}(K)$ is induced by the isomorphism $\phi : K \to K\Phi$. Now K is generated by $x^{n+1}x^{-n}$ or $x^{-(n+1)}x^{n}$ (each right regular) for some $n \geqq 2$. Similarly $K\Phi$ is generated by $a^{p+1}a^{-p}$ or $a^{-(p+1)}a^{p}$ for some $p \geqq 2$. If K is generated by $x^{n+1}x^{-n}$, then $(x^{n+1}x^{-n})\,\phi = a^{p+1}a^{-p}$ or $a^{-(p+1)}a^{p}$. But $y^{m} \in K$ and $y^{m}\,\sigma\, x\,\sigma\, x^{n+1}x^{-n}$, so $y^{m}\phi\,\sigma\,(x^{n+1}x^{-n})\,\phi\,\sigma\, a^{\pm 1}$. Hence $\langle y^{m}\phi\rangle\,\sigma = \langle a\rangle\,\sigma$. A similar result is obtained if $K = \langle x^{-(n+1)}x^{n}\rangle$. Thus $\langle a\rangle\,\sigma = (\langle y^{m}\rangle\,\Phi)\,\sigma \subseteq (\langle y\rangle\,\Phi)\,\sigma = \langle u\rangle\,\sigma$, and so $\langle a\rangle\,\sigma = \langle u\rangle\,\sigma$.

In very case, then, $\langle a\rangle\,\sigma = \langle u\rangle\,\sigma$, so that $(\langle z\rangle\,\Phi)\,\sigma = \langle c\rangle\,\sigma = \langle u\rangle\,\sigma = \langle a\rangle\,\sigma = (\langle x\rangle\,\Phi)\,\sigma$.

Corollary 3.9. *If S is E-unitary and ϕ_E is an isomorphism, let $x, y \in S$. Then*

$$\langle x\rangle\,\sigma = \langle y\rangle\,\sigma \quad \textit{if and only if} \quad (\langle x\rangle\,\Phi)\,\sigma = (\langle y\rangle\,\Phi)\,\sigma\,.$$

Proof. Suppose $\langle x\rangle\,\sigma = \langle y\rangle\,\sigma$. Then $x\,\sigma\,w$ for some $w \in \langle y\rangle$, and therefore $fx = fw = z$, say, for some $f \in E_S$. If $z \in E_S$, then since S is E-unitary, $x, w \in E_S$. Clearly in this case $\langle x\rangle\,\Phi, \langle y\rangle\,\Phi \subseteq E_T$ and $(\langle x\rangle\,\Phi)\,\sigma = (\langle y\rangle\,\Phi)\,\sigma$. Otherwise Lemma 3.8 applies and

$$(\langle x\rangle\,\Phi)\,\sigma = (\langle z\rangle\,\Phi)\,\sigma = (\langle w\rangle\,\Phi)\,\sigma \subseteq (\langle y\rangle\,\Phi)\,\sigma\,.$$

The converse inclusion follows similarly and so

$$(\langle x\rangle\,\Phi)\,\sigma = (\langle y\rangle\,\Phi)\,\sigma\,.$$

The converse implication follows similarly (for by Theorem 3.4, T is also E-unitary).

Recall from [15, Lemma 5.5] that in an E-unitary inverse semigroup S, the natural morphism $\sigma^{\#}$ of S upon S/σ induces a *lattice* morphism Σ_S of $\mathscr{L}\mathscr{F}(S)$. the lattice of *full* inverse subsemigroups of S, upon $\mathscr{L}(S/\sigma)$.

Theorem 3.10. *If S is E-unitary and ϕ_E is an isomorphism then there is a lattice isomorphism Ψ of S/σ upon T/σ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathscr{L}\mathscr{F}(S) \arrow[r, "\Phi"] \arrow[d, "\Sigma_S"'] & \mathscr{L}\mathscr{F}(T) \arrow[d, "\Sigma_T"] \\
\mathscr{L}(S/\sigma) \arrow[r, "\Psi"] & \mathscr{L}(T/\sigma)
\end{tikzcd}$$

Fig. 1.

Proof. By Theorem 3.4, T is E-unitary and so Σ_T is a morphism. For convenience, put $G = S/\sigma$, $H = T/\sigma$.

If $K \in \mathscr{L}(G)$, define

$$K\Psi = (K\sigma^{-1})\,\Phi\Sigma_T\,,$$

(where, of course, $K\sigma^{-1} = \{s \in S : s\sigma \in K\}$. If $L \in \mathscr{L}(H)$, define

$$L\Theta = (L\sigma^{-1})\,\Phi\Sigma_S\,,$$

(where, now $L\sigma^{-1} = \{t \in T : t\sigma \in L\}$, the symbol σ standing for σ_S or σ_T as required).

In order that $(K\Psi)\,\Theta = K$, we require $[(K\sigma^{-1})\,\Phi]\,\Sigma_T\sigma^{-1} = (K\sigma^{-1})\,\Phi$, that is $t\sigma \in ((K\sigma^{-1})\,\Phi)\,\sigma$ implies $t \in (K\sigma^{-1})\,\Phi$. (In other words we require that $(K\sigma^{-1})\,\Phi$ be *unitary*).

