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FUNCTIONAL SEPARATION OF INDUCTIVE LIMITS AND REPRESENTATION OF PRESHEAVES BY SECTIONS PART FOUR: REPRESENTATION OF PRESHEAVES BY SECTIONS

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INTRODUCTION

In this paper some representation theorems for certain presheaves are proven. They state that in the covering space of the presheaf in question there is a closure such that the set of all continuous sections over any open set, endowed with the topology of pointwise convergence, is precisely the set of those that canonically correspond to the sets of the presheaf, and that the natural maps of the spaces of the presheaf onto the spaces of the sections are homomorphisms. In the final section we find out when there is even a topology with the above mentioned properties. That gives us a representation theorem in terms of topological spaces.

This is the fourth and last part of the paper "Functional Separation of Inductive Limits and Representation of Presheaves by Sections". The basic definitions and notation were introduced at the beginning of Part One which together with the other foregoing parts is very often reffered to. If we refer, say, to 3.2.7 or 0.5, we mean Remark 3.2.7 of the second section of Part Three or Definition 0.5 at the beginning of Part One, respectively.

4. REPRESENTATION OF PRESHEAVES BY SECTIONS

1. PRELIMINARY LEMMAS

4.1.1. Notation. A. The set of all open nonempty subsets of a topological space X is denoted by $\mathscr{B}(X)$. If $x \in X$, we put $\mathscr{B}x = \{U \in \mathscr{B}(X) \mid x \in U\}$. Throughout this chapter the inverse inclusion order in $\mathscr{B}(X)$ is denoted by $\leqq$ (given U, V open, then

$U \leqq V$ iff $V \subset U$). If $\mathscr{X} = (X, t)$ is a uniform (proximal, ...) space, then $\operatorname{cl} \mathscr{X} = (X, \operatorname{cl} t)$, where $\operatorname{cl} t$ is the closure in X generated by t.

B. A presheaf over X from a category $\mathfrak{K}$ is an inductive family $\mathscr{S} = \{S_U | \varrho_{UV} | \cdot \langle \mathscr{B}(X) \leqq \rangle\}$ from $\mathfrak{K}$ over $\langle \mathscr{B}(X) \leqq \rangle$ (see 0.2). We denote it by $\mathscr{S} = \{S_U | \varrho_{UV} | X\}$.

C. Given an inductive category $\mathfrak{K}$ (see 0.5) and a presheaf $\mathscr{S} = \{S_U | \varrho_{UV} | X\}$ over X from $\mathfrak{K}$, $x \in X$, then we put $\mathscr{S}_x = \{S_U | \varrho_{UV} | \langle \mathscr{B}x \leqq \rangle\}$. As $\langle \mathscr{B}x \leqq \rangle$ is right directed, $\mathscr{S}_x$ is a presheaf from $\mathfrak{K}$ in the sense of 0.2. Thus there is $\mathscr{I}_x = \varinjlim \mathscr{S}_x$, and the $\mathfrak{K}$-object $\mathscr{I}_x$ is called a *stalk* over x. By 0.4, for each $U \in \mathscr{B}x$ there is a canonical $\mathfrak{K}$-morphism $\xi_{Ux} : S_U \to \mathscr{I}_x$.

D. Let an i.c. category $\mathfrak{L}$ with the union property (see 0.19) and a presheaf $\mathscr{S} = \{\mathscr{X}_U | \varrho_{UV} | X\}$ from $\mathfrak{L}$ be given. For every $x \in X$ we have the stalk $\mathscr{I}_x = \varinjlim \mathscr{S}_x$, and there is an $\mathfrak{L}$-object $\mathscr{P} = \bigcup\{\mathscr{I}_x \mid x \in X\}$ (see 0.19), which is called a *covering space* of $\mathscr{S}$. If $U \subset X$ is open then the section over U in $\mathscr{P}$ is map $r : U \to \mathscr{P}$ such that $r(x) \in \mathscr{I}_x$ for all $x \in U$. Recall that $\mathscr{P}$ is an object from $\mathfrak{L}$, thus $\mathscr{P}$ is also from CLOS or from SEM or from PROX (see 0.10). Regarding $\mathscr{P}$ as an element of that of these to which $\mathscr{P}$ belongs, we have $\mathscr{P} = (P, t)$, where $P = |\mathscr{P}|$ is a set and t is a closure or semiuniformity or proximity, respectively (see 0.9) .Thus r is a map of U into the set P.

E. If $U \in \mathscr{B}(X)$, $a \in |\mathscr{X}_U| = X_U$, we have $\xi_{Ux}(a) \in |\mathscr{I}_x| = I_x$ for all $x \in U$. Setting $\tilde{a}(x) = \xi_{Ux}(a)$ for $x \in U$, we get a section $\tilde{a}$ over U. Putting $A_U = \{\tilde{a} \mid a \in X_U\}$, $p_U(a) = \tilde{a}$, we get a set A_U of sections corresponding canonically to X_U. The map $p_U : X_U \to A_U$ is onto. It is $1-1$ iff the following condition is fulfilled:

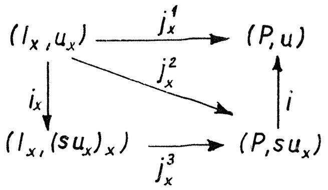
COND. *If $a, b \in X_U$ so that there is an open cover $\mathscr{V}$ of U with $\varrho_{UV}(a) = \varrho_{UV}(b$: for all $V \in \mathscr{V}$, then $a = b$.*

(The proof is straight forward.)

If $x \in U$, $\tilde{a} \in A_U$, we put $\eta_{Ux}(\tilde{a}) = \xi_{Ux}(a)$, where $\tilde{a} = p_U(a)$. We get a map η_{Ux}) $: A_U \to I_x$. Clearly, if p_U is $1-1$, we have $\eta_{Ux} = \xi_{Ux} \circ p_U^{-1} : A_U \to \mathscr{I}_x$.

F. Given $\mathscr{S} = \{\mathscr{X}_U | \varrho_{UV} | X\}$ from $\mathfrak{L}$, where $\mathfrak{L}$ is one of the categories mentioned in 0.5, and the covering space $\mathscr{P} = \bigcup\{\mathscr{I}_x \mid x \in X\}$ of $\mathscr{S}$, it is known that $P = |\mathscr{P}| = \bigcup\{I_x = |\mathscr{I}_x| \mid x \in X\}$. If in every I_x we have a closure (topology, ...) s_x, then by ss_x we denote the closure (topology, ...) inductively defined in P by the canonical embeddings $j_x : (I_x, s_x) \to P$. Further, if s_x^* is the closure (topology, ...) of $\mathscr{I}_x = \varinjlim \mathscr{S}_x$, then $\mathscr{P} = (P, ss_x^*)$ (see 0.19). If u is a closure (topology, ...) in P – for example, if $\mathfrak{L} = \text{UNIF}$, then u is a uniformity in P –, $x \in X$, then we denote by u_x the closure (topology, ...) in I_x projectively defined by the canonical embeddings $j_x : I_x \to (P, u)$. The following statement holds: *The identical map $i : (P, su_x) \to (P, u)$ is an $\mathfrak{L}$ – morphism (for example, if u is a uniformity, then i is uniformly continuous) and $(su_x)_x = u_x$.*

Proof. Look at the commutative diagram

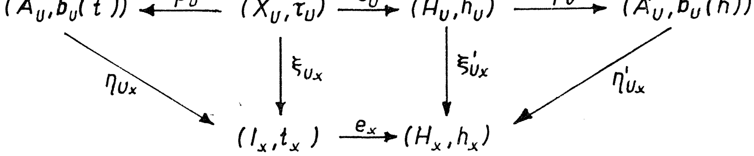


Here $i(i_x, i_x^{-1})$ is continuous iff the same is true for $ij_x^2 = j_x^1(j_x^3 i_x = j_x^2, j_x^1 i_x^{-1} = ij_x^3)$. But the latter holds by the definition of $u_x(su_x, (su_x)_x$ and the continuity of i).

If u is a closure (topology, ...) in P, $U \in \mathscr{B}(X)$, then by $b_U(u)$ and $\tau_U(u)$ we denote the closure (topology, ...) projectively defined in A_U and in X_U by the canonical maps $\{\eta_{Ux} : A_U \to (I_x, u_x) \mid x \in U\}$ and $\{\xi_{Ux} : X_U \to (I_x, u_x) \mid x \in U\}$, respectively. If t is a closure in P and $U \in \mathscr{B}(X)$, then $\Gamma(U, t)$ is the set of all continuous sections $r : U \to (P, t)$ (see 4.1.1D).

4.1.2. Notation. Let $\mathscr{T} = \{\mathscr{T}_U | \varrho_{UV} | X\}$ be from an i.c. category $\mathfrak{L}$, let $\mathscr{H} = \{(H_U, h_U) | r_{UV} | X\}$ be a hull of $\mathscr{T}$ from TOP (see 2.1.2B and 0.11). We put $\mathscr{S} = \operatorname{cl} \mathscr{T} = \{\mathscr{X}_U = (X_U, \tau_U) | \varrho_{UV} | X\}$ (see 4.1.1A, 0.9). Let $\mathscr{I}_x = (I_x, t_x^*) = \varinjlim \mathscr{S}_x$, $\mathscr{R}_x = (H_x, h_x^*) = \varinjlim \mathscr{H}_x$ be the stalks of $\mathscr{P}_{\mathscr{S}} = \bigcup\{\mathscr{I}_x \mid x \in X\} = (P_{\mathscr{S}}, st_x^*)$, $P_{\mathscr{H}} = \bigcup\{\mathscr{R}_x \mid x \in X\} = (\mathscr{P}_{\mathscr{H}}, sh_x^*)$, respectively, where $P_{\mathscr{S}} = \bigcup\{I_x, x \in X\}$, $P_{\mathscr{H}} = \bigcup\{H_x \mid x \in X\}$ (see 4.1.1C, E). Recall that $\varinjlim \mathscr{T}_x \in \mathfrak{L}$ while $\mathscr{I}_x, \mathscr{R}_x \in$ CLOS. Given $U \in \mathscr{B}(X)$, $x \in U$, let A_U and A'_U be the sets of the sections over U in $\mathscr{P}_{\mathscr{S}}$ and in $\mathscr{P}_{\mathscr{H}}$, that canonically correspond to X_U and H_U, respectively. Let $p_U : X_U \to A_U$, $p'_U : H_U \to A'_U$, $\xi_{Ux} : X_U \to I_x$, $\xi'_{Ux} : H_U \to H_x$ be the canonical maps (see 4.1.1C, E). As $\mathscr{H}$ is a hull of $\mathscr{S}$ (see 2.1.2B) so there is a $1-1$ continuous map $e_x : \mathscr{I}_x \to \mathscr{R}_x$ for every $x \in X$. If t and h are closures in $P_{\mathscr{S}}$ and in $P_{\mathscr{H}}$, then $b_U(t)$ and $b'_U(h)$ denote the closures projectively defined in A_U and A'_U by the maps $\eta_{Ux} : A_U \to \mathscr{I}_x$ and $\eta'_{Ux} : A'_U \to \mathscr{R}_x$, respectively (see 1.4.1F). The next commutative diagram shows the situation.

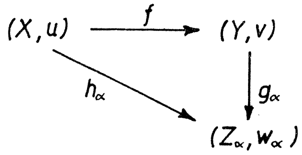
(4.1.3)



The maps p_U, p'_U are onto (see 0.5). Further, p_U and p'_U are $1-1$ if $\mathscr{T}$ and $\mathscr{H}$, respectively, fulfil the condition COND from 4.1.1E; e_U are continuous, open and $1-1$ maps into (H_U, h_U), hence homeomorphisms of $(X_U, m\tau_U)$ into (H_U, h_U) ($m\tau_U$ is the topological modification of τ_U — see 2.1.2B, 0.9), η_{Ux}, η'_{Ux} are continuous; e_x is $1-1$ for all $x \in U$.

The following lemma will be useful:

4.1.4. Lemma. A. *Let A be a set and for every $\alpha \in A$ let us have a commutative diagram of closure spaces and maps such that v is projectively defined by $\{g_\alpha : Y \to \to (Z_\alpha, w_\alpha) \mid \alpha \in A\}$. Then f is continuous iff all the h_α are.*



B. *Let $U \subset X$ be open. Suppose ξ'_{Ux} in the diagram* 4.1.3 *are continuous for all $x \in U$. If the closure t_x in I_x coincides with that projectively defined by $e_x : I_x \to \to (H_x, h_x)$, then ξ_{Ux} in* 4.1.3 *are continuous for all $x \in U$. Thus, if for some $x \in X$, the closure h_x is coarser than h^*_x (i.e. ξ'_{Ux} in* 4.1.3 *is continuous for all $U \in \mathscr{B}(X)$) and the closure t_x is projectively defined by $e_x : I_x \to (H_x, h_x)$, then ξ_{Ux} from* 4.1.3 *is continuous for all $U \in \mathscr{B}x$.*

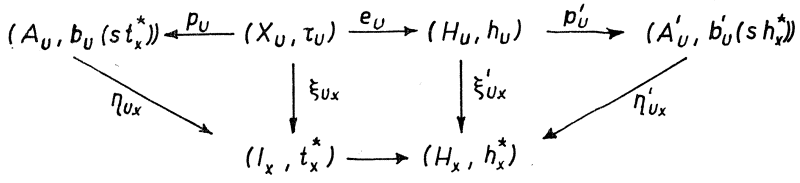
Proof. A: f is continuous iff so is $g_\alpha f = h_\alpha$ for all $\alpha \in A$. But h_α is continuous for such α. B: The continuity of $\xi'_{Ux} e_U = e_x \xi_{Ux}$ yields that of ξ_{Ux}.

4.1.5. Lemma. *With the same symbols as in* 4.1.2, *let us consider the diagram* 4.1.3 *for $U \in \mathscr{B}(X)$, $x \in U$.*

A. *If $a, b \in X_U$, then $p_U(a) = p_U(b)$ iff $p'_U\, e_U(a) = p'_U\, e_U(b)$. Thus p_U is $1-1$ if p'_U is (therefore if* COND *from* 4.1.1E *holds for $\mathscr{H}$, then p_U are $1-1$).*

B. *Assume that ξ_{Ux}, ξ'_{Ux}, e_x in* 4.1.3 *are continuous for all $x \in U$ (which, by* 4.1.4B, *holds if, for all $x \in U$, h_x is coarser than h^*_x and t_x is projectively defined by $e_x : I_x \to (H_x, h_x)$). Then the maps p_U, p'_U in* 4.1.3 *are continuous.*

C. *If e_U is a homeomorphism into (H_U, h_U) (which holds if (X_U, τ_U) is topological – see* 2.1.2B) *together with p'_U, then so is p_U (recall that e_U maps X_U into H_U while p'_U maps H_U onto A'_U). Especially, setting $t = st^*_x$, $h = sh^*_x$ (see* 4.1.1E), *we get from* 4.1.3



where all the maps are continuous for all $U \in \mathscr{B}(X)$, $x \in U$.