So let $t \in T$ be such that $t\;\sigma\;v$ for some $v \in (K\sigma^{-1})\,\Phi$. Then $\langle t\rangle\,\Phi^{-1} = \langle s\rangle$ and $\langle v\rangle\,\Phi^{-1} = \langle u\rangle$ for some $s, u \in S$. By Corollary 3.9, applied to Φ^{-1}, $\langle s\rangle\,\sigma = \langle u\rangle\,\sigma \subseteq$ $\subseteq (K\sigma^{-1})\,\sigma = K$. Thus $\langle s\rangle \subseteq K\sigma^{-1}$, whence $t \in \langle t\rangle \subseteq (K\sigma^{-1})\,\Phi$, as required.

Hence $(K\Psi)\,\Theta = K$ for all $K \in \mathscr{L}(G)$. Similarly $(L\Theta)\,\Psi = L$ for all $L \in \mathscr{L}(H)$. Therefore $\Theta = \Psi^{-1}$.

If $K_1, K_2 \in \mathscr{L}(G)$, $K_1 \subseteq K_2$, then clearly $K_1\Psi \subseteq K_2\Psi$. So Ψ and similarly Θ are order-preserving, that is Ψ is an order-isomorphism (lattice isomorphism) of $\mathscr{L}(G)$ upon $\mathscr{L}(H)$.

To show the diagram commutes, let $A \in \mathscr{L}\mathscr{F}(S)$. Now $A \subseteq (A\sigma)\,\sigma^{-1} = (A\Sigma_S)\,\sigma^{-1}$, so

$$A\Phi\Sigma_T \subseteq ((A\Sigma_S)\,\sigma^{-1})\,\Phi\Sigma_T = (A\Sigma_S)\,\Psi\,.$$

Conversely, let $h \in (A\Sigma_S)\,\sigma^{-1}\Phi\Sigma_T$, that is $h = t\sigma$ for some $t \in (A\Sigma_S)\,\sigma^{-1}\Phi$. Let $\langle t\rangle\,\Phi^{-1} = \langle s\rangle$, so that $s \in (A\Sigma_S)\,\sigma^{-1}$. Then $s\;\sigma\;a$ for some $a \in A$. By Corollary 3.9, $(\langle s\rangle\,\Phi)\,\sigma = (\langle a\rangle\,\Phi)\,\sigma \subseteq (A\Phi)\,\sigma$ and so $h \in \langle t\rangle\,\sigma \subseteq A\Phi\Sigma_T$. Therefore $A\Phi\Sigma_T =$ $= A\Sigma_S\Psi$ and the diagram commutes.

We now apply this theorem to the class of quasi-free inverse semigroups. Recall [29, p. 71] that an inverse semigroup is called *quasi-free* if it is isomorphic to the quotient of a free inverse semigroup by an *idempotent-determined* congruence (that is, a congruence which ([8]) identifies no non-idempotent with an idempotent).

Result 3.11. [29, Theorem 5.3]. *Every quasi-free inverse semigroup is E-unitary. An inverse semigroup S is quasi-free if and only if* a) *S/σ is a free group and* b) *S has a set M of generators such that $\sigma^\# \mid M$ is injective and $M\sigma^\#$ generates S/σ freely.*

More precisely, if $Q = \mathscr{F}\mathscr{I}_X/\varrho$ for some idempotent-determined congruence ϱ, then $M = X\varrho^\#$ is a generating set satisfying b), and $M\sigma^\#$ generates $Q/\sigma \cong \mathscr{F}\mathscr{G}_X$ freely. We call $|X|$ the *rank* of Q. Note that M consists of non-idempotents.

The importance of quasi-free inverse semigroups stems from the fact ([29,Theorem 5.3]) that every inverse semigroup is an idempotent-separating image of such a semigroup.

Theorem 3.12. *The class of quasi-free inverse semigroups of given rank is closed under lattice isomorphisms.*

Proof. Let Q be quasi-free of rank χ, say, and let Φ be a lattice isomorphism of Q upon an inverse semigroup T.

Let M be a generating set of the above kind, so that M consists of non-idempotents. By Corollary 3.2, ϕ_E is an isomorphism. Hence by Theorem 3.4, T is E-unitary and by Theorem 3.10, there is an isomorphism Ψ of $\mathscr{L}(Q/\sigma)$ upon $\mathscr{L}(T/\sigma)$ such that $A\Phi\Sigma_T = A\Sigma_Q\Psi$ for each $A \in \mathscr{L}\mathscr{F}(Q)$. Since Q/σ is free on χ generators so is T/σ, by a theorem of Sadovskiĭ [30].

In fact if $\chi \neq 1$, Ψ is *induced* by an isomorphism ψ of Q/σ upon T/σ.

For each $m \in M$, let $\langle m \rangle \Phi = \langle m' \rangle$ for some $m' \in T$, and put $M' = \{m' : m \in M\}$. Then M' generates T. If $m \in M$, then $\langle E_Q, m \rangle \in \mathscr{L}\mathscr{F}(Q)$ and so $(\langle E_Q, m\rangle \Phi)\Sigma_T = (\langle E_Q, m \rangle \Sigma_Q)\Psi = (\langle m \rangle \sigma^{\#})\Psi = \langle m\sigma^{\#}\rangle \Psi$, that is $\langle m'\sigma^{\#}\rangle = \langle E_T, m'\rangle \Sigma_T = (\langle E_Q, m\rangle \Phi)\Sigma_T = \langle m\sigma^{\#}\rangle \Psi$.

If $\chi = 1$, that is Q is elementary, then so is T and by Result 3.11, T is clearly quasi-free.

If $\chi \neq 1$, then $\langle m'\sigma^{\#}\rangle = \langle m\sigma^{\#}\rangle \psi = \langle (m\sigma^{\#})\psi\rangle$, where ψ induces Φ. But T/σ, being free, is certainly torsion-free, so $m'\sigma^{\#} = (m\sigma^{\#}\psi)^{\pm 1}$ for all m in M. Since $M\sigma^{\#}$ generates Q/σ freely, $(M\sigma^{\#})\psi$ generates T/σ freely. Hence $M'\sigma^{\#}$ generates T/σ freely. (See, for example [9, Chapter 7]). If $m_1'\sigma^{\#} = m_2'\sigma^{\#}$, $m_1, m_2 \in M$, then $(m_1\sigma^{\#})\psi = [(m_2\sigma^{\#})\psi]^{\pm 1}$, that is $m_1\sigma^{\#} = (m_2\sigma^{\#})^{\pm 1}$. Since $M\sigma^{\#}$ generates Q/σ freely, $m_1\sigma^{\#} = m_2\sigma^{\#}$, and since $\sigma^{\#} \mid M'$ is injective, $m_1 = m_2$. Thus $\sigma^{\#} \mid M'$ is injective.

Hence by Result 3.11, T is quasi-free of rank χ.

Some quasi-free inverse semigroups, for example free inverse semigroups [14], free groups [30] and the bicyclic semigroup [17] are in fact *determined* (in most cases strongly) by their lattices of inverse subsemigroups. The author conjectures that the same is true for all quasi-free inverse semigroups. It is however, possible for two E-unitary inverse semigroups to have isomorphic semilattices and isomorphic maximum group images without being isomorphic. (See [20, Theorem 1.3]).

4. FREE PRODUCTS OF GROUPS

If G and H are groups then following [21] we will denote by G inv H their free product *in the category of inverse semigroups*, and by G gp H their free product *in the category of groups*. The inverse semigroup G inv H and the group G gp H are characterized by the usual universal properties in their respective categories.

It was proved by Sadovskiĭ [31] that G gp H is determined by its lattice of subgroups when G and H are non-trivial. (If one group is trivial then G gp H is isomorphic with the other). In fact Holmes [10] and Aršinov [1] proved that each lattice isomorphism of G gp H upon a group K is induced by a unique isomorphism of G gp H upon K.

We now apply the results of Section 3 to show that for any groups G and H,

$G \operatorname{inv} H$ is determined by its lattice of inverse subsemigroups. Whether $G \operatorname{inv} H$ is *strongly* determined by $\mathscr{L}(G \operatorname{inv} H)$ is not known.

Explicit descriptions of $G \operatorname{inv} H$ were given independently by McAlister [21] and by Knox [18] to which the reader is referred for all details. We present a brief summary of the relevant results. (For properties of $G \operatorname{gp} H$, see [9]).

Result 4.1. *Let G and H be groups. Then*
(i) $G \operatorname{inv} H$ *is completely semisimple, and so* $\mathscr{D} = \mathscr{J}$,
(ii) $G \operatorname{inv} H$ *is fundamental, except when G and H are finite and either G or H (but not both) is trivial,*
(iii) $G \operatorname{inv} H$ *is E-unitary and* $(G \operatorname{inv} H)/\sigma \cong G \operatorname{gp} H$.

This result gives the global properties of $G \operatorname{inv} H$. To exhibit the fine structure some definitions are required.

A word in G and H is said to *reduced* if no two adjacent letters are in the same group G or H and no letter is idempotent. The empty word is defined to be reduced.

Then $G \operatorname{gp} H$ has as its set of elements the reduced words in G and H [9] with the empty word as identity.

If $w = w_1 \ldots w_n$ is a reduced word, the expression is unique and n is the length of w, denoted by $l(w)$.

For $n \geqq 2$, let $\operatorname{pre}(w) = \{w_1, w_1 w_2, \ldots, w_1 \ldots w_{n-1}\}$; let $\operatorname{pre}(w) = \Box$ if $w \in G \cup H$. (We are considering G and H to be subsets of $G \operatorname{gp} H$ – both 1_G and 1_H are identified with the identity 1 of $G \operatorname{gp} H$).

A finite non-empty subset X of $G \operatorname{gp} H$ is a *filter* [21, p. 12] if

(i) $w \in X$ implies $\operatorname{pre}(w) \subseteq X$,

and

(ii) $X \cap G \neq \Box$ and $X \cap H \neq \Box$ imply $1 \in X$.

(This definition is slightly different from, but equivalent to McAlister's.) For example if w is a non-empty reduced word then $\operatorname{pre}(w) \cup \{w\}$ is a filter; if $l(w) \geqq 2$, so is $\operatorname{pre}(w)$ itself.