Proof. If $a, b \in X_U$, $p_U(a) = p_U(b)$, then $\xi_{Ux}(a) = \xi_{Ux}(b)$ for all $x \in U$. Thus $\xi'_{Ux}\, e_U(a) = e_x\, \xi_{Ux}(a) = e_x\, \xi_{Ux}(b) = \xi'_{Ux}\, e_U(b)$ for all $x \in U$, hence $p'_U\, e_U(a) = = p'_U\, e_U(b)$. Conversely, if $p'_U\, e_U(a) = p'_U\, e_U(b)$ then $e_x\, \xi_{Ux}(a) = \xi'_{Ux}\, e_U(a) =$

$= \xi'_{Ux}\, e_U(b) = e_x\, \xi_{Ux}(b)$ for all $x \in U$, so $\xi_{Ux}(a) = \xi_{Ux}(b)$ for all $x \in U$ which shows that $p_U(a) = p_U(b)$.

If h_x is coarser than h_x^* and t_x is projectively defined by $e_x : I_x \to (H_x, h_x)$, then clearly $\xi'_{Ux} : (H_U, h_U) \to (H_x, h_x)$ and $e_x : (I_x, t_x) \to (H_x, h_x)$ are continuous while $\xi_{Ux} : (X_U, \tau_U) \to (I_x, t_x)$ is continuous by 4.1.4B.

The closure $b_U(st_x)$ is projectively defined by $\{\eta_{Ux} : A_U \to (I_x, t_x) \mid x \in U\}$ (see 4.1.2B), so p_U is continuous by 4.1.4A. The same argument works for p'_U.

If p'_U and e_U are homeomorphisms, then p_U^{-1} exists. Further, p_U^{-1} is continuous iff so is $k_U = p'_U \circ e_U \circ p_U^{-1}$, for e_U is a homeomorphism into (H_U, h_U) (see 0.15). As e_x is continuous and $b'_U(sh_x)$ is projectively defined by $\{\eta'_{Ux} : A'_U \to (H_x, h_x) \mid$ $\mid x \in U\}$, the continuity of k_U follows from 4.1.4A. The proof is thereby complete.

4.1.6. Remark. If (H_U, h_U) from 4.1.3 is a compact topological space and if $(A'_U, b'_U(h))$ is a Hausdorff topological space, then p'_U is a homeomorphism if it is $1-1$ and continuous. Clearly, $(A'_U, b'_U(h))$ is Hausdorff and topological if so is (H_x, h_x) for all $x \in U$. It can happen that (H_x, h_x^*) from 4.1.5 is not topological or not Hausdorff, but that there is a Hausdorff topology $h_x^\#$ in H_x coarser than h_x^*. Then we can projectively define the topology $b'_U(sh_x^\#)$ in A'_U by the maps $\{\eta'_{Ux} : A'_U \to (H_x, h_x^\#) \mid$ $\mid x \in U\}$, getting a Hausdorff topological space $(A'_U, b'_U(sh_x^\#))$. So we get

4.1.7. Corollary. *The same symbols as in* 4.1.2 *are used. For every $x \in X$ let $h_x^\#$ be a Hausdorff topology in H_x coarser than h_x^* (such an $h_x^\#$ exists provided (H_x, h_x^*) is f.s. – see* 1.1.2*). Let t be a closure in P such that for any $U \in \mathscr{B}(X)$, $x \in U$, the maps $\xi_{Ux} : (X_U, \tau_U) \to (I_x, t_x)$ and $e_x : (I_x, t_x) \to (H_x, h_x^\#)$ are continuous (by* 4.1.4B, *this holds provided $t = st_x^*$ or $t = st_x^\#$, where $t_x^\#$ is projectively defined in I_x by $e_x : I_x \to (H_x, h_x^\#)$). Let us consider the diagram*

$$\begin{array}{ccccccc} (A_U, b_U(t)) & \xleftarrow{\;p_U\;} & (X_U, \tau_U) & \xrightarrow{\;e_U\;} & (H_U, h_U) & \xrightarrow{\;p'_U\;} & (A'_U, b'_U(sh_x^\#)) \\ & \searrow^{\eta_{Ux}} & \downarrow{\scriptstyle \xi_{Ux}} & & \downarrow{\scriptstyle \xi'_{Ux}} & \swarrow_{\eta'_{Ux}} & \\ & & (I_x, t_x) & \xrightarrow{\;e_x\;} & (H_x, h_x^\#) & & \end{array} \tag{4.1.8}$$

(here $b_U(t) = b_U(st_x)$ for $(st_x)_x = t_x$ – see 4.1.1F*). Every map here is continuous and $(A'_U, b'_U(sh_x^\#))$ is topological and Hausdorff. If p'_U is a homeomorphism (which holds provided (H_U, h_U) is compact and $\mathscr{H}$ fulfils* COND *from* 4.1.1E *– then p'_U is $1-1$) together with e_U (which holds provided (X_U, τ_U) is topological – recall that e_U maps X_U into H_U), then so is p_U as well. Especially, $p_U : (X_U, \tau_U) \to$ $\to (A_U, b_U(st_x^\#))$ is a homeomorphism if (H_U, h_U) is compact, (X_U, τ_U) topological and* COND *holds for $\mathscr{H}$.*

Proof follows directly from 4.1.5.

4.1.9. Remark. Let $\mathscr{S} = \{(X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ be from UNIF (see 0.5), let $\mathscr{H} = \{(H_U, h_U) \,|r_{UV}|\, X\}$ be a compact hull of $\mathscr{S}$ so that $\mathscr{H} \subset$ TOP (see 2.1.2B, C, D), As (H_U, h_U) are compact, there is a unique uniformity n_U in H_U so that $\mathrm{cl}\, n_U = h_U$ (see 0.9) and $\mathscr{H}' = \{(H_U, n_U) \,|r_{UV}|\, X\} \subset$ UNIF. We put $(H_x, n_x^*) = \varinjlim \mathscr{H}'_x$, $(I_x, s_x^*) = \varinjlim \mathscr{S}_x$ for all $x \in X$ (n_x^* is a uniformity). By 4.1.1E, we can take the semi-iunformities ss_x^*. sn_x^* in $P_{\mathscr{S}} = \bigcup\{I_x \mid x \in X\}$ and $P_{\mathscr{H}} = \bigcup\{H_x, x \in X\}$, respectively. Then the uniformities $b_U(ss_x^*)$, $b'_U(sn_x^*)$ can be made as in 4.1.2E. We get

(D)
$$\begin{array}{ccccccc}
(A_U, b_U(ss_x^*)) & \xleftarrow{p_U} & (X_U, \tau_U) & \xrightarrow{e_U} & (H_U, n_U) & \xrightarrow{p'_U} & (A'_U, b'_U(sn_x^*)) \\
 & \searrow^{\eta_{Ux}} & \downarrow{\scriptstyle \xi_{Ux}} & & \downarrow{\scriptstyle \xi'_{Ux}} & \swarrow_{\eta'_{Ux}} & \\
 & & (I_x, s_x^*) & \xrightarrow{e_x} & (H_x, n_x^*) & &
\end{array}$$

But we cannot get the statements of 4.1.5 or 4.1.7 in terms of UNIF unless e_U and p'_U are uniform embeddings (see 0.15). For example, if the complete hulls (H_U, c_U) of (X_U, τ_U) are compact (i.e. $(H_U, \mathrm{cl}\, c_U)$ are compact; c_U is a uniformity), we can set $n_U = c_U$ (in this case $e_U : (X_0, \tau_U) \to (H_U, n_U)$ are uniform embeddings). If $\mathscr{R}_x = (H_x, n_x^*)$ is f.s. by $U(\mathscr{R}_x \to R)$ then there is a separated uniformity $n_x^\#$ in H_x. Replacing n_x^*, $b'_U(sn_x^*)$ in (D) by $n_x^\#$, $b'_U(sn_x^\#)$, we get that $p_U : (X_U, n_U) \to (A_U, b_U(sn_x^\#))$ is a uniform embedding if so is $p'_U : (H_U, n_U) \to (A'_U, b'_U(sn_x^\#))$ (the both maps are onto).

We have been dealing with the question whether the map p_U is a homeomorphism. If this should be true, then p_U must be $1-1$, which holds if so is p'_U. The map p'_U is $1-1$ iff $\mathscr{H}$ fulfils COND from 4.1.1E. If this is not the case then we cannot use the same tools as above. Nevertheless, we still can deal with the question whether the identity $i_U : (X_U, \tau_U) \to (X_U, \tau_U(t))$, where $\tau_U(t)$ is defined in 4.1.1F, is a homeomorphism. We do it in the next remark.

4.1.10. Remark. Let a presheaf $\mathscr{S} = \{(X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ from CLOS and its hull $\mathscr{H} = \{(H_U, h_U) \,|r_{UV}|\, X\}$ from TOP be given. As usual, let $\mathscr{I}_x = (I_x, t_x^*) = \varinjlim \mathscr{S}_x$, $\mathscr{R}_x = (H_x, h_x^*) = \varinjlim \mathscr{H}_x$, $\mathscr{P}_{\mathscr{S}} = (P_{\mathscr{S}}, st_x^*)$, $\mathscr{P}_{\mathscr{H}} = (P_{\mathscr{H}}, sh_x^*)$, A_U, A'_U be the stalks, the covering spaces and the sections in the covering spaces of $\mathscr{S}$ and $\mathscr{H}$, respectively. If $t(h)$ is a closure in $P_{\mathscr{S}}(P_{\mathscr{H}})$, then we can projectively define the closure $\tau_U(t)$ $(h_U(h))$ in $A_U(A'_U)$ by the maps $\{\xi_{Ux} : X_U \to (I_x, t_x) \mid x \in U\}$ $(\{\xi'_{Ux} : H_U \to (H_x, h_x) \mid x \in U\})$ – see 4.1.1E – as the following commutative diagram shows (i_U, i'_U are identities):

(4.1.11)
$$\begin{array}{ccccccc}
(X_U, \tau_U(t)) & \xleftarrow{i_U} & (X_U, \tau_U) & \xrightarrow{e_U} & (H_U, h_U) & \xrightarrow{i'_U} & (H_U, h_U(h)) \\
 & \searrow^{\xi_{Ux}} & \downarrow{\scriptstyle \xi_{Ux}} & & \downarrow{\scriptstyle \xi'_{Ux}} & \swarrow_{\xi'_{Ux}} & \\
 & & (I_x, t_x) & \xrightarrow{e_x} & (H_x, h_x) & &
\end{array}$$

Assume that e_x, ξ_{Ux}, ξ'_{Ux} are continuous for all $x \in U$ (by 4.1.1,B this holds if ξ'_{Ux} is continuous – in particular, if every h_x is coarser than h^*_x – and if t_x is projectively defined by $e_x : I_x \to (H_x, h_x)$). Then clearly i_U, i'_U are continuous by the definition of $\tau_U(t)$, $h_U(h)$. If e_U is a homeomorphism (this holds if (X_U, τ_U) is topological – recall that e_U maps X_U into H_U) together with i'_U (this holds in particular if (H_U, h_U) is compact and $(H_U, h_U(h))$ Hausdorff), then so is i_U. Especially, if for every $x \in U$ there is a Hausdorff topology $h^\#_x$ in H_x coarser than h^*_x (this holds if (H_x, h^*_x) is f.s.) and if $t^\#_x$ is projectively defined in I_x by $e_x : I_x \to (H_x, h^\#_x)$, then $i_U : (X_U, \tau_U) \to$ $\to (X_U, \tau_U(st^\#_x))$ is a homeomorphism if (H_U, h_U) is compact and (X_U, τ_U) topological.

Proof. The continuity of i_U^{-1} can be proved as the continuity of p_U^{-1} in 4.1.5. If (X_U, τ_U) is topological then e_U is a homeomorphism (see 2.1.2B). If (H_U, h_U) is compact, then $i'_U : (H_U, h_U) \to (H_U, h_U(sh^\#_x))$ is a homeomorphism, from which our statement easily follows.

4.1.12. Lemma. *If i'_U in* 4.1.11 *is continuous or open (i.e., $i'_U(M)$ is $h_U(h)$-open if M is h_U-open), then $p'_U : (H_U, h_U) \to (A'_U, b'_U(h))$ is continuous or open, respectively (we do not assume that p'_U is* $1-1$). *Thus if i'_U is a homeomorphism, then $(A'_U, b'_U(h))$ is topological and if p'_U is* $1-1$ *then it is a homeomorphism.*

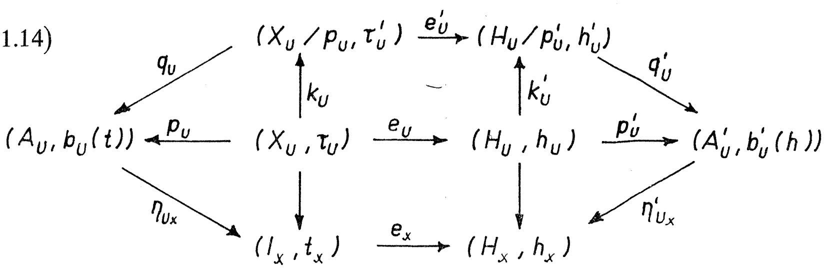
Proof. Openness: Given $a \in H_U$, a finite set $F = \{x_1, \ldots, x_n\} \subset U$ and some h_{x_i} – nbds N_i of $\xi_{Ux_i}(a)$ in H_{x_i}, $i = 1, \ldots, n$, then put $N_F = \bigcap\{\xi'^{-1}_{Ux_i} N_i \mid i = 1, \ldots, n\}$. Then N_F and $M_F = \bigcap\{\eta'^{-1}_{Ux_i} N_i \mid i = 1, \ldots, n\}$ is $h_U(h)$ – nbd and $b'_U(h)$ – nbd of a, respectively, and $p'_U N_F \subset M_F$. If $q \in M_F$ then there is $a \in H_U$ with $p'_U(a) = q$. As $\eta'_{Ux_i} q \in N_i$, $i = 1, \ldots, n$, we get $\xi'_{Ux_i}(a) = \eta'_{Ux_i} p'_U(a) = \eta'_{Ux_i} q \in N_{x_i}$ for $i =$ $= 1, \ldots, n$, so $p'_U N_F = M_F$. Let N be a τ_U – open set. Since $i'_U(N)$ is open, there is a family $\mathscr{F}$ of finite subsets $F \subset U$ such that $i'_U(N) = \bigcup\{N_F \mid F \in \mathscr{F}\}$, where N_F are the sets described above. Then $p'_U(N) = \bigcup\{p'_U N_F = M_F \mid F \in \mathscr{F}\}$, which is open. To prove the continuity of p'_U, look at the commutative diagram

$$\begin{array}{ccc} (H_U, h_U) & \xrightarrow{i'_U} & (H_U, h_U(h)) \\ \downarrow{\scriptstyle p'_U} & & \downarrow{\scriptstyle \xi'_{Ux}} \\ (A'_U, b'_U(h)) & \xrightarrow{\eta'_{Ux}} & (H_x, h_x) \end{array}$$

Here the continuity of i'_U implies the continuity of $\xi'_{Ux} i'_U = \eta'_{Ux} p'_U$ for all $x \in U$, so p'_U is continuous.