Denote the set of filters by $\mathscr{F}_1$. Put $\mathscr{F} = \mathscr{F}_1 \cup \{e, f\}$, where $e = e_G$, $f = e_H$, and define

$$e > X \quad \text{if} \quad X \cap G \neq \Box, \qquad f > X \quad \text{if} \quad X \cap H \neq \Box$$

and

$$X \geqq Y \quad \text{if} \quad X \subseteq Y, \quad X, Y \in \mathscr{F}_1 .$$

Then [21, Lemma 2.5] $\mathscr{F}$ is a semilattice having $\mathscr{F}_1$ as an ideal. If $X \in \mathscr{F}_1$, let $\varepsilon(X) = \prod_{x \in X} xefx^{-1}$. Since $G \operatorname{inv} H$ is generated by G and H, $\varepsilon(X)$ is an element, in fact an idempotent of $G \operatorname{inv} H$. (Again we consider G and H to be subsets of $G \operatorname{inv} H$. Each non-empty reduced word $w_1 \ldots w_n$ can then be considered an element of $G \operatorname{inv} H$ in the obvious way.)

Result 4.2. *Let G and H be groups. Then*

(i) *every idempotent of $G \operatorname{inv} H$ can be expressed uniquely in one of the forms e, f or $\varepsilon(X)$, where X is a filter,*

and

(ii) *$\mathscr{F} \cong E(G \operatorname{inv} H)$, the isomorphism taking e to e, f to f and the filter X to $\varepsilon(X)$.*

Further if $X, Y \in \mathscr{F}_1$, then

(iii) *$\varepsilon(X) \mathscr{D} \varepsilon(Y)$ if and only if $X = wY$ for some reduced word w. In that case $|X| = |Y|$.*

For example if w is a non-empty reduced word in G and H, then $wefw^{-1}$ corresponds to the filter $\operatorname{pre}(w) \cup \{w\}$. (So if $g \in G$, $gefg^{-1}$ corresponds to $\{g\}$ and $(gefg^{-1})(ef)$ to $\{1, g\}$.)

Since $\mathscr{F} \cong E(G \operatorname{inv} H)$ we may write $\mathscr{D}$ again for the equivalence on $\mathscr{F}$ induced by the restriction of $\mathscr{D}$ to the idempotents of $G \operatorname{inv} H$. Then $X \mathscr{D} Y$ if and only if $X = wY$ for some reduced word w.

Now put $S = G \operatorname{inv} H$ and for the remainder of the section let Φ be a lattice isomorphism of $\mathscr{L}(S)$ upon $\mathscr{L}(T)$ for some inverse semigroup T. Since G and H generate S, $G\Phi$ and $H\Phi$ generate T (and $G\Phi$ and $H\Phi$ are groups). If both G and H are trivial, then $G \operatorname{inv} H$ is the semilattice $\{e, f, ef\}$, $e \parallel f$, which by Result 1.1 is determined by its lattice of inverse subsemigroups. From now on G or H, but not both, may be trivial.

From Result 4.2 it is readily seen that $E(J_{ef}) = \{aefa^{-1} : a \in G \cup H\}$ and that the correspondences $g \to gefg^{-1}$, $g \in G$, and $h \to hefh^{-1}$, $h \in H$ are each one-to-one. Moreover $J \leqq J_{ef}$ for all $J \in \mathscr{J}(S)$, $J \neq J_e$, J_f (since if $X \in \mathscr{F}$, then $x \in X$, by (i) of the definition, whence $\varepsilon(X) \leqq \varepsilon(\{x\}) = xefx^{-1} \mathscr{J} ef$). By Lemma 3.1 the weak isomorphism ϕ_E is therefore an isomorphism on $E(S \setminus G \cup H)$. But $e \parallel f$, so $e\phi \parallel f\phi$ and $(ef)\phi = (e\phi)(f\phi)$. Thus $J_{(ef)\phi} < J_{e\phi}, J_{f\phi}$ and so $\phi_{\mathscr{J}}$ is an order-isomorphism of $\mathscr{J}(S)$ upon $\mathscr{J}(T)$. By Proposition 1.12, ϕ_E is therefore an isomorphism. We have proved the first part of the following.

Proposition 4.3. *The map ϕ_E of E_S upon E_T is an isomorphism.*
Therefore (i) *T is E-unitary.*
Further (ii) *T is completely semisimple*
and (iii) *T is fundamental.*

Proof. (i) follows from Theorem 3.4, (ii) from Result 1.2 and (iii) from Theorem 1.15, each applied to Result 4.1.

At this stage we can, using Theorem 3.10, deduce also that $\mathscr{L}(S/\sigma) \cong \mathscr{L}(T/\sigma)$ whence by the results quoted earlier, there is an isomorphism of S/σ upon T/σ inducing the lattice isomorphism of $\mathscr{L}(S/\sigma)$ upon $\mathscr{L}(T/\sigma)$. Hence $T/\sigma \cong G \operatorname{gp} H$. There seems however little progress to be made by pursuing this approach further and we will therefore use alternative techniques to achieve our goal.