4.1.13. Notation. If the hull $\mathscr{H}$ of $\mathscr{S}$ does not satisfy COND from 4.1.1E then the map $p'_U : H_U \to A'_U$ is not $1-1$. If not even $\mathscr{S}$ satisfies COND, then p_U is not $1-1$, either. But we can make the factorspace $(X_U/p_U, \tau'_U)$ of (X_U, τ_U) or $(H_U/p'_U, h'_U)$ of (H_U, h_U) by the equivalence $\{a, b \in X_U$, then $a \sim b$ iff $p_U(a) = p_U(b)\}$ or $\{p, q \in H_U$, then $p \mathrel{\tilde{\sim}} q$ iff $p'_U(p) = p'_U(q)\}$, endowed with the closure τ'_U or h'_U inductively defined in X_U/p_U or H_U/h'_U by the canonical map $k_U : (X_U, \tau_U) \to X_U/p_U$ or $k'_U : (H_U, h_U) \to$ $\to H_U/p'_U$, respectively. By 4.1.5, if $a, b \in X_U$, then $a \sim b$ iff $e_U(a) \mathrel{\tilde{\sim}} e_U(b)$. Therefore, there is a $1-1$ map $e'_U : X_U/p_U \to H_U/p'_U$ such that $e'_U k_U = k'_U e_U$. Further, there are canonical maps $q_U : X_U/p_U \to A_U$, $q'_U : H_U/p'_U \to A'_U$ which are $1-1$. If moreover t and h are closures in $P_{\mathscr{S}}$ and in $P_{\mathscr{H}}$ (so that we can make $b_U(t)$, $b'_U(h)$) respectively, we have the following commutative diagram:

(4.1.14)



Here e'_U is continuous. Indeed, the continuity of $e'_U \circ k_U = k'_U \circ e_U$ yields that of e'_U.

A subset M of H_U is called *saturated* if $\{p \in H_U |$ there is $q \in M$ such that $p \mathrel{\tilde{\sim}} q\} =$ $= M$.

4.1.15. Proposition. *Suppose that there is a closure h in $P_{\mathscr{H}}$ such that the identity $i'_U : (H_U, h_U) \to (H_U, h_U(h))$ is open and continuous (this holds if (H_U, h_U) is compact and if there is a Hausdorff topology $h^\#_x$ in every stalk H_x, coarser than h^*_x, particularly, if the stalks (H_x, h^*_x) are f.s. — then we can put $h = sh^\#_x$ — see 4.1.10). If $e_U(X_U)$ is saturated, then e'_U is open. Further, the map q'_U is $h'_U - b'_U(h)$ continuous.*

Proof. Look at 4.1.14, where h is the closure from the assumption. If $B \subset X_U/p_U$ is τ'_U-open then $C = k_U^{-1}(B)$ is τ_U-open and $e_U(C)$ is open in $(e_U(X_U)$, ind $h_U)$ — see 0.14. There is an h_U-open set D such that $e_U(C) = D \cap e_U(X_U)$. Clearly, $E = \{q \in H_U |$ there is $p \in D$ such that $p \mathrel{\tilde{\sim}} q\}$ is saturated. Moreover, $E \cap e_U(X_U) = e_U(C)$. Indeed, if $q \in e_U(X_U) \cap E$ then there is $p \in D$ with $p \mathrel{\tilde{\sim}} q$. We have $p \in e_U(X_U)$, for q is from the saturated set $e_U(X_U)$. There are $a, b \in X_U$ with $e_U(a) = p$, $e_U(b) = q$. We have $a \in C$ for $p \in D$. From $p \mathrel{\tilde{\sim}} q$ we get by 4.1.6 that $a \sim b$. Thus $k_U(b) = k_U(a) \in B$, so $b \in C$. Therefore $q = e_U(b) \in e_U(C)$.

Now we prove that E is open. If $q \in E$, then there is $p \in D$ such that $p'_U(p) = p'_U(q)$. There is an open h — nbd N of p such that $N \subset D$ for D is open (recall that (H_U, h_U) is to pological). Let h be the closure mentioned in the assumptions, for which $i'_U :$ $: (H_U, h_U) \to (H_U, h_U(h))$ is open. By 4.1.12, $p'_U : (H_U, h_U) \to (A_U, b'_U(h))$ is open and continuous. Thus $M = p'_U(N)$ is $b'_U(h)$-open, $L = p'^{-1}_U(M)$ is h_U-open, $q \in L$, so L is an h'_U-nbd of q and $L \subset E$ as desired. Further, $e'_U(B) = k'_U\, e_U(C) = q'^{-1}_U p'_U\, e_U(C)$. The first equality and $k'_U\, e_U(C) \subset q'^{-1}_U p'_U\, e_U(C)$ is clear. To prove the other inclusion, take $a \in$ $\in q'^{-1}_U p'_U\, e_U(C)$. There is $b \in e_U(C)$ with $p'_U(b) = q'_U(a)$. If there were $k'_U(b) \neq a$ then we should have $p'_U(b) = q'_U\, k'_U(b) \neq q'_U(a)$, for q'_U is $1-1$. This contradicts $p'_U(b) = = q'_U(a)$, so $a \in k'_U\, e_U(C)$ as desired. Furthermore, $p'_U(E \cap e_U(X_U)) = p'_U(E) \cap p'_U\, e_U(X_U)$. Indeed if a is from the right hand side, then there is $u \in E$ and $v \in e_U(X_U)$ with $p'_U(u) = = p'_U(v) = a$. As $u \sim v$ and E with $e_U(X_U)$ are saturated, we get $u \in E$, $u \in e_U(X_U)$, hence $a \in p'_U(E \cap e_U(X_U))$, which proves the inclusion $\supset$ while $\subset$ is clear. Finally, $q'_U : (H_U/p'_U, h'_U) \to (A'_U, b'_U(h))$ is continuous for so is $q'_U \circ k'_U = p'_U$ in 4.1.14. (By 4.1.12 p'_U is continuous.) Thus $p'_U(E)$ is $b'_U(h)$ open for p'_U is open. Further, $q'^{-1}_U\, p'_U(E)$ is h'_U-open for q'_U is continuous. Thus $e'_U(B) = q'^{-1}_U p'_U\, e_U(C) = q'^{-1}_U\, p'_U(E \cap e_U(X_U)) = = q'^{-1}_U(p'_U E \cap p'_U\, e_U(X_0)) = q'^{-1}_U p'_U E \cap q'^{-1}_U p'_U\, e_U(X_U)$. We have $q'^{-1}_U p'_U\, e_U(X_U) = = e'_U\, k_U(X_U) = e'_U(X_U/p_U)$. Indeed, if a is from the left hand side, then there is $b \in X_U$ with $q'_U(a) = p'_U\, e_U(b)$. Since q'_U is $1-1$, we get $e'_U\, k_U(b) = a$ for $q'_U e'_U\, k_U(b) = = p'_U\, e_U(b) = q'_U(a)$. This proves the inclusion $\subset$ from the left equality, while the others are clear. Thus $e'_U(B) = q'^{-1}_U p'_U E \cap e'_U(X_U/p_U)$. The porposition is proved.

4.1.16. Corollary. *Let (H_U, h_U) be compact and $e_U(X_U)$ saturated. Suppose that there is a closure h in $P_{\mathscr{H}}$ such that $(A'_U, b'_U(h))$ and $(H_U, h_U(h))$ are Hausdorff and topological and that $p'_U : (H_U, h_U) \to (A'_U, h'_U(h))$ is continuous. (This holds if there is a Hausdorff topology $h^{\#}_x$ in every H_x coarser than h^*_x, particularly if (H_x, h^*_U) are f.s. — then we can put $h = sh^{\#}_x$). If t is a closure in $P_{\mathscr{S}}$ such that $\xi_{Ux} : (X_U, \tau_U) \to (I_x, t_x)$ and $e_x : (I_x, t_x) \to (H_x, h_x)$ are continuous for all $x \in U$ (in particular, if $t_x = t^*_x$ or if t_x is projectively defined by $e_x : I_x \to (H_x, h_x)$ for all $x \in U$) then $q_U : (X_U/p_U, m\tau'_U) \to (A_U, m\, b_U(t))$ is a homeomorphism (see* 0.9).

Proof. Look at 4.1.14, where h is the closure mentioned in the assumptions. By [1, Chap. 1, sec. 10(6), Cor. 1 of Prop. 8, p. 97], $(H_U/p'_U, mh'_U)$ is Hausdorff and hence compact. By 4.1.15, q'_U is $1-1$ and continuous, hence an $mh'_U - b'_U(h)$ homeomorphism. Further, $i'_U : (H_U, h_U) \to (H_U, h_U(h))$ is a homeomorphism, hence by 4.1.15 so is $e'_U : (X_U/p_U, m\tau'_U) \to (H_U/p'_U, mh'_U)$ as well. The continuity of all ξ_{Ux} for all $x \in U$ gives that of p_U. From this the continuity of g_U follows. As e_x and η_{Ux} are continuous for all $x \in U$, we get the continuity of q_U^{-1} as in 4.1.5. It remains to prove that ξ_{Ux} and e_x are continuous if $t_x = t^*_x$ or if t_x is projectively defined by $e_x : I_x \to \to (H_x, h_x)$. But this follows from 4.1.4, which completes the proof.

Suppose that the hull $\mathscr{H} = \{(H_U, h_U) |r_{UV}| X\}$ of $\mathscr{S} = \{(X_U, \tau_U) |\varrho_{UV}| X\}$ is an $\mathscr{E}$ – compact hull ($\mathscr{E}^\beta$ – compact hull) of $\mathscr{S}$ by a strongly separating family $\mathscr{E} = = \{F_U \subset C((X_U, \tau_U) \to Q) \mid U \in \mathscr{B}(X)\}$ ($\mathscr{E} = \{\mathscr{A}_U \subset C^*((X_U, \tau_U) \to C) \mid U \in \mathscr{B}(X)\}$) (see 2.1.6, 2.2.6; Q is the compact unit interval and C is the field of complex numbers). If the maps p_U and p'_U are not $1-1$, then we can use 4.1.16 if $e_U(X_U)$ are saturated. This means that if $\varphi \in H_U$, $a \in X_U$, $\varphi \sim e_U(a)$, then there is $b \in X_U$ with $e_U(b) = \varphi$. Here H_U are the sets Q^{F_U} (the sets $ML(\mathscr{A}_U \to C)$ of all continuous multiplicative linear functionals on $\mathscr{A}_U$) and $r_{UV} = \varrho^{**}_{UV}$.

4.1.17. Remark. The saturatedness of $e_U(X_U)$ is equivalent to the following condition K: "Given $\varphi \in H_U$, $a \in X_U$ and an open cover $\mathscr{V}$ of U such that φ coincides with $e_U(a)$ on $M_{\mathscr{V}} = \{\varrho^*_{UV}F_V \mid V \in \mathscr{V}\}$ (on $N_{\mathscr{V}} = \{\varrho^*_{UV}\mathscr{A}_V \mid V \in \mathscr{V}\}$) – or equivalently $\varrho^{**}_{UV}\,\varphi(f_V) = f_V\,\varrho_{UV}(a)$ for all $V \in \mathscr{V}$ and all $f_V \in F_V$ $(f_V \in \mathscr{A}_V)$ – then there is $b \in X_U$ with $\varphi = e_U(b)$." Let $\mathscr{S}$ fulfil COND from 4.1.1E so that all the p_U are $1-1$.

A. Since $e_U(a) \sim \varphi = e_U(b)$ implies $a = b$, we conclude that if $e_U(X_U)$ is saturated, then $\varphi \sim e_U(a)$ iff $\varphi = e_U(a)$. Thus $e_U(X_U)$ is saturated iff $\varphi \sim e_U(a)$ implies $\varphi = = e_U(a)$.

B. If $\mathscr{H}$ is the $\mathscr{E}$ – hull of $\mathscr{S}$ then the following are equivalent:

1) p'_U is $1-1$,

2) $e_U(X_U)$ is saturated,

3) $M_{\mathscr{V}} = F_U$ for any open cover $\mathscr{V}$ of U.

If $\mathscr{H}$ is the $\mathscr{E}^\beta$ – hull of $\mathscr{S}$, the conditions

4) $\tilde{N}_{\mathscr{V}}$ is norm dense in $\mathscr{A}_U$ ($\tilde{N}_{\mathscr{V}}$ is the smallest subalgebra of $\mathscr{A}_U$ such that $N_{\mathscr{V}} \subset \tilde{N}_{\mathscr{V}}$),

5) for any open cover $\mathscr{V}$ of U and any $a \in X_U$ there is a unique extension φ of the restriction $e_U(a)/N_{\mathscr{V}}$ of $e_U(a) \in ML(\mathscr{A}_U \to C)$ to the whole $\mathscr{A}_U$ (if this is the case then $\varphi = e_U(a)$) satisfy $4 \Rightarrow 1 \Rightarrow 5 \Leftrightarrow 2$. Further, each of the conditions 3 and 4 implies COND; thus $3 \Rightarrow 4$ and $4 \Rightarrow 1$ even without assuming COND beforehand.

Proof. If $\varphi, \psi \in H_U$ then $\mathscr{S} \sim \psi$ means that $\xi'_{Ux}(\varphi) = \xi'_{Ux}(\psi)$ for all $x \in U$. Thus for every $x \in U$ there is an open nbd $V_x \subset U$ of x such that $\varrho^{**}_{UV_x}\,\varphi(f_x) = \varphi(\varrho^*_{UV_x}f_x) = = \varrho^{**}_{UV_x}\,\psi(f_x) = \psi(\varrho^*_{UV_x}f_x)$ for all $x \in X$ and all $f_x \in F_{V_x}$. So $\varphi = \psi$ on $M_{\mathscr{V}}$ (on $N_{\mathscr{V}}$), where $\mathscr{V} = \{V_x \mid x \in X\}$.

Let COND hold for $\mathscr{S}$. If $a, b \in X_U$, $e_U(a) \sim e_U(b)$ then there is an open cover $\mathscr{V}$ of U such that $\varrho^{**}_{UV}\,e_U(a)(f) = f\,\varrho_{UV}(a) = \varrho^{**}_{UV}\,e_U(b)(f) = f\,\varrho_{UV}(b)$ for all $V \in \mathscr{V}$ and all $f \in F_V$ $(f \in \mathscr{A}_V)$. Let $V \in \mathscr{V}$. As $\mathscr{E}(\mathscr{E}^\beta)$ is separating, we get $\varrho_{UV}(b) = \varrho_{UV}(a)$. So $\varrho_{UV}(a) = \varrho_{UV}(b)$ for all $V \in \mathscr{V}$ which, by COND, gives $a = b$.

Let $e_U(X_U)$ be saturated, $a \in X_U$, $\varphi \in H_U$, $\varphi \sim e_U(a)$. Then there is $b \in X_U$ with $\varphi = e_U(b)$, so $a = b$ and $\varphi = e_U(a)$. Conversely, if $\varphi \sim e_U(a)$ implies $\varphi = e_U(a)$ then $e_U(X_U)$ is saturated, which proves A.

B: Let $\mathscr{H}$ be the $\mathscr{E}$ – hull of $\mathscr{S}$. As $\varphi \sim \psi$ iff $p'_U(\varphi) = p'_U(\psi)$ we get $1 \Rightarrow 2$. If $\varphi \sim \psi$ then there is $\mathscr{V}$ such that $\varphi = \psi$ on $M_{\mathscr{V}}$. If $M_{\mathscr{V}} = F_U$ we have $\varphi = \psi$ which proves $3 \Rightarrow 1$ (here we have not used COND). Let $f \in F_U - M_{\mathscr{V}}$. Take $a \in X_U$ and set $\varphi(g) = g(a)$ for all $g \in F_U - \{f\}$ $\varphi(f) = c$ where $c = 0$ if $f(a) \neq 0$ and $c = 1$ if $f(a) = 0$. Then φ is a map of F_U into Q, hence $\varphi \in H_U$. As $\varphi = e_U(a)$ on $M_{\mathscr{V}}$, we have $\varphi \sim e_U(a)$. Since $\varphi \neq e_U(a)$, we have $\varphi \neq e_U(b)$ for all $b \in X_U$, so $e_U(X_U)$ is not saturated which gives $2 \Rightarrow 3$. If $\mathscr{H}$ is the $\mathscr{E}^\beta$ – hull of $\mathscr{S}$, $\varphi \sim \psi$, then there is an open cover $\mathscr{V}$ of U with $\varphi = \psi$ on $N_{\mathscr{V}}$. As φ, ψ are continuous maps of $\mathscr{A}_U$ into C, we have $\varphi = \psi$ if $\varphi \sim \psi$ and if $\tilde{N}_{\mathscr{V}}$ is dense in $\mathscr{A}_U$. This proves $4 \Rightarrow 1$ (here we have not used COND). Given $a \in X_U$, an open cover $\mathscr{V}$ of U and an extension $\varphi \in \in \mathrm{ML}(\mathscr{A}_U \to C)$ of $e_U(a) \mid N_{\mathscr{V}}$, then $e_U(a) \sim \varphi$. If p'_U is $1-1$, we have $\varphi = e_U(a)$, so $1 \Rightarrow 5$. If $\varphi \in H_U$, $a \in X_U$, $\varphi \sim e_U(a)$, then there is $\mathscr{V}$ with $\varphi = e_U(a)$ on $N_{\mathscr{V}}$.

If the extension of $e_U(a)$ from $N_{\mathscr{V}}$ is unique, we have $\varphi = e_U(a)$, hence $e_U(X_U)$ is saturated, so $5 \Rightarrow 2$. Given $a \in X_U$ and $\mathscr{V}$ such that $e_U(a)/N_{\mathscr{V}}$ has an extension $\varphi \in \mathrm{ML}(\mathscr{A}_U \to C)$, $\varphi \neq e_U(a)$, then $\varphi \in H_U$, $\varphi \sim e_U(a)$, so $e_U(X_U)$ is not saturated and $2 \Rightarrow 5$ follows. The remark is proved.

2. REPRESENTATION THEOREMS

4.2.1. Definition. Let a presheaf $\mathscr{S} = \{S_U | \varrho_{UV} | X\}$, $U \in \mathscr{B}(X)$ and an open cover $\mathscr{V}$ of U be given. A family $\mathscr{H} = \{a_V \in S_V \mid V \in \mathscr{V}\}$ is called *$\mathscr{V}$-smooth* if $\varrho_{VV \cap W}(a_V) = \varrho_{WV \cap W}(a_W)$ for all $V, W \in \mathscr{V}$ with $V \cap W \neq \emptyset$. $\mathscr{S}$ is called *projective* if for every $U \in \mathscr{B}(X)$, any open cover $\mathscr{V}$ of U and any $\mathscr{V}$-smooth family $\mathscr{H}$ there is $a \in S_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$.

4.2.2. Theorem. *Let $\mathscr{S}' = \{\mathscr{X}'_U | \varrho_{UV} | X\}$ be a presheaf from an i.c. category $\mathfrak{L}$ such that $\mathscr{S} = \mathrm{cl}\, \mathscr{S}' = \{\mathscr{X}_U = (X_U, \tau_U) | \varrho_{UV} | X\}$ is T_1* (*see* 2.1.2A), *which is endowed with a strongly separating family $\mathscr{E} = \{F_U \subset C^*(\mathscr{X}'_U \to R \mid \mathfrak{L}) \mid U \in \mathscr{B}(X)\}$* (*see* 1.1.5) *so that all the ϱ^*_{UV} send F_V into F_U* (*see* 4.1.5A). *Further, let every $x \in X$ have a filter base Ax of open* nbds *of x such that*

(1) *$\langle Ax \leqq \rangle$ is well ordered* (*see* 4.1.1A),

(2) a) *the family $\mathscr{E}_x = \{F_U \mid U \in Ax\}$ is leftward smooth;*

b) *either $\mathscr{E}_x$ is connected* (*see* 1.1.5A) *or x is of a countable local character and ϱ^*_{UV} maps F_V onto F_U for any $U, V \in Ax$, $U \leqq V$;*

(3) *if $U \subset X$ is open and if $\mathscr{V}$ is an open cover of U then $F_U = \bigcup\{\varrho^*_{UV} F_V \mid V \in \mathscr{V}\}$.*

If $\mathscr{P} = (P, t)$ is the covering space of $\mathscr{S}$ and A_U the set of the sections in $\mathscr{P}$ which corresponds canonically to X_U (*see* 4.1.1D, E, F), *then:*

(a) *For any $x \in X$ the stalk $\mathscr{I}_x = (I_x, t^*_x) = \varinjlim \mathscr{S}_x$ of $\mathscr{P}$* (*see* 4.1.1C) *is f.s. by $C^*(\mathscr{I}_x \to R)$. Thus there is a separated topology $t^\#_x$ in I_x coarser than t^*_x* (*see* 1.1.2), *The topology, projectively defined by any separating family $Dx \subset C^*(\mathscr{I}_x \to R)$. may be taken as $t^\#_x$.*

(b) *For any open $U \subset X$ the map $p_U : (X_U, m\tau_U) \to (A_U, b_U(st^\#_x))$ is a homeomorphism* (*see* 4.1.1E; *$m\tau_U$ is the topological modification of τ_U – see* 0.9; *$st^\#_x$ and $b_U(st^\#_x)$ are closures in $P = \bigcup\{I_x \mid x \in X\}$ and in A_U, respectively – see* 4.1.1F).

(c) *Let $\mathscr{S}$ be projective* (*see* 4.2.1). *There is a separated closure $\hat{t}$ in P such that $b_U(\hat{t}) = b_U(st^\#_x)$ (thus $p_U : (X_U, m\tau) \to (A_U, b_U(\hat{t}))$ is a homeomorphism), and $\Gamma(U, \hat{t}) = A_U$ for any open U* (*see* 4.1.1F).

(d) *There is a separated topology $\tilde{t}$ in P with $A_U \subset \Gamma(U, \tilde{t})$ and $b_U(\tilde{t}) = b_U(st^\#_x)$ for all $U \in \mathscr{B}(X)$, so that each canonical map $p_U : (X_U, \tau_U) \times U \to (P, \tilde{t})$ is continuous* (*the joint continuity of p_U is meant – see* [9, Ch. 7, p. 233] *– we have $\hat{p}_U(a, x) = \xi_{Ux}(a) = (p_U(a))(x)$ if $U \in \mathscr{B}(X)$, $a \in X_U$, $x \in U$ – see* 4.1.1C, D). *Further, if the topology $t^\#_x$ in every stalk is metrisable and X is metriasble then so is $\tilde{t}$.*

Proof. Let $\mathcal{T} = \{\mathscr{C}_U = (C_U, t_U) | \varrho_{UV}^{**} | X\}$ be the $\mathscr{E}$-hull of $\mathscr{S}$ – see 2.1.6. By 2.1.4, $\mathcal{T}_x = \mathcal{T}_{Ax}$ is the $\mathscr{E}_x$ – compact hull of $\mathscr{S}_{Ax}$ by $\mathscr{E}_x = \{F_U \mid U \in Ax\}$. If x has countable Local character then there is a countable filter base $Bx \subset Ax$ of nbds of x. By Th. 2.1.7, $(H_x, h_x^*) = \varinjlim \mathcal{T}_x$ is f.s. (we put $\mathcal{T}_x = \mathcal{T}_{Ax}$) – see 4.1.1A, C, and 1.1.1.

By 1.1.2, there is a Hausdorff topology $h_x^\#$ in H_x coarser than t_x^*. For each open U let A'_U be the set of the sections in the covering space of $\mathcal{T}$, which corresponds canonically to C_U, and for $x \in U$ let ξ'_{Ux} and η'_{Ux} be the canonical maps of C_U and A'_U into H_x, respectively. As in 4.1.2B, let $b'_U(sh_x^\#)$ be the topology projectively defined in A'_U by $\{\eta'_{Ux} : A'_U \to (H_x, h_x^\#) \mid x \in U\}$ (see 4.1.1F). By 4.1.6, 4.1.9, $b'_U(sh_x^\#)$ is Hausdorff, and from the condition (3) and 4.1.17B it follows that $p'_U : \mathscr{C}_U \to$ $\to (A'_U, b'_U(sh_x^\#))$ is $1-1$, hence it is a homeomorphism. By 4.1.5, all the p_U are $1-1$. As $\mathcal{T}$ is a hull of $\mathscr{S}$, the canonical embeddings $e_U : (X_U, m\tau_U) \to \mathscr{C}_U$ are homeomorphisms (see 2.1.1B). Now, let $t_x^\#$ be the topology projectively defined in I_x by $e_x : I_x \to (H_x, h_x^\#)$ – see 4.1.7. Then the maps $\xi_{Ux} : X_U \to I_x$ are $\tau_U - t_x^\#$ continuous, so they are also $m\tau_U - t_x^\#$ continuous. By 4.1.7, the map $p_U : (X_U, m\tau_U) \to$ $\to (A_U, b_U(st_x^\#))$ is a homeomorphism. The statements (a), (b) are proved.

If $U \in \mathscr{B}(X)$, $x \in U$, $a \in X_U$, $\alpha = \xi_{Ux}(a)$, we set graph $(a; U) = \{\xi_{Uy}(a) \mid y \in U\}$. If $\alpha \in P$, $\alpha \in I_x$ we set $H(\alpha) = \{\text{graph}\,(a; U) \cup N \mid U \in \mathscr{B}(X),\ a \in X_U$ with $x \in U$, $\xi_{Ux}(a) = \alpha;\ N$ is a $t_x^\#$-nbd of $\alpha\}$, $K(\alpha) = \{\bigcup\{I_y \mid y \in U,\ y \neq x\} \cup N \mid U \in \mathscr{B}(X)$, $a \in X_U$ with $x \in U$, $\xi_{Ux}(a) = \alpha$; N is a $t_x^\#$-nbd of $\alpha\}$. Then $H(\alpha), K(\alpha)$ are filter bases round α in P. They make a separated closures $\hat{t}, \tilde{t}$ in P. If $t = \hat{t}, \tilde{t}$ then clearly $t_x = t_x^\#$ (t_x is the closure induced in I_x by t) for all $x \in X$ (so $b_U(t) = b_U(st_x^\#)$, and $A_U \subset$ $\subset \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$. By [11, Chap. 2, Sec. 4, Prop. 2.4.3, p. 608], $\Gamma(U, \hat{t}) = A_U$ which proves (c). Clearly $\tilde{t}$ is a topology and (d) holds (compare also with [11, Chap. 2, Sec. 3, Prop. 2.3.4, p. 607]).

Let every $t_x^\#$ be metrisable by a metric d_x and let X be metrisable by D'. Given $\alpha, \beta \in P$, $\alpha \in I_x$, $\beta \in I_y$, we set $D(\alpha, \beta) = D'(x, y)$ if $x \neq y$, $D(\alpha, \beta) = d_x(\alpha, \beta)$ if $x = y$. Clearly D is a metric in P which makes $\tilde{t}$. The theorem is proved.

4.2.3. Theorem. *Let $\mathscr{S}' = \{\mathscr{X}'_U | \varrho_{UV} | X\}$ be a presheaf from an i.c. category $\mathfrak{L}$ such that $\mathscr{S} = \text{cl}\,\mathscr{S}' = \{\mathscr{X}_U | \varrho_{UV} | X\}$ is T_1 (see 2.1.2A). Suppose that for each open $U \subset X$ we have a Banach algebra $\mathscr{A}_U \subset C^*(\mathscr{X}'_U \to C) \mid \mathfrak{L}\}$ (see 0.11, 2.1.2) with the sup-norm which separates points from closed sets of $\mathscr{X}_U$, so that $\overset{*}{\varrho}_{UV}$ maps $\mathscr{A}_V$ into $\mathscr{A}_U$ if $V \subset U$. Let every $x \in X$ have a filter base Ax of open* nbds *such that $\langle Ax \leqq \rangle$ is well ordered (see 4.1.1A) and*

(1) a) *$\overset{*}{\varrho}_{UU+1}\mathscr{A}_{U+1}$ is norm-dense in $\mathscr{A}_U$ if $U \in Ax$ ($U + 1$ is the follower of U in $\langle Ax \leqq \rangle$);*

b) *either x is of countable Local character and $\overset{*}{\varrho}_{UV}(\mathscr{A}_V)$ is norm-dense in $\mathscr{A}_U$ for all $U, V \in Ax$, $U \leqq V$, or the family $\mathscr{E}_x = \{\mathscr{A}_U \mid U \in Ax\}$ is connected (see 1.1.5) and $\mathscr{A}_U$ is symmetric (see 2.2.2B) for all $U \in Ax$;*

(2) *if $U \subset X$ is open and if $\mathscr{V}$ is an open cover of U then the smallest algebra in $\mathscr{A}_U$ containing $\bigcup\{\overset{*}{\varrho}_{UV}\mathscr{A}_V \mid V \in \mathscr{V}\}$ is norm dense in $\mathscr{A}_U$.*

Then the statements (a)–(d) *of Th.* 4.2.2 *hold.*

Proof. Put $\mathscr{E} = \{\mathscr{A}_U \mid U \in \mathscr{B}(X)\}$ and let $\mathscr{T} = \{\mathscr{C}_U = (\mathscr{T}_U, t_U) \,|\varrho^*_{UV}|\, X\}$ be the $\mathscr{E}^\beta$ – hull of $\mathscr{S}$ by $\mathscr{E}$ – see 2.2.5B, 2.2.6. If $x \in X$ then by Th. 2.2.7 (or by 2.2.8 if x is of countable local character), $(H_x, h^*_x) = \varinjlim \mathscr{T}_x$ is f.s. (here $\mathscr{T}_x = \mathscr{T}_{Ax}$), so by 1.1.2 there is a Hausdorff topology $h^\#_x$ in H_x coarser than h^*_x. By 4.1.6, $b'_U(sh^\#_x)$ is Hausdorff (see 4.1.2B). From the condition (2) and 4.1.17B it follows that $p'_U : \mathscr{C}_U \to \to (A'_U, b'_U(sh^\#_x))$ is 1–1 so it is a homeomorphism (see 4.1.1E, 4.1.2). The rest of the proof is the same as in Th. 4.2.2.