We now show that the properties of J_{ef}, given above, are preserved in $J_{(ef)\phi} = (J_{ef})\,\phi_{\mathscr{J}}$. For convenience put $k = (ef)\,\phi = (e\phi)(f\phi)$. Then $E(J_k) = E(J_{ef})\,\phi_E$.

From the description of $G \operatorname{inv} H$ as a "P-semigroup" in [21, Theorem 4.3] it is easily seen that J_{ef} has trivial subgroups. Thus the same is true of J_k. If $a \in G\Phi \cup H\Phi$, then $aka^{-1}\,\mathscr{D}\,k$, for $a^{-1}(aka^{-1})\,a = (a^{-1}a)\,k = k$. If $aka^{-1} = k$, then $ak\,\mathscr{H}\,k$ so $ak = k$ and $a \in \{e\phi, f\phi\}$ since T is E-unitary. Thus if $a, b \in G\Phi$ and $aka^{-1} = bkb^{-1}$, so that $(b^{-1}a)\,k(b^{-1}a)^{-1} = k$, then $a = b$. Hence the map $a \to aka^{-1}$, $a \in G\Phi$ is an injection, taking $e\phi$ to k. There is a similar injection $b \to bkb^{-1}$ of $H\Phi$ into $E(J_k)$, taking $f\phi$ to k.

Let $d \in E(D_k)$, $d \neq k$. Thus $d = bkb^{-1}$ for some $b \in T \setminus E_T$. Since $T = \langle G\Phi, H\Phi\rangle$ we may write $b = b_1 \dots b_n$, where each $b_i \in G\Phi \cup H\Phi$ and alternate b_i's are in different subgroups $G\Phi$ or $H\Phi$. Suppose $n \geqq 2$, and without loss of generality $b_1 \in G\Phi$, $b_2 \in H\Phi$, so that $b_1(e\phi) = b_1$ and $(f\phi)\,b_2 = b_2$. Then $(b_1kb_1^{-1})(bkb^{-1}) = (b_1e\phi\, f\phi\, b_1^{-1})(b_1b_2 \dots b_nkb^{-1}) = b_1e\phi f\phi\, b_2 \dots b_nkb^{-1} = bkb^{-1}$. So $b_1kb_1^{-1} \geqq bkb^{-1} = d$ and $b_1kb_1^{-1}, d \in E(J_k)$ whence by complete semisimplicity of T, $d = b_1kb_1^{-1}$.

Thus $E(J_k) = \{aka^{-1} : a \in G\Phi \cup H\Phi\}$. It is now apparent that the composite map $\theta_G : g \mapsto gefg^{-1} \mapsto g_1kg_1^{-1} \mapsto g_1$, where $g_1kg_1^{-1} = (gefg^{-1})\,\phi_E$, is a bijection of G upon $G\Phi$ taking e to $e\phi$. A bijection $\theta_H : h \mapsto h_1$ of H upon $H\Phi$ is defined similarly.

A technical lemma is now required.

Lemma 4.4. *For all $g \in G$, $\langle g\rangle\,\Phi = \langle g_1\rangle$. Further g and g_1 have the same order.*

Proof. Let $g \in G$, and $(gefg^{-1})\,\phi = g_1kg_1^{-1}$. Since $\langle g_1\rangle$ is a cyclic group, so is $\langle g_1\rangle\,\Phi^{-1}$, generated by a, say, $a \in G$. Consider the inverse subsemigroup $\langle g_1, k\rangle$ of T. Every idempotent of $\langle g_1, k\rangle$ has the form $e\phi$ or $(g_1^{\alpha_1}kg_1^{-\alpha_1}) \dots (g_1^{\alpha_n}kg_1^{-\alpha_n})$ for some integers α_i, $1 \leqq i \leqq n$ $(n \geqq 1)$. As above, each $g_1^{\alpha_i}kg_1^{-\alpha_i} \in E(J_k)$ and no product of distinct idempotents of J_k is again in J_k. Hence

$$E\langle g_1, k\rangle \cap J_k = \{g_1^{\alpha}kg_1^{-\alpha} : \alpha \in \mathbb{Z}\}\,.$$

Similarly $E\langle a, ef\rangle \cap J_{ef} = \{a^{\beta}efa^{-\beta} : \beta \in \mathbb{Z}\}$, in S.

But, recalling that ϕ_E is $\mathscr{J}$-preserving, $(E\langle a, ef\rangle \cap J_{ef})\,\phi_E = E\langle a, ef\rangle\,\phi_E \cap J_k = E\langle a, ef\rangle\,\Phi \cap J_k = E\langle g_1, k\rangle \cap J_k$, so $\{a^{\beta}efa^{-\beta} : \beta \in \mathbb{Z}\}\,\phi_E = \{g_1^{\alpha}kg_1^{-\alpha} : \alpha \in \mathbb{Z}\}$. Since the correspondences $a^{\beta} \mapsto a^{\beta}efa^{-\beta}$ and $g_1^{\alpha} \mapsto g_1^{\alpha}kg_1^{-\alpha}$ are one-to-one, $|\{a^{\beta} : \beta \in \mathbb{Z}\}| = |\{g_1^{\alpha} : \notin \in \mathbb{Z}\}|$, that is a and g_1 have the same order.