4.2.4. Corollary. *Given a presheaf* $\mathscr{S} = \{\mathscr{X}'_U|\varrho_{UV}|\, X\}$ *from* UNIF *such that all the* $\mathscr{X}'_U$ *are separated* (*see* 0.5, 0.17), *suppose that every* $x \in X$ *has a filter base* Ax *of open* nbds *such that* $\langle Ax \leqq \rangle$ *is well ordered* (*see* 4.1.1A) *and*

(1) a) ϱ_{UU+1} *is a uniform embedding of* $\mathscr{X}'_U$ *into* $\mathscr{X}'_{U+1}$ *for all* $U \in Ax$ (*see* 0.15);

b) *if* $\mathscr{E} = \{F_U = U^*(\mathscr{X}'_U \to R) \mid U \in \mathscr{B}(X)\}$ – *see* 0.14, *then either the family* $\mathscr{E}_x$ *is connected or* ϱ_{UV} *is a uniform embedding of* $\mathscr{X}'_U$ *into* $\mathscr{X}'_V$ *for all* $U, V \in Ax$, $U \leqq V$ *and* x *is of countable local character;*

(2) *if* $U \subset X$ *is open and* $\mathscr{V}$ *is an open cover of* U *then* $\bigcup\{\varrho^*_{UV}F_V \mid V \in \mathscr{V}\} = F_U$ (*this holds if* $\varrho_{UV} : \mathscr{X}'_U \to \mathscr{X}'_V$ *are uniform embeddings for all* $V \subset U$ – *see* 1.3.4b).

Then the statements (a)–(d) *of Th.* 4.2.2 *hold.*

The conditions (1b) *and* (2) *may be replaced by the following ones:*

(1b′) *If* $\mathscr{E} = \{\mathscr{A}_U = U^*(\mathscr{X}'_U \to C) \mid U \in \mathscr{B}(X)\}$ (*see* 0.14; *C is the field of complex numbers*), *then either* $\mathscr{E}_x = \{\mathscr{A}_U \mid U \in Ax\}$ *is connected, or* ϱ_{UV} *is a uniform embedding of* $\mathscr{X}'_U$ *into* $\mathscr{X}'_V$ *for all* $U, V \in A_x$, $U \leqq V$, *and* x *is of countable local character.*

(2′) *The smallest algebra in* $\mathscr{A}_U$ *containing* $\bigcup\{\varrho^*_{UV}\mathscr{A}_V \mid V \in \mathscr{V}\}$ *is norm-dense in* $\mathscr{A}_U$ *for every open* $U \subset X$ *and any open cover* $\mathscr{V}$ *of* U.

((1b) *is equivalent to* (1b′) *for* $\mathscr{A}_U = \{f + ig \mid f, g \in F_U\}$ *for all* U; *notice that* (2) *yields* (2′) *but* (2) *does not follow from* (2′).)

Proof. By 1.3.4B, 1.3.7B, the conditions of Th. 4.2.2 are fulfilled for $\mathscr{S}$ and $\mathscr{E}$. If (1b′) and (2′) hold then by the same argument Th. 4.2.3 works.

4.2.5. Corollary. *Let* $\mathscr{S} = \{\mathscr{X}_U|\varrho_{UV}|\, X\}$ *be a normal and* T_1 *presheaf from* TOP (*see* 2.1.2A, 0.5). *Suppose that every* $x \in X$ *has a filter base* Ax *of open* nbds *such that* $\langle Ax \leqq \rangle$ *is well ordered and*

(1) a) ϱ_{UU+1} *is a homeomorphism of* $\mathscr{X}_U$ *into* $\mathscr{X}_{U+1}$ *and* $\varrho_{UU+1}(\mathscr{X}_U)$ *is closed in* $\mathscr{X}_{U+1}$ *for all* $U \in Ax$;

b) *if we put* $\mathscr{E} = \{F_U = C^*(\mathscr{X}_U \to R) \mid U \in \mathscr{B}(X)\}$ *then either* $\mathscr{E}_x = \{F_U \mid U \in Ax\}$ *is fully connected, or* (1a) *is fulfilled for any pair* $U, V \in Ax$, $U \subset V$, *instead of for* $U, U + 1$ *only and* x *is of countable local character;*

(2) $F_U = \bigcup\{\varrho^*_{UV}F_V \mid V \in \mathscr{V}\}$ *for every* $U \in \mathscr{B}(X)$ *and any open cover* $\mathscr{V}$ *of* U.

This holds if $\varrho_{UV} : \mathscr{X}_U \to \mathscr{X}_V$ is a homeomorphism into $\mathscr{X}_V$ for all $V \subset U$ — see 1.3.4*a*.) *Then the statement of Th.* 4.2.2 *holds.*

The conditions (1b) *and* (2) *may be replaced by the following ones*:

(1b′) *If $\mathscr{E} = \{\mathscr{A}_U = C^*(\mathscr{X}_U \to C) \mid U \in \mathscr{B}(X)\}$ then either $\mathscr{E}_x = \{\mathscr{A}_U \mid U \in Ax\}$ is connected or* (1a) *is fulfilled for any pair $U, V \in Ax$, $U \leqq V$, instead of for U, $U + 1$ only.* ((1b) *is equivalent to* (1b′), *C is the field of complex numbers.*)

(2′) *The smallest algebra in $\mathscr{A}_U$ containing $\bigcup\{\varrho^*_{UV}\mathscr{A}_V \mid V \in \mathscr{V}\}$ is norm-dense in $\mathscr{A}_U$ for every $U \in \mathscr{B}(X)$ and any open cover $\mathscr{V}$ of U.*

Proof. By 1.3.1A, 1.3.7A, the conditions of Th. 4.2.2 are fulfilled for $\mathscr{S}$ and $\mathscr{E}$. If we have (1b′), (2′) then by the same argument 4.2.3 works.

4.2.6. Corollary. *Let $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ be a locally compact presheaf from* TOP. *Suppose that every $x \in X$ has a filter base Ax of open* nbds *such that $\langle Ax \leqq \rangle$ is well ordered and*

(1) a) *if $U, V \in Ax$, $U \leqq V$, then ϱ_{UV} is a homeomorphism into $\mathscr{X}_V$ such that the filter base $B_{UV} = \{\varrho_{UV}(X_U - K) \mid K \subset X_U \text{ compact}\}$ either has no cluster point or has a limit point in $\mathscr{X}_V$;*

b) *if $\mathscr{E} = \{\mathscr{A}_U = \mathscr{L}^\infty_U = \{f \in C(\mathscr{X}_U \to C) \mid f \text{ has a limit at infinity}\} \mid U \in \mathscr{B}(X)\}$ —* (*see* 2.3.1), *then either $\mathscr{E}_x = \{\mathscr{A}_U \mid U \in Ax\}$ is connected, or x is of countable Local character;*

(2) *the algebra generated in $\mathscr{A}_U$ by $\bigcup\{\varrho^*_{UV}\mathscr{A}_V \mid V \in \mathscr{V}\}$ is norm-dense in $\mathscr{A}_U$ in the usual sup-norm for all open U and any open cover $\mathscr{V}$ of U.*

Then the statement of Th. 4.2.2 *holds.*

Proof. By 2.3.2, 2.3.5A, B, the conditions of Th. 4.2.3 hold for $\mathscr{S}$, $\mathscr{E}$.

4.2.7. Corollary. *Let $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ be a compact presheaf from* CLOS *which fulfils* COND *from* 4.1.1D. *Suppose that each $x \in X$ has a filter base Ax of open* nbds *so that $\langle Ax \leqq \rangle$ is well ordered and*

(1) a) *$\varrho_{U+1\,U+2}$ is $1-1$ on $\varrho_{U\,U+1}(X_U)$ for all $U \in Ax$ ($U + 1$ is the follower of U in $\langle Ax \leqq \rangle$);*

b) *either x is of countable Local character, or the family $\mathscr{E}_x = \{F_U = C(\mathscr{X}_U \to R) \mid U \in Ax\}$ fulfils the following condition*: "*Given $U \in Ax$ such that the predecessor $U - 1$ of U in $\langle Ax \leqq \rangle$ does not exist, $V \in Ax$, $V < U$, and a thread $\{f_W \in F_W \mid W \in Ax,\ V \leqq W < U\}$ through $\mathscr{E}_x$, then there is $f \in F_{U+1}$ with $\varrho^*_{W\,U+1} f = f_W$ for all $W \in Ax$, $V \leqq W < U$.* (*This always holds for countable Ax.*)

Then the statement of Th. 4.2.2 *holds.*

Proof. By 3.1.3, $(I_x, t^*_x) = \varprojlim \mathscr{S}_{Ax}$ is f.s. Thus if $t^\#_x$ is a Hausdorff topology in I_x coarser than t^*_x, $U \in \mathscr{B}(X)$, then it easily follows from 4.1.5 that $p_U : \mathscr{X}_U \to$

$\to (A_U, b_U(st_\chi^\#))$ is continuous, hence it is a homeomorphism, which proves that the statements (a), (b) of 4.2.2 hold for $\mathscr{S}$. The rest can be proved as in 4.2.2.

4.2.8. Theorem. *Let $\mathscr{S}' = \{\mathscr{X}'_U | \varrho_{UV} | X\}$ be a presheaf from an i.c. category $\mathfrak{L}$ such that $\mathscr{S} = \mathrm{cl}\, \mathscr{S}' = \{\mathscr{X}_U = (X_U, \tau_U) | \varrho_{UV} | X\}$ is T_1, which is endowed with a strongly separating family $\mathscr{E}' = \{F_U \subset C_U = C^*(\mathscr{X}'_U \to C \mid \mathfrak{L}) \mid U \in \mathscr{B}(X)\}$ such that all ϱ^*_{UV} send F_V into F_U. Let every $x \in X$ have a filter base Ax of open* nbds *so that $\langle Ax \leqq \rangle$ is well ordered (see* 4.1.1A) *and*

(1) a) *$\varrho^*_{UU+1} F_{U+1}$ is norm-dense in F_U if $U \in Ax$ ($U + 1$ is the follower of U in $\langle Ax \leqq \rangle$);*

b) *either* (i): *x is of countable local character and $\varrho^*_{UV}(F_V)$ is norm-dense in F_U for all $U, V \in Ax$, $U \leqq V$,*

or (ii): *the family $\mathscr{E}_x = \{F_U \mid U \in Ax\}$ is connected (see* 1.1.5) *and for every $U \in Ax$, F_U is either a symmetric Banach algebra or an algebra of real functions over the field of real numbers, complete in the sup-norm;*

(2) *if $U \subset X$ is open and if $\mathscr{V}$ is an open cover of U then $M_\mathscr{V} = \{\varrho^*_{UV} F_V \mid V \in \mathscr{V}\}$ is norm-dense in F_U. Then the statements* (a)–(d) *of Th.* 4.2.2 *hold.*

Proof. Let us denote by $\mathscr{A}_U$ the symmetric algebraic hull of F_U and set $\mathscr{E} = \{\mathscr{A}_U \mid U \subset X \text{ open}\}$. Let $\mathscr{T} = \{(\mathscr{F}_U, t_U) \mid \varrho^{**}_{UV} | X\}$ be the $\mathscr{E}^\beta$-hull of $\mathscr{S}$ by $\mathscr{E}$ – see 2.2.5B, 2.2.6. By 2.2.5A, $\varrho^*_{UV}(\mathscr{A}_V)$ is norm-dense in $\mathscr{A}_U$ whenever so is $\varrho^*_{UV}(F_V)$ in F_U. Thus if $x \in X$ then $(H_x, h^*_x) = \varinjlim \mathscr{T}_{Ax}$ is f.s. by Th. 2.2.8 if the case (i) occurs, and by Th. 2.2.7 if (ii) occurs. Putting $S = T = X_U$, $F = F_U$, $G = \bigcup\{\varrho^*_{UV}(F_V) \mid V \in \mathscr{V}\}$, h = identity in 2.2.5A we get that the symmetric algebraic hull $\mathscr{A}(G)$ is norm-dense in $\mathscr{A}_U$. Thus also the smallest algebra Z generated by G is dense in $\mathscr{A}_U$ as $\mathscr{A}(G)$ is the norm-closure of Z. But $G \subset N_\mathscr{V} = \bigcup\{\varrho^*_{UV}(\mathscr{A}_V) \mid V \in \mathscr{V}\}$, so the smallest algebra generated by $N_\mathscr{V}$ is dense in $\mathscr{A}_U$ which yields that $p'_U : \mathscr{F}_U \to A'_U$ is 1–1 (see 4.1.1E, 4.1.2). Henceforward the proof proceeds as that of Th. 4.2.3 and of Th. 4.2.2.

4.2.9. Corollary. *Let $\mathscr{S} = \{(X_U, \tau_U) | \varrho_{UV} | X\}$ be a presheaf from* CLOS *such that there is a metric d_U in every X_U which generates τ_U, and that all $\varrho_{UV} : (X_U, \tau_U) \to (X_V, \tau_V)$ are homeomorphisms into (X_V, τ_V). Assume that every $x \in X$ has countable local character. Let (P, st^*_x) be the* CLOS*-covering space of $\mathscr{S}$ – see* 4.1.1D *and let $\mathscr{S}$ satisfy* COND *from* 4.1.1E. *Then there is a metric v in P such that for the topology t generated in P by v we have*

a) *the map $p_U : (X_U, \tau_U) \to (A_U, b_U(t))$ is a homeomorphism for every U (see* 4.1.1E);

b) *the statements* (c), (d) *of Th.* 4.2.2 *hold, if we write t instead of $st^\#_x$.*

Proof. We have $\mathscr{I}_x = \varinjlim \mathscr{S}_x$ for all $x \in X$ (here $\mathscr{S}_x = \mathscr{S}_{Ax}$ and $\mathscr{I}_x$ is the stalk of $\mathscr{P} = (P, st^*_x)$ – see 4.1.1D). As Ax is countable, the metric D_x in every $I_x = |\mathscr{I}_x|$ can be defined by 3.4.1.

If $a, b \in P$ then there are $x, y \in X$ with $a \in I_x$, $b \in I_y$. We set $v(a, b) = 1$ if $x \neq y$ and $v(a, b) = D_x(a, b)$ if $x = y$. Clearly v is a metric in P.

Let $U \subset X$ be open. By 3.4.1B, we may assume that U is the smallest element of $\langle Ax \leqq \rangle$ for all $x \in U$. As $\mathscr{S}$ fulfils COND, (see 4.1.1.E) p_U^{-1} exists. The continuity of $\xi_{Ux} : (X_U, d_U) \to (I_x, D_x)$ or of $\xi_{Ux}^{-1} : (\xi_{Ux}(X_U), D_x) \to (X_U, d_U)$ for all $x \in U$ yields that of $p_U : (X_U, d_U) \to (A_U, b_U(t))$ (see 4.1.4A) or of $p_U^{-1} : (A_U, b_U(t)) \to \to (X_U, d_U)$, respectively (we have $p_U^{-1} = \xi_{Ux}^{-1} \circ \eta_{Ux}$ for all $x \in U$, where $\eta_{Ux} : : (A_U, b_U(t)) \to (I_x, D_x)$, with continuous ξ_{Ux}^{-1}, η_{Ux} and $\eta_{Ux}(A_U) = \xi_{Ux}(X_U)$), which proves (a), while (b) can be proved as in 4.2.2.