By definition $gefg^{-1} = (g_1kg_1^{-1})\,\phi_E^{-1}$ and so $gefg^{-1} = a^mefa^{-m}$, that is $g = a^m$, for some $m \in \mathbb{Z}$, $m \neq 0$.

Therefore $\langle g\rangle \subseteq \langle a\rangle$ and $\langle g\rangle\,\Phi \subseteq \langle a\rangle\,\Phi = \langle g_1\rangle$.

Conversely, $\langle g\rangle\,\Phi = \langle b\rangle$ for some $b \in G\Phi$. By similar arguments $g_1 = b^n$ for some $n \in \mathbb{Z}$, $n \neq 0$ and so $\langle g_1\rangle \subseteq \langle b\rangle = \langle g\rangle\,\Phi$. Hence $\langle g\rangle\,\Phi = \langle g_1\rangle$. Since therefore $\langle g\rangle = \langle a\rangle$, g and g_1 have the same order.

Proposition 4.5. *The maps θ_G and θ_H are isomorphisms of G upon $G\Phi$, H upon $H\Phi$, inducing the lattice isomorphisms $\Phi \mid \mathscr{L}(G)$, $\Phi \mid \mathscr{L}(H)$ respectively.*

Proof. Again denoting $g\theta_G$ by $g_1(g \in G)$ we show first that $g_1^{-1} = (g^{-1})_1$. Clearly we may assume $g \neq e$. Let $g_1^{-1} = u_1$ for some $u \in G$. (So $(uefu^{-1})\,\phi = g_1^{-1}kg_1$; we must show $u = g^{-1}$). Now in T,

$$k(g_1^{-1}kg_1) = g_1^{-1}((g_1kg_1^{-1})\,k)\,g_1 \mathscr{D} (g_1kg_1^{-1})\,k\,,$$

that is $(ef\,.\,uefu^{-1})\,\phi\,\mathscr{D}\,(gefg^{-1}\,.\,ef)\,\phi$. Since ϕ_E and ϕ_E^{-1} are $\mathscr{D}$-preserving, in S we have $ef\,.\,uefu^{-1}\,\mathscr{D}\,gefg^{-1}\,.\,ef$, or in terms of filters $\{1, u\}\,\mathscr{D}\,\{1, g\}$, whence $\{1, u\} = x\{1, g\}$ for some reduced word x, using Result 4.2 and the comments immediately following it.

If $x = u$, then $ug = 1$, that is $u = g^{-1}$, as required. Otherwise $x = 1$ and $u = g$. In that case $g_1^{-1} = g_1$, so that $g_1^2 = 1$. By the lemma g also has order 2, and so $u = g = g^{-1}$ again.

Now let $a, b \in G$. If a or b is idempotent clearly $a_1b_1 = (ab)_1$, so suppose $a, b \in G \setminus \{e\}$, and put $a_1b_1 = v_1$ for some $v \in G$. (So $(vefv^{-1})\,\phi = a_1b_1kb_1^{-1}a_1^{-1}$; we must show $v = ab$.) If $v = e$, then $a_1b_1 = e\phi$ and $a_1 = b_1^{-1} = (b^{-1})_1$, by the above, whence $a = b^{-1}$ and $ab = e = v$.

Assume then, that $v \neq e$. By methods similar to those used above,

$$k(a_1ka_1^{-1})\,(a_1b_1kb_1^{-1}a_1^{-1})\,\mathscr{D}\,(a_1^{-1}ka_1)\,k(b_1kb_1^{-1})\,,$$

that is

$$(ef\,.\,aefa^{-1}\,.\,vefv^{-1})\,\phi\,\mathscr{D}\,(a^{-1}efa\,.\,ef\,.\,bkb^{-1})\,\phi\,,$$

yielding $ef\,.\,aefa^{-1}\,.\,vefv^{-1}\,\mathscr{D}\,a^{-1}efa\,.\,ef\,.\,bkb^{-1}$, in S.

In terms of filters, then,

$$\{1, a, v\} = y\{a^{-1}, 1, b\} \text{ for some reduced word } y\,.$$

If a) $y = 1$, then $\{a, v\} = \{a^{-1}, b\}$. If $v = b$, then $a_1b_1 = v_1 = b_1$, a contradiction since $a \neq 1$. Otherwise $a = b$ and $v = a^{-1}$. Then $a_1^2 = a_1b_1 = v_1 = a_1^{-1}$, that is $a_1^3 = 1$. By the lemma, $a^3 = 1$, and $v = a^{-1} = a^2 = ab$.

If b) $y = a$, then $v = ab$.

If c) $y = v$, then $\{1, a\} = \{va^{-1}, vb\}$. If $va^{-1} = 1$, that is $v = a$, then $a = ab$, a contradiction. Otherwise $va^{-1} = a$ and $vb = 1$. Consider the situation $a = b$ first. Clearly $v = a^2$, that is $a_1^2 = a_1b_1 = v_1 = (a^2)_1$ (and since this is the last case to consider in this situation, $a_1^2 = (a^2)_1$ for all $a \in G$). In the general case, then, $v = a^2$ implies $a_1b_1 = v_1 = (a^2)_1 = a_1^2$, whence $a_1 = b_1$, $a = b$ and $v = ab$ as required.