3. TOPOLOGISATION

In Theorems 4.2.1 – 4.2.6 we proved the existence of such a closure $\hat{t}$ in P that the maps $p_U : \mathrm{cl}\, \mathscr{X}_U \to (A_U, b_U(\hat{t}))$ are homeomorphisms and the sets $\Gamma(U, \hat{t})$ of all continuous sections over U in $(P, \hat{t})$ are precisely the sets A_U for all $U \in \mathscr{B}(X)$. In this section we seek conditions for the existence of a topology in P with the mentioned properties.

4.3.1. Definition. A subset M of a closure space X is called a *zero set* if there is an $f \in C(X \to R)$ such that $M = f^{-1}(0)$. We say that X has the zero property (ZP) if each $x \in X$ is a zero set. If $x \in X$ is a zero set, $f \in C(X \to R)$ with $f(x) = 0$, $f > 0$ on $X - \{x\}$, then f is called the *x – function on X*.

Clearly, we get the same if we replace $C(X \to R)$ by $C(X \to Q)$ in the definition (Q being the compact unit interval).

An inductive family $\mathscr{S} = \{X_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from CLOS is said to have ZP if every X_α has ZP.

A presheaf $\mathscr{S} = \{S_U | \varrho_{UV} | X\}$ from a category $\mathfrak{K}$ is said to have the *unique Continuation property* (UCP) if X is locally connected and the maps ϱ_{UV} are 1 – 1 for every connected U and any $V \subset U$.

4.3.2. Lemma. A. *A necessary condition for a closure space (X, t) to have ZP is that every $x \in X$ has a countable set Sx of t-nbds of x with $\bigcap\{N \mid N \in Sx\} = \{x\}$. This condition is also sufficient if (X, t) is topological and completely regular.*

B. ZP is hereditary.

Proof. If (X, t) has ZP, $a \in X$, then there is an a – function f on X. If $N_k = \{x \in X \mid f(x) < 2^{-k}\}$, $Sa = \{N_k \mid k = 1, 2, \ldots,\}$, then Sa has the desired properties. If (X, t) is topological and completely regular, $a \in X$, $Sa = \{N_1, N_2, \ldots\}$, then for every $j = 1, 2, \ldots$ there is $f_j \in C((X, t) \to Q)$ with $f_j \geqq 0$, $f_j(a) = 0$, $f_j = 1$ on $X - N_j$ as we may assume that N_j are open. Then $f = \sum_{j=1}^{\infty} 2^{-j} f_j$ is the desired a – function, which proves A; B is clear.

4.3.3. Lemma. *Let (X, t) be normal and have* ZP.

A. *Given a nonnegative continuous function g on (X, t), a closed set $Y \subset X$ and $a \in Y$ such that $Y \cap \{x \in X \mid g(x) = 0\} = \{a\}$, then there is an a – function f on X with $f \geqq g, f = g$ on Y.*

B. *Let a nonnegative continuous function g on (X, t), $a \in X$, $\varepsilon > 0$, $\delta \geqq 0$ be given. Let (X, t) be T_1. Then there is an a – function f on X such that $f \geqq$ max (g, δ) on $M(g, \varepsilon) = \{x \in X \mid |g(x) - g(a)| \geqq \varepsilon\}$.*

C. *Given a closed set $Y \subset X$, $a \in Y$ and an a – function g on Y, then there is an a – function f on X with $f = g$ on Y. Moreover, if h is a nonnegative continuous function on X and $g \geqq$ max (h, δ) on $\{x \in Y \mid |h(x) - h(a)| \geqq \varepsilon\}$ then there is an a – function f' on X such that $f' = g$ on Y and $f' \geqq$ max (h, δ) on $M = \{x \in X \mid |h(x) - h(a)| \geqq \varepsilon\}$.*

Proof. A. By 4.3.2B, there is an a – function f_1 on $N = \{x \in X \mid g(x) = 0\}$, so there is a nonnegative continuous function f_2 on X with $f_2 = f_1$ on N, $f_2 = 0$ on Y. Then $f = g + f_2$ is the desired function.

B. Let (X, t) be T_1. As $M = M(g, \varepsilon)$ is closed, max (g, δ) continuous on it and $a \notin M$, there is a nonnegative continuous function f_1 on X with $f_1(a) = 0$, $f_1 = $ max (g, δ) on M since $Y = M \cup \{a\}$ is closed. If $N = \{x \in X \mid f_1(x) = 0\}$, then $Y \cap N = \{a\}$. Applying A to Y and f_1 we get the function we desired.

C. There is a continuous nonnegative extension f_1 of g to the whole X. Applying A to f_1 and Y we get the function f we wanted. If h is the function mentioned in C, we can put $f_1' = $ max (f, h, δ) on M. As $f_1' = g$ on $M \cap Y$, there is a continuous nonnegative function f_2' on X with $f_2' = f_1'$ on M and $f_2' = g$ on Y. Applying A to $M \cup Y$ and f_2' we get the desired function

4.3.4. Definition. Let a presheaf $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \mid \varrho_{UV} \mid X\}$ from CLOS and its covering space $\mathscr{P} = (P, st_x^*)$ with the stalks $\mathscr{I}_x = (I_x, t_x^*)$ over $x \in X$ be given.

A. If $x, y \in X$, $\alpha \in I_x$, $\beta \in I_y$ then α and β are said to be *relative* ($\alpha \simeq \beta$) if there is a connected $U \in \mathscr{B}(X)$ with $x, y \in U$ and $a \in X_U$ so that $\xi_{U,x}(a) = \alpha$, $\xi_{U,y}(a) = \beta$. (The relation $\simeq$ is not necessarily an equivalence. Clearly, if $x = y$ and $\alpha \simeq \beta$ then $\alpha = \beta$.)

B. If $M \subset X$, we put $P_M = \bigcup\{I_x \mid x \in M\}$. If f is a function on P_M, $N \subset M$, $x \in M$, we set $f_N = f/P_N, f_x = f_{\{x\}}$. An M – function with respect to the relation $\simeq$ is defined to be a nonnegative function f on P_M such that

1) f_x is continuous on every stalk $\mathscr{I}_x$ with $x \in M$,

2) if $\gamma, \delta \in P_M$, $\gamma \simeq \delta$, then $f(\gamma) = f(\delta)$ (it means the function $\varphi(x) = f \circ \xi_{Ux}(a)$ is constant on $M \cap U$ for every connected $U \in \mathscr{B}(X)$ and any $a \in X_U$).

If $\alpha \in P_M$ then an M – function f with respect to $\simeq$ is called the *M – function for α with respect to* $\simeq$, if f_x is a γ – function in $\mathscr{I}_x$ (see 4.3.1) whenever $x \in M$, $\gamma \in I_x$, $\gamma \simeq \alpha$.

If there is no danger of misunderstanding we write only M – function, M – function for α, leaving out the words "with respect to the relation $\simeq$".

C. $\mathscr{S}$ is said to be *connectedly projective* if the following proposition holds: "Given a connected $U \in \mathscr{B}(X)$, an open cover $\mathscr{V}$ of U and a $\mathscr{V}$ – smooth family $\{a_V \in X_V \mid V \in \mathscr{V}\}$ (see 4.2.1), then there is $a \in X_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$."

4.3.5. Remark. A. If f, g are M-functions for α, then so are $\max(f, g)$, $\min(f, g)$.

B. The relation $\simeq$ is an equivalence in P if $\mathscr{S}$ is connectedly projective and either all ξ_{UV} are $1-1$ or the topology in X is made by an order and ϱ_{UV} are $1-1$ for connected U.

C. If all the ϱ_{UV} are $1-1$ then $\mathscr{S}$ is not projective (see 4.2.1).

D. If $\mathscr{S}$ fulfils COND from 4.1.1E and $\simeq$ is an equivalence then ϱ_{UV} are $1-1$ for all connected $U \in \mathscr{B}(X)$.

Proof. B. If $\alpha \simeq \beta \simeq \gamma$, $\alpha \in I_x$, $\beta \in I_y$, $\gamma \in I_z$, then there are connected $U, V \in \mathscr{B}(X)$ provided $x \in U$, $y \in U \cap V$, $z \in V$ and $a \in X_U$, $b \in X_V$ with $\xi_{Ux}(a) = \alpha$, $\xi_{Uy}(a) = \xi_{Vy}(b) = \beta$, $\xi_{Vz}(b) = \gamma$. Setting $U \cap V = W$, $a' = \varrho_{UW}(a)$, $b' = \varrho_{VW}(b)$, we have $\xi_{Wy}(a') = \beta = \xi_{Wy}(b')$. If all the ϱ_{UV} are $1-1$ we get $a' = b'$. If the topology in X is made by an order then W is connected, so again $a' = b'$. As $Z = U \cup V$ is connected and $\mathscr{S}$ is connectedly projective, there is $c \in X_Z$ with $\varrho_{ZU}(c) = a$, $\varrho_{ZV}(c) = b$. Since $\xi_{Zx}(c) = \alpha$, $\xi_{Zz}(c) = \gamma$ we have $\alpha \simeq \gamma$.

C. If $\mathscr{S}$ is projective, $U, V \in \mathscr{B}(X)$, $U \cap V = \emptyset$, $a \in X_U$, $b, c \in X_V$, $b \neq c$, $Z = U \cup V$, then there are $d, e \in X_Z$ with $\varrho_{ZV}(d) = b$, $\varrho_{ZV}(e) = c$, $\varrho_{ZU}(d) = \varrho_{ZU}(e) = a$. As $b \neq c$, we have $d \neq e$, so ϱ_{ZU} is not $1-1$.

D. If $U \in \mathscr{B}(X)$ is connected, $V \subset U$ open, $a, b \in X_U$, $\varrho_{UV}(a) = \varrho_{UV}(b)$, $x \in U$, $y \in V$, $\alpha = \xi_{Ux}(a)$, $\beta = \xi_{Ux}(b)$, $\gamma = \xi_{Uy}(a) = \xi_{Uy}(b)$ then $\alpha \simeq \gamma \simeq \beta$, hence $\alpha = \beta$, as $\alpha \simeq \beta$ (see 4.3.4A). Thus $\xi_{Ux}(a) = \xi_{Ux}(b)$ for all $x \in U$. By COND, $a = b$ which proves D. A is clear.

4.3.6. Lemma. *Let* $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ *be a presheaf from* CLOS *with* UCP (*see* 4.3.1), *which is endowed with a family* $\mathscr{E} = \{F_U \subset C(\mathscr{X}_U \to R) \mid U \in \mathscr{B}(X)\}$ *such that* $\mathscr{S}$ *fulfils the conditions* P1–P3 aqd $\mathscr{E}$ *the conditions* F1–F3 *below*:

P1: *Every* $x \in X$ *has a well ordered filter base* $\langle Ax \leqq \rangle$ *of its open* nbds.

P2: *If* $\mathscr{P} = (P, st_x^*)$ *is the covering space of* $\mathscr{S}$ *then the relation* $\simeq$ *is an equivalence in* P.

P3: *Given a component* M *of* X, $x, y \in M$ *and* $W \in Ax$, $W \subset M$, *then for* $B(W, y) = \{U \in \mathscr{B}(X) \mid U$ *connected*, $W \subset U$, $y \in U\}$ *we have* $\mathscr{X}_W = \varinjlim \{\mathscr{X}_U|\varrho_{UV}| \langle B(W, y), \leqq\rangle\}$ (*notice that* $B(W, y)$ *is not always right directed*: *the inductive limit is meant in the sense of* 0.12).

F1: *If* $x \in X$, $V \in A_x$, $a \in X_V$ *then there is* $W \in Ax$ *with* $W \subset V$ *such that there is a* $\varrho_{VW}(a)$ – *function* $f \in F_W$ (*see* 4.3.1; *this holds namely if* $\mathscr{S}_{Ax}$ *has* ZP *and* $F_V \supset C^*(\mathscr{X}_V \to R)$ *for any* $x \in X$ *and any* $V \in Ax$).

F2: *Given* $x \in X$, $V \in Ax$, $a \in X_V$ *and an* a *– function* $f \in F_V$, *then there is a* $\varrho_{VV+1}(a)$ *– function* $g \in F_{V+1}$ *with* $\varrho^*_{VV+1} g = f$ *(*$V + 1$ *is the follower of* V *in* $\langle Ax \leqq \rangle$*; this holds namely if* $\mathscr{S}_{Ax}$ *is normal, has* ZP, $F_V \supset C^*(\mathscr{X}_V \to R)$ *and* $\varrho_{VV+1} X_V$ *is* τ_{V+1} *– closed for every* $x \in X$ *and any* $V \in Ax$ *– see* 4.3.3C).

F3: *Either x is of countable local character or the following holds: Given* $U \in Ax$ *such that the predecessor* $U - 1$ *of* U *in* $\langle Ax \leqq \rangle$ *does not exist,* $W \in Ax$ *with* $U \subset W$, $a \in X_W$ *and a thread* $\mathscr{F} = \{f_V \mid V \in Ax,\ U \subset V \subset W\}$ *through* $\mathscr{E}$ *such that every* f_V *is a* $\varrho_{WV}(a)$ *– function on* X_V, *then there is a* $\varrho_{WU}(a)$ *– function* f *on* X_U *with* $\varrho^*_{VU} f = f_V$ *for all* $V \in Ax$ *with* $U \subset V \subset W$.

F4: *If* $x \in X$, $V \in Ax$, $U \in \mathscr{B}(X)$ *connected with* $V \subset U$, *then* ϱ^*_{UV} *sends* F_V *into* F_U.

F5: *Let* M *be a component of* X, $x, y \in M$, $W \in Ax$. *If* $\mathscr{F} = \{f_V \mid V \in B(W, y)\}$ *is a thread through* $\mathscr{E}$ *(see* P2) *ihen* $\varinjlim \mathscr{F} \in F_W$. (F4, F5 *hold if* $F_V = C^*(\mathscr{X}_V \to R)$ *or* $F_V = C(\mathscr{X}_V \to Q)$ *for all* $V \in \mathscr{B}(X)$.)

Let M *be a component of* X, $x \in M$, $\alpha \in I_x$. *Then*

A: *There is an* M *– function* f *for* α *such that* $\{f_z \circ \xi_{U,z} \mid U \in Az\}$ *is a thread through* $\mathscr{E}_{Az}$ *for all* $z \in M$. *Further, given an* α *– function* g *on* I_x *such that* $\{g \circ \xi_{V,x} \mid V \in Ax\}$ *is a thread through* $\mathscr{E}_{Ax}$, *then there is a unique* M *– function* f *for* α *with* $f_x = g$. *Furthermore, given* $\varepsilon > 0$, $\delta \geqq 0$, $y \in M$, $\beta \in I_y$ *and an* M*–function* g' *with* $f \geqq \max(g', \delta)$ *on* $\{\gamma \in I_y \mid |g'(\gamma) - g'(\beta)| \geqq \varepsilon\}$, *then* $f \geqq \max(g', \delta)$ *on* $\{\gamma \in P_M \mid |g'(\gamma) - g'(\beta)| \geqq \varepsilon\}$.