In every case, therefore, $a_1b_1 = (ab)_1$. Thus θ_G is an isomorphism of G upon $G\Phi$. That θ_H is an isomorphism follows similarly.

Using Lemma 4.4 in conjunction with Lemma 3.3 of [17], θ_G and θ_H induce $\Phi \mid \mathscr{L}(G)$ and $\Phi \mid \mathscr{L}(H)$ respectively.

By the universal properties of S $(= G \operatorname{inv} H)$, there is a unique morphism θ of S upon T extending θ_G and θ_H. For all $a \in G \cup H$, $(aefa^{-1})\,\theta = (a\theta)(ef)(a\theta)^{-1} = a_1 k a_1^{-1} = (aefa^{-1})\,\phi$.

In fact we will prove $\theta \mid E_S = \phi_E$. Let $\alpha = \theta\phi_E^{-1} \mid E_S$. Then $\alpha : E_S \to E_S$ and for all $d \in E_S$, $d\theta = (d\alpha)\,\phi$. Since ϕ_E is an isomorphism of E_S upon E_T and since θ is a morphism, α is a well-defined surjective endomorphism of E_S. Moreover α is $\mathscr{D}$-preserving (since ϕ_E^{-1} is).

Proposition 4.6. *The endomorphism α of E_S is in fact the identity. Thus $\theta \mid E_S = \phi_E$.*

Proof. Clearly α induces a surjective endomorphism, which we may again call α, of $\mathscr{F}_1$, the semilattice of filters. We show $X\alpha = X$ for every filter X, by induction on $|X|$.

If $|X| = 1$, that is $X = \{a\}$ for some $a \in G \cup H$, then as above $(aefa^{-1})\,\theta = (aefa^{-1})\,\phi$, or $(aefa^{-1})\,\alpha = aefa^{-1}$. In $\mathscr{F}_1$, then, $\{a\}\,\alpha = \{a\}$.

Suppose now that $n \geqq 2$ and that $X\alpha = X$ for all filters X with less than n elements. Let X be a filter with n elements.

Since for all words $w \in X$, $\operatorname{pre}(w) \subseteq X$, we have $X = \bigcup_{w \in X} \operatorname{pre}(w) \cup \{w\}$, where each set $\operatorname{pre}(w) \cup \{w\} \in \mathscr{F}_1$. If $|\operatorname{pre}(w) \cup \{w\}| < n$ for every $w \in X$ then by the hypothesis $(\operatorname{pre}(w) \cup \{w\})\,\alpha = \operatorname{pre}(w) \cup \{w\}$ for each $w \in X$. Thus $X\alpha = X$, since α is a morphism. So we may assume $X = \operatorname{pre}(w) \cup \{w\}$ for some reduced word w of length n. Let $w = w_1 \dots w_n$: thus $X = \{w_1, w_1w_2, \dots, w_1 \dots w_n\}$.

Now $w_1^{-1}X = \{1, w_2, \dots, w_2 \dots w_n\}$, again a filter, and $X \mathscr{D} w_1^{-1}X$, so $X\alpha \mathscr{D} (w_1^{-1}X)\,\alpha$. But by hypothesis again, $(w_1^{-1}X)\,\alpha = w_1^{-1}X$, since $w_1^{-1}X = \{1\} \cup \{w_2, \dots, w_2 \dots w_n\}$, so $X\alpha \mathscr{D} w_1^{-1}X \mathscr{D} X$. Thus $|X\alpha| = |X|$. Moreover since $\operatorname{pre}(w)$ is a filter contained in X and $|\operatorname{pre}(w)| = n - 1$, then $\operatorname{pre}(w) = \operatorname{pre}(w)\,\alpha \subseteq X\alpha$. Thus $X\alpha = \operatorname{pre}(w) \cup \{y\}$ for some reduced word y. It remains to show $y = w$.

By similar arguments we have $X \cup \{1\} \mathscr{D} w_1^{-1}(X \cup \{1\}) = w_1^{-1}X \cup \{w_1^{-1}\}$, again a filter, and so $X\alpha \cup \{1\} \mathscr{D} (w_1^{-1})\,\alpha \cup \{w_1^{-1}\}\,\alpha = w_1^{-1}X \cup \{w_1^{-1}\}$. (Note that since $|X\alpha \cup \{1\}| = |X \cup \{1\}|$ nad $1 \notin X$, then $1 \notin X\alpha$ also.)

For some reduced word z, therefore,

$$X\alpha \cup \{1\} = z\{1, w_1, \dots, w_1 \dots w_j, \dots, w_1 \dots w_n\}\,.$$

If $z = 1$, then $X\alpha = X$.

Otherwise $1 = zw_1 \dots w_j$ for some j, $1 \leqq j \leqq n$. In that case $z = w_j^{-1} \dots w_1^{-1}$.