B: *Let* $S(\alpha)$ *be the set of all the* M *– functions for* α *such that* $\{f \circ \xi_{U,x} \mid U \in Ax\}$ *is a thread through* $\mathscr{E}_{Ax}$. *If for every* $V \in Ax$ *the set* F_V *is closed under maximum and minimum of two functions from* F_V, *then so is* $S(\alpha)$.

C: *If* $y \in M$ *then there is a unique* $\beta = h_{xy}(\alpha) \in I_y$ *with* $\alpha \simeq \beta$. *The map* $h_{xy} : \mathscr{I}_x \to \mathscr{I}_y$ *is a homeomorphism. This statement follows only form* P2, P3.

Proof. Let $U \in Ax$ be such that there is $a \in X_U$ with $\xi_{U,x}(a) = \alpha$. By F1, we can assume that there is an a – function $f_U \in F_U$. As P1, F1–F3 hold, a thread $\mathscr{F} = \{f_V \mid V \in Ax,\ V \subset U\}$ through $\mathscr{E}_{Ax}$ can be made by induction similarly as in 1.1.7, so that every $f_V \in \mathscr{F}$ be a $\varrho_{UV}(a)$ – function on X_V. Then $g = \varinjlim \mathscr{F}$ is an α – function on I_x and $\{g \circ \xi_{V,x} \mid V \in Ax\}$ is a thread through $\mathscr{E}_{Ax}$.

Now, let g be any α – function on I_x such that $\{g_V = g \circ \xi_{V,x} \mid V \in Ax\}$ is a thread through $\mathscr{E}_{Ax}$. By the maximality principle, there is a maximal set $N \subset M$ and an N – function h for α with $h_x = g$ (see 4.3.4B) such that $\{h^z_V = h_z \circ \xi_{V,z} \mid V \in Az\}$ is a thread through $\mathscr{E}_{Az}$ for every $z \in N$. If $y \in M - N$, $Vy \in Ay$ with $Vy \subset M$, $a \in X_{Vy}$, $z \in N$, then by P3, there is a connected $U \in \mathscr{B}(X)$ with $Vy \subset U$, $z \in U$ and $b \in X_U$ so that $\varrho_{UVy}(b) = a$ (such aVy exists, for the components of X are open since X is locally connected as $\mathscr{S}$ has UCP – see 4.3.1). Further, there is $Vz \in Az$ with $Vz \subset U$. We can put

$$f^{U,z,b}_{Vy}(a) = h^z_U(b) = h_z \circ \xi_{U,z}(b) \tag{E}$$

which defines the function $f_{Vy}^{U,z,b}(h_U^z)$ on $\varrho_{UVy}(X_U)\,(X_U)$. As $h_U^z = \varrho_{UVz}^* h_{Vz}^z$ and $h_{Vz}^z \in F_{Vz}$, we have $h_U^z \in F_U$ by F4. We shall prove that $f_{Vy}^{U,z,b}$ does not depend on U, z, b. If $U' \in \mathscr{B}(X)$ is connected, $v \in N$ and $c \in X_{U'}$ with $\varrho_{U'Vy}(c) = a$, then for $\gamma = \xi_{U,z}(b)$, $\delta = \xi_{U',v}(c)$, $\eta = \xi_{U,y}(b) = \xi_{U,y}(c)$ we have $\gamma \simeq \eta \simeq \delta$. By P2, $\gamma \simeq \delta$, so $h_z(\gamma) = h_v(\delta)$. Thus by (E), $f_{Vy}^{U',v,c}(a) = h_v(\delta) = h_z(\gamma) = f_{Vy}^{U,z,b}(a)$ as desired.

Thus we can define a function f_{Vy} on X_{Vy} in this way: If $a \in X_{Vy}$ then we can take an arbitrary $z \in N$. By P3, there is a connected $U \in \mathscr{B}(X)$ with $z \in U$, $Vy \in U$ and $b \in X_U$ with $\varrho_{UVy}(b) = a$. Then we can put $f_{Vy}(a) = f_{Vy}^{U,z,b}(a) = h_U^z(b)$. We have just shown that this choice does not depend on U, b and z.

Now we shall show that $\mathscr{F}(Vy, z) = \{h_U^z \mid U \in B(Vy, z)\}$ is a thread through $\mathscr{E}_{B(Vy,z)}$. By (E), if $U, U' \in B(Vy, z)$, $U \subset U'$ then $\varrho_{U'U}^* h_U^z = h_U^z \circ \varrho_{U'U} = h_z \circ \xi_{U,z} \circ \varrho_{U'U} = h_z \circ \xi_{U'z} = h_{U'}^z$ as desired.

Now we show that $f_{Vy} \in F_{Vy}$. By (E), $f_{Vy} \circ \varrho_{UVy} = h_U^z \in F_U$ for all $U \in B(Vy, z)$ (notice that we have $a = \varrho_{UVy}(b)$ in (E)), which together with P3 gives $f_{Vy} = \varprojlim \mathscr{F}(Vy, z)$. As $\mathscr{F}(Vy, z)$ is a thread through $\mathscr{E}_{B(Vy,z)}$, we get $f_{Vy} \in F_{Vy}$ by F5.

Now we show that $\mathscr{F}y = \{f_V \mid V \in Ay\}$ is a thread through $\mathscr{E}_{Ay}$. If $V, W \in Ay$, $W \subset V$, U open and connected with $V \subset U$, $a \in X_V$, $b \in X_U$, $\varrho_{UV}(b) = a$, then by (E), $f_V(a) = h_U^z(b) = f_W(\varrho_{VW}(a)) = \varrho_{VW}^* f_W(a)$, so $\varrho_{VW}^* f_W = f_V$ as desired. If $f' = \varprojlim \mathscr{F}y$, $f = g$ on $N, f = f'$ on I_y, then f is an $N \cup \{y\}$ – function for α, so $M = N$.

In order to prove that f is unique we need to prove the statement C. By P3, there is a connected $U \subset M$ with $x, y \in U$ and $a \in X_U$ with $\xi_{U,x}(a) = \alpha$. Then $\beta = \xi_{U,y}(a) \simeq \alpha$. If $\gamma \in I_y$, $\gamma \simeq \alpha$ then $\gamma \simeq \alpha \simeq \beta$, so $\gamma \simeq \beta$, hence $\gamma = \beta$. We set $h_{xy}(\alpha) = \beta$. Since $h_{xy}(\alpha)$ is unique and the relation $\simeq$ is symmetric, we get $h_{yx} \circ h_{xy} =$ identity. Inter changing x and y we get $h_{xy} \circ h_{yx} =$ identity, so h_{xy} is a $1-1$ map onto I_y. Further, $h_{xy} : \mathscr{I}_x \to \mathscr{I}_y$ is continuous iff so is $h_V = h_{xy} \circ \xi_{V,x} = \mathscr{X}_V \to \mathscr{I}_y$ for all $V \in Ax$. If $V \in Ax$, then by P3, h_V is continuous iff so is $g_U = h_V \circ \varrho_{UV} : \mathscr{X}_U \to \mathscr{I}_y$ for all $U \in B(V, y)$. But $g_U = \xi_{U,y}$ and it is continuous, so C follows.

Now the uniqueness of f follows from the following statement: "If g, h are two M – functions such that there is $z \in M$ with $g = h$ on I_z, then $g = h$ on P_M. "Indeed, if $y \in M$, $\alpha \in I_y$, then by C, there is $\beta \in I_z$ with $\alpha \simeq \beta$, so $g(\alpha) = g(\beta) = h(\beta) = h(\alpha)$ as desired.

The last statement we need is S: "Given M – functions f, g, f_1, g_1 and $z \in M$ such that $f_1 \geqq g_1$ on $\{\gamma \in I_z \mid f(\gamma) \geqq g(\gamma)\}$, then $f_1 \geqq g_1$ on $\{\gamma \in P_M \mid f(\gamma) \geqq g(\gamma)\}$." Indeed, if $\gamma \in P_M$ with $f(\gamma) \geqq g(\gamma)$ then by C, there is $\beta \in I_z$ with $\gamma \simeq \beta$. As $f(\beta) = f(\gamma) \geqq g(\gamma) = g(\beta)$, we have $f_1(\beta) \geqq g_1(\beta)$, so $f_1(\gamma) \geqq g_1(\gamma)$ as desired. Now the last statement of (A) follows from the fact that $|g' - g'(\beta)|$, $\varepsilon, f, \max(g', \delta)$ are M – functions, and from S.

We prove B. If $f, g \in S(\alpha)$, $h = \max(f, g)$, then by 4.3.5A, h is an M – function for α. If $V \in Ax$ then $h_V = h \circ \xi_{V,x} = \max(f \circ \xi_{V,x}, g \circ \xi_{V,x}) \in F_V$, so $\{h_V \mid V \in Ax\}$ is a thread through $\mathscr{E}_{Ax}$. By A, there is a unique $h' \in S(\alpha)$ with $h' = h$ on I_x, so $h' = h$ which completes the proof.

4.3.7. Remark. A. In 4.3.6 we assume that $\mathscr{S}$ has UCP. It means that X is locally connected and that ϱ_{UV} are $1-1$ for connected U. But in the proof we never used the fact that these ϱ_{UV} are $1-1$ or that M are components. Thus it can be easily seen that the conditions of 4.3.6 can be weakened. Instead of UCP we can only assume that every $x \in X$ has a connected open nbd K so that the conditions P1–P3, F1–F5 of 4.3.7 hold for K instead of M, and in the statement as well as in the proof replace M by K.

Another weak point of the foregoing lemma is the condition P2 which is very often not fulfilled (see 4.3.5). Nevertheless, in this point we can take the lemma as a method for formulating many other similar ones for we can take another relation $\simeq$ (for instance "$\alpha \simeq \beta$ iff there is a relatively compact $U \in \mathscr{B}(X)$ and $a \in X_U$ with $\alpha = \xi_{U,x}(a)$, $\beta = \xi_{U,y}(a)$", which is an equivalence if $\mathscr{S}$ is projective and if ϱ_{UV} are $1-1$ for relatively compact U) and adopt the conditions, the statement and the proof to it. In this way we can get many similar lemmas. This method can be expressed in a one piece which we do in part C of the remark. Instead of UCP the following notion will be useful there: "Given a nonempty set $\mathscr{D}(X) \subset \mathscr{B}(X)$, $\alpha \in I_x$, $\beta \in I_y$, then α and β are $\mathscr{D}(X)$ – relative if there is $U \in \mathscr{D}(X)$ and $a \in X_U$ with $\xi_{U,x}(a) = \alpha$, $\xi_{U,y}(a) = \beta$."

Instead of P2 we may assume that the $\mathscr{D}(X)$ – relation is an equivalence in P. Likewise as in 4.3.4 we can define the M – functions for α and the M – functions with respect to the $\mathscr{D}(X)$ – relation.

B. A question arises when the $\mathscr{D}(X)$ – relation is an equivalence. To answer it, the following property can be useful, which can be called the "$\mathscr{D}(X)$ – projectivity" and which is formulated as follows: "Given $U \in \mathscr{D}(X)$, an open cover $\mathscr{V}$ of U and a $\mathscr{V}$ – smooth family $\{a_V \in X_V \mid V \in \mathscr{V}\}$ then there is $a \in X_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$". Likewise as in 4.3.5 we can prove this statement: Let $\mathscr{S}$ be $\mathscr{D}(X)$ – projective. Then the $\mathscr{D}(X)$ – relation is an equivalence if either ϱ_{UV} are $1-1$ for all the $U \in \mathscr{D}(X)$ and $V, W \in \mathscr{D}(X)$, $V \cap W \neq \emptyset$ implies $V \cup W$, $V \cap W \in \mathscr{D}(X)$, or if all the ϱ_{UV} are $1-1$ and either $\mathscr{S}$ is projective or $V \cup W \in \mathscr{D}(X)$ if $V, W \in \mathscr{D}(X)$, $V \cap W \neq \emptyset$.

C. The generalization of 4.3.6 proceeds like this: Let the family $\mathscr{E}$ and the bases Ax from 4.3.6 fulfil the conditions P1, F1–F3 and P2′: There is an open cover $\mathscr{K}$ of X such that for every $K \in \mathscr{K}$ there is a nonempty set $\mathscr{D}(K) \subset \mathscr{B}(K)$ so that the $\mathscr{D}(K)$ – relation (see 4.3.7A) in P_K is an equivalence,

P3′: Given $y, z \in K$, $W \in Ay$, $W \subset K$, then

$$\mathscr{X}_W = \varliminf \{\mathscr{X}_U | \varrho_{UV} | \langle B_K(W, z) = \{U \in \mathscr{D}(K) \mid W \subset U \subset K,\ z \in U\} \leqq \rangle\},$$

F4′: If $K \in \mathscr{K}$, $x \in K$, $V \in Ax$, $U \in \mathscr{D}(K)$ with $V \subset U$, then ϱ^*_{UV} sends F_V into F_U,

F5′: Given $y, z \in K$, $W \in Ay$ then $\varliminf \mathscr{F} \in F_W$ for any thread $\mathscr{F}$ through $\mathscr{E}_{B_K(W,z)}$.

Given $K \in \mathscr{K}$, $x \in K$, $\alpha \in I_x$, then there is a unique K – function f^K for α with respect to the $\mathscr{D}(K)$ – relation (see 4.3.4) such that $\{f^K \circ \xi_{U,z} \mid U \in Az\}$ is a thread through $\mathscr{E}_{Az}$ for all $z \in K$, and that the statements 4.3.6A, B, C hold if we replace the word "M – function" by "K – function" and the equivalence $\simeq$ by the $\mathscr{D}(K)$ – equivalence in P_K.

The proof of 4.3.7C follows directly from that of 4.3.6.

The purpose of the foregoing remark is to extend every α – function g on $\mathscr{I}_x$ to a K – function for α, where K is an open nbd of x (which may depend on α, g).

If we have $F_U = C^*(\mathscr{X}_U \to R)$ or $F_U \subset C(\mathscr{X}_U \to Q)$ for all $U \in \mathscr{B}(X)$, $x, y \in K$ then the homeomorphism h_{xy} carries every α – function on $\mathscr{I}_x$ onto an $h_{xy}(\alpha)$ – function on $\mathscr{I}_y$; thus g can be extended to a K – function with the help of those h_{xy}, so that we need not use the conditions F4 F5 (F4′, F5′) from 4.3.6 (4.3.7C). If F_U are defined in this way, we can assume directly that $\mathscr{I}_x$ and $\mathscr{I}_y$ are homeomorphic under a homeomorphism h_{xy} so that $h_{xz} = h_{yz} \circ h_{xy}$, $h_{xy} \circ h_{yx} =$ identity for all $x, y, z \in K$. Clearly, a sufficient condition for this is "$\mathscr{X}_W = \varinjlim \{\mathscr{X}_U | \varrho_{UV}| \langle\{U \in \mathscr{B}(X) \mid y \in U\} \leqq \rangle\}$ for all sufficiently small $W \in Ax$ and any $y \in K$". By 1.4.5, this is fulfilled if P3 or P3′ from 4.3.6 or from 4.3.7, respectively, holds.