If $j < n$, $X\alpha = \{w_j^{-1} \dots w_1^{-1}, \dots, w_j^{-1}, w_{j+1}, \dots, w_{j+1} \dots w_n\}$. However, this is impossible, for w_j and w_{j+1} lie in different subgroups G and H, so that $X\alpha \cap G \neq \square$ and $X\alpha \cap H \neq \square$ yet $1 \notin X\alpha$, contradicting the defining property (ii) of filters.

If $j = n$, that is $z = w_n^{-1} \dots w_1^{-1}$, then

$$\{w_1, \dots, w_1 \dots w_{n-1}, y\} = X\alpha = \{w_n^{-1} \dots w_1^{-1}, w_n^{-1} \dots w_2^{-1}, \dots, w_n^{-1}\}\,.$$

By comparing lengths of words $w_1 = w_n^{-1}, w_1 \dots w_{n-1} = w_n^{-1} \dots w_2^{-1}$ and $y = (w_n^{-1} \dots w_2^{-1}) w_1^{-1} = (w_1 \dots w_{n-1}) w_n = w$.

Therefore $X\alpha = X$ and the result follows by induction.

Since ϕ_E is an isomorphism, θ is therefore idempotent-separating.

Hence if neither G nor H is trivial, θ is an isomorphism of $G \operatorname{inv} H$ upon T, for by Result 4.1, $G \operatorname{inv} H$ is in this case fundamental and so admits no non-trivial idempotent-separating congruences.

Suppose H (or similarly G) is trivial. Then since $S(=G \operatorname{inv} \{f\}$ here) is generated by G and f, every element s of S can be written in the form dg for some $d \in E_S$ and $g \in G$. If $s\theta \in E_T$, that is $(d\theta)(g\theta) \in E_T$, then $g\theta \in E_T$, since T is E-unitary. But $\theta / G = \theta_G$, an isomorphism, so $g = e$ and $s \in E_S$. Thus θ is idempotent – determined, that is $E_T\theta^{-1} = E_S$. But then $\theta\theta^{-1} \subseteq \sigma$, whence $\theta\theta^{-1} \subseteq \sigma \cap \mathscr{H} \subseteq \sigma \cap \mathscr{R}$ (since θ is idempotent-separating). But S in E-unitary and so $\theta\theta^{-1} = \iota$, that is θ is an isomorphism. (Note – in this case if the non-trivial subgroup is infinite $G \operatorname{inv} H$ is in fact again fundamental. Otherwise, $G \operatorname{inv} H$ has a kernel isomorphic with $G \operatorname{gp} H$.)

This completes the proof of our main theorem:

Theorem 4.7. *If G and H are any groups, then $G \operatorname{inv} H$ is determined up to isomorphism by its lattice of inverse subsemigroups.*

The author does not know whether $G \operatorname{inv} H$ need in general be *strongly* determined by $\mathscr{L}(G \operatorname{inv} H)$. If, however, G and H are *torsion-free* this is the case, as we now show.

Corollary 4.8. *If G and H are torsion-free groups, then every lattice isomorphism of $G \operatorname{inv} H$ upon an inverse semigroup T is induced by a unique isomorphism of $G \operatorname{inv} H$ upon T.*

Proof. Let $\Phi : \mathscr{L}(G \operatorname{inv} H) \to \mathscr{L}(T)$ be an isomorphism. When G and H are torsion-free, the subgroups G and H of $S(= G \operatorname{inv} H)$ are the only non-trivial $\mathscr{H}$-classes [21, Corollary 4.6], whence $G\Phi$ and $H\Phi$ are the only non-trivial $\mathscr{H}$-classes of T. The partial one-to-one map ϕ defined in Corollary 1.7 is therefore a bijection, of $S \setminus (G \cup H)$ upon $T \setminus (G\Phi \cup H\Phi)$. But $\theta \mid E_S = \phi_E = \phi \mid E$, so for all $s \in G \cup H$. $(s\theta)(s\theta)^{-1} = (ss^{-1})\,\theta = (ss^{-1})\,\phi = (s\phi)(s\phi)^{-1}$. Thus $s\theta \mathrel{\mathscr{R}} s\phi$ and similary $s\theta \mathrel{\mathscr{L}} s\phi$ Since $H_{s\theta}$ is trivial, $s\theta = s\phi$. By Proposition 1.6, therefore $\langle s \rangle\, \Phi = s\theta$ for all $s \in S$, $s \notin G \cup H$. But from Proposition 4.5 the same is true if $s \in G$ or $s \in H$. By Lemma 3.3 of [17], the isomorphism induces Φ.

If ψ is any isomorphism inducing Φ, let $g \in G$. Then $(gefg^{-1})\,\psi = (gefg^{-1})\,\phi = (gefg^{-1})\,\theta$, by Corollary 1.7, that is $(g\psi)\,k(g\psi)^{-1} = (g\theta)\,k(g\theta)^{-1}$ (where $k = (ef)\,\phi$), whence by the arguments preceding Lemma 4.4, $g\psi = g\theta$. Hence $\psi \mid G = \theta_G$ and similarly $\psi \mid H = \theta_H$. But θ extends θ_G and θ_H uniquely, so $\theta = \psi$.

The arguments of the last paragraph hold in general so *if Φ is induced by an isomorphism it is induced by a unique one.*

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