4.3.8. Proposition. *Let $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) |\varrho_{UV}| X\}$ be a presheaf from* CLOS *which fulfils* COND *from* 4.1.1E. *Suppose that the following conditions hold:*

1) *For every $x \in X$ and any $\alpha \in I_x$ there is a nonempty set $S(\alpha)$ whose elements are pairs (V, f), where V is an open* nbd *of x and f is a function on P_V (see* 4.3.4B), *which satisfies a)–d):*

a) *If $(V, f) \in S(\alpha)$ then for any open* nbd *$W \subset V$ of x we have $(W, f/W) \in S(\alpha)$.*

b) *If (V, f), $(W, g) \in S(\alpha)$ then there is an open* nbd *$Z \subset V \cap W$ of x with* $(Z, \max(f, g))$, $(Z, \min(f, g)) \in S(\alpha)$.

c) *Given $(V, g) \in S(\alpha)$, $y \in V$, $\beta \in I_y$, $\varepsilon > 0$, $\delta \geqq 0$, then there is $(W, f) \in S(\beta)$ with $W \subset V$ such that $f \geqq \max(g, \delta)$ on $\{\gamma \in P_W \mid |g(\gamma) - g(\beta)| \geqq \varepsilon\}$.*

d) *If $(K, f) \in S(\alpha)$ then f_x is an α – function on $\mathscr{I}_x$ and for every $y, z \in K$ there is a homeomorphism h_{yz} of $\mathscr{I}_y$ onto $\mathscr{I}_z$ with $h_{yz} \circ h_{zy} =$ identity such that for $\beta \in I_y$, $\gamma \in I_z$ we have $f(\beta) = f(\gamma)$ if $\gamma = h_{yz}(\beta)$, and that for $a \in X_K$ there is an open* nbd *$V \subset K$ of x such that $y, z \in V$, $\beta = \xi_{K,y}(a)$, $\gamma = \xi_{K,z}(a)$ implies $\gamma = h_{yz}(\beta)$ (clearly, this holds if there is an open cover $\mathscr{D}(K)$ of K such that the $\mathscr{D}(K)$ – relation (see* 4.3.4, 4.3.7A) *is an equivalence in P_K and that* P3′ *from* 4.3.7C *holds).*

2) *There is an open cover $\mathscr{V}$ of X such that every $V \in \mathscr{V}$ has* ZP *(see* 4.3.1).

3) *In every stalk $I_x \subset P$ there is a closure $t_x^\#$ with the following properties:*

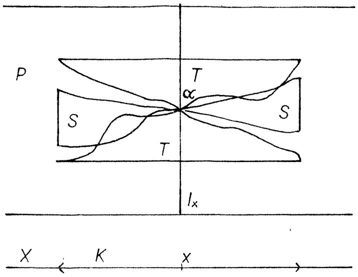
a) *All the canonical maps $p_U : \mathscr{X}_U \to (A_U, b_U(st_x^\#))$ are homeomorphisms.*

b) *Given $\alpha \in I_x$ and a $t_x^\#$* – nbd *L of x, then there is $(V, g) \in S(\alpha)$ and $\varepsilon > 0$ such that $M_\varepsilon = \{\gamma \in I_x \mid g(\gamma) < \varepsilon\} \subset L$.*

Then there is a Hausdorff topology t in P such that all the canonical maps $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ are homeomorphisms and $A_U \subset \Gamma(U, t)$ (see 4.1.1F) *for all $U \in \mathscr{B}(X)$.*

If every $V \in \mathscr{V}$ is a normal topological space, X has countable local character and $\mathscr{S}$ is $\mathscr{D}(X)$ – projective (see 4.3.7B) or projective, then $\Gamma(U, t) = A_U$ for all $U \in \mathscr{D}(X)$ or for all $U \in \mathscr{B}(X)$, respectively. Further, (I_x, t_x) is functionally separated by $Lx = \{f \mid$ there is $\alpha \in I_x$ and V with $(V, f) \in S(\alpha)\}$ ($t_x = t/I_x$ – see 0.14).

Proof. Let $\alpha \in I_x$, $\varepsilon > 0$. By (1), there is $(K, f) \in S(\alpha)$. By 1a and 2, there is an x – function on K. We put $S(\alpha \mid f < g \mid K) = \{\gamma \in P_K \mid f(\gamma) < g\, p(\gamma)\}$, $S(\alpha \mid g < f < \varepsilon \mid K) = \{\gamma \in P_K \mid g\, p(\gamma) < f(\gamma) < \varepsilon\}$ (p is the canonical projection of P onto X, i.e., if $x \in P$, $\alpha \in I_x$, then $p(\alpha) = x$). If $(K, f_1) \in S(\alpha)$ and if g_1 is another x – function on K, we set $N(\alpha \mid f, g, K, \varepsilon, f_1, g_1) = S(\alpha \mid f < g \mid K) \cup S(\alpha \mid g_1 < f_1 < \varepsilon \mid K) \cup \{\alpha\}$. It is sketched in the following picture (S and T is the first and the second set in the union respectively):



We can easily see from 1a, 1b that the set $N(\alpha)$ of all $N(\alpha \mid f, g, K, \varepsilon, f_1, g_1)$ is a filter base round α. This holds for every $\alpha \in P$ so that we get a closure t in P. We shall show that the sets from $N(\alpha)$ are t – open, henceforth t is a topology.

A: Let $\beta \in S(\alpha \mid f < g \mid K)$, $\beta \in I_y$. Then by 1d, $0 \leqq f(\beta) < g(y)$. We have to find $(K', f_1) \in S(\beta)$ and a y – function g_1 on K' so that $\gamma \in P_{K'}, f_1(\gamma) < g_1\, p(\gamma)$ implies $f(\gamma) < g\, p(\gamma)$. There is $\varepsilon > 0$ such that $f(\beta) + \varepsilon < g(y)$. By 1c, there is $(K', f_1) \in S(\beta)$ with $K' \subset K$ and with $f_1 \geqq f$ on $\{\gamma \in P_{K'} \mid f(\gamma) \geqq f(\beta) + \varepsilon\}$. By 2 and 1a, we may assume that there is a y – function g_1 on K' with $g_1 < f(\beta) + \varepsilon < g$ on K'. Thus if $\gamma \in P_{K'}$ and $f(\gamma) \geqq g\, p(\gamma)$, then $g_1\, p(\gamma) < f(\beta) + \varepsilon < g\, p(\gamma) \leqq f(\gamma) \leqq f_1(\gamma)$, so $g_1\, p(\gamma) \leqq f_1(\gamma)$ which shows that for all $\gamma \in P_{K'}$ with $f_1(\gamma) < g_1\, p(\gamma)$ we have $f(\gamma) < g\, p(\gamma)$ as desired.

B: Now we want to find $\delta > 0$, $(K', f_1') \in S(\beta)$ and a y – function g_1' on K' (K' is from A) such that $f(\gamma) < g\, p(\gamma)$ if $\gamma \in P_{K'}$, $g_1'\, p(\gamma) < f_1'(\gamma) < \delta$. We set $\delta = \varepsilon$ where ε is from A. In A we have found $(K', f_1) \in S(\beta)$ with $\varepsilon \leqq f(\beta) + \varepsilon < g\, p(\gamma) \leqq f(\gamma) \leqq f_1(\gamma)$ if $\gamma \in P_{K'}$, $f(\gamma) \geqq g\, p(\gamma)$. Thus if $\gamma \in P_{K'}$, $f_1(\gamma) < \varepsilon$ then $f(\gamma) < g\, p(\gamma)$. Hence $S(\beta \mid g_1' < f_1 < \varepsilon \mid K') \subset S(\alpha \mid f < g \mid K)$ for any y – function g_1' on K' as desired.

C: Let $\beta \in S(\alpha \mid g < f < \varepsilon \mid K)$, $\beta \in I_y$. We shall find $(K', f_1) \in S(\beta)$ and a y – function g_1 on K' such that $g\, p(\gamma) < f(\gamma) < \varepsilon$ if $\gamma \in P_{K'}$, $g_1\, p(\gamma) < f_1(\gamma) < \varepsilon$.

As $0 \leqq g(y) < f(\beta) < \varepsilon$, there is $\eta > 0$ with $f(\beta) + \eta < \varepsilon$, $g(y) + \eta < f(\beta)$. By 1c, there is $(K', f_1) \in S(\beta)$ with $K' \subset K$, $f_1 \geqq \max(f, \varepsilon)$ on $\{\gamma \in P_{K'} \mid |f(\gamma) - f(\beta)| \geqq \eta\}$. By 1a, we may assume $g < f(\beta) - \eta$ on K'. Now, if $\gamma \in P_{K'}$, $f(\gamma) \geqq \varepsilon$ then $f(\beta) + \eta < \varepsilon \leqq f(\gamma) \leqq f_1(\gamma)$, so $f_1(\gamma) \geqq \varepsilon$. Thus $f(\gamma) < \varepsilon$ if $\gamma \in P_{K'}$, $f_1(\gamma) < \varepsilon$. If $f(\gamma) \leqq g\ p(\gamma)$, $\gamma \in P_{K'}$ then $f(\gamma) \leqq f(\beta) - \eta$ so $f_1(\gamma) \geqq \varepsilon$, hence $f_1(\gamma) < \varepsilon$ implies $g\ p(\gamma) < f(\gamma)$. Thus $S(\beta \mid g_1 < f_1 < \varepsilon \mid K') \subset S(\alpha \mid g < f < \varepsilon \mid K)$ for any y – function g_1 on K'.

D: Now we will find a y – function g_1' on K' such that $\gamma \in P_{K'}$, $f_1(\gamma) < g_1'\ p(\gamma)$ implies $g\ p(\gamma) < f(\gamma) < \varepsilon$ (K' f_1, ε are from C). Take a y – function g_1 on K' with $g_1 < \varepsilon$. Then for $\gamma \in P_{K'}$ with $f_1(\gamma) < g_1\ p(\gamma)$ we have $f_1(\gamma) < \varepsilon$, so $g\ p(\gamma) < f(\gamma) < \varepsilon$ by C. Thus $S(\beta \mid f_1 < g_1 \mid K') \subset S(\alpha \mid g < f < \varepsilon \mid K)$ which completes the proof of the openness of the t – nbds of the points of P. Thus t is a topology.

It easily follows from 1c that every $f \in Lx$ is t_x – continuous. Thus (I_x, t_x) is functionally separated by Lx, hence t_x is Hausdorff. Thus so is t as well.

Let $\alpha \in P$, $\alpha \in I_x$. By 3b, the topology t_x induced in I_x by t is finer than the closure $t_x^{\#}$, so all the $p_U^{-1} : (A_U, b_U(t)) \to \mathscr{X}_U$ are continuous. If $U \in \mathscr{B}(X)$ then $\xi_{U,x} : \mathscr{X}_U \to (I_x, t_x)$ are continuous for all $x \in U$ as t_x is coarser than t_x^* (recall that if $x \in X$, $\alpha \in I_x$, $(V, f) \in S(\alpha)$, then f_x is an α – function on $\mathscr{I}_x$, hence it is t_x^* – continuous – see 1d). By 4.1.4A or 4.1.5, all the $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ are continuous, so they are homeomorphisms. The definition of t directly implies $A_U \subset \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$.

Let every $V \in \mathscr{V}$ be normal, and let X have countable local character. Given $U \in \mathscr{B}(X)$, $x \in U$, $a \in X_U$ and a continuous section $r : U \to (P, t)$ such that $\alpha = r(x) = \xi_{U,x}(a)$, then $N = \{y \in U \mid r(y) = \xi_{U,y}(a)\} \neq \emptyset$ as $x \in N$. The set $\{\xi_{U,y}(a) \mid y \in U\}$ is closed in (P_U, t) as t is Hausdorff. Thus N is closed in U. We want to show that $N = U$ if U is connected. We will prove the statement S: "Every $y \in N$ has an open nbd $W \subset U$ such that $r(z) = \xi_{U,z}(a)$ in W" (here U need not be connected).

Let $y \in N$. There is $(K, f) \in S(\beta)$, where $\beta = r(y)$. We may assume $K \subset U$. Suppose on the contrary that in every open nbd $W \subset K$ of y there is x_W with $r(x_W) \neq \xi_{U,x_W}(a)$. The function $g = f \circ r$ is continuous on K and by 1d, there is an open nbd $V \subset K$ of y such that $f^b(z) = f \circ \xi_{V,z}(b)$ is constant on V for every $b \in X_V$. By 1a, we may assume $V = K$. Thus by 1d, $g(z) = 0$ iff $r(z) = \xi_{U,z}(a)$, so $\{z \in K \mid \xi_{U,z}(a) = r(z)\} = \{z \in K \mid g(z) = 0\}$. There is $V \subset \mathscr{V}$ with $y \in V$. We may assume $K \subset V$. As X has countable local character, we may assume by 1d that there is a sequence $\{x_n\}$ in K which tends to y so that $g(x_n) > 0$ for all n. As g is continuous on the closed set $M = \{x_n\} \cup \{y\}$ so by the normality of V and 4.3.3C, there is a y – function h on V with $h = g$ on M. Then $r(x_n) \notin N(\beta \mid f, h, K, \varepsilon\ f, h)$ for every n and any $\varepsilon > 0$. Thus $r : U \to (P, t)$ is not continuous, which is the desired contradiction. Thus every $y \in N$ has an open nbd $V \subset K$ of y with $r(z) = \xi_{U,z}(a)$ in V, so N is open and closed hence $N = U$ if U is connected.

Now, if $\mathscr{S}$ is $\mathscr{D}(X)$ – projective and $U \in \mathscr{D}(X)$ then for every $r \in \Gamma(U, t)$ there is $a \in X_U$ and $x \in U$ with $r(x) = \xi_{U,x}(a)$. Indeed, for every $x \in U$ there is an open nbd $Ux \subset U$ of x and $a_x \in X_{Ux}$ with $\xi_{Ux,x}(a_x) = r(x)$. By the statement S, for every

$x \in X$ there is an open nbd $Vx \subset Ux$ of x with $\xi_{Ux,z}(a_x) = r(z)$ for all $z \in Vx$. Then $\mathscr{V} = \{Vx \mid x \in U\}$ is an open cover of U. If $V \in \mathscr{V}$, we set $a_V = \varrho_{U_zV}(a_x)$. If $V, W \in \mathscr{V}$, $Z = V \cap W \neq \emptyset$, $z \in Z$, $b = \varrho_{VZ}(a_V)$, $c = \varrho_{WZ}(a_W)$ then $\xi_{Z,z}(b) = r(z) = \xi_{Z,z}(c)$. As $\mathscr{S}$ fulfils COND from 4.1.1E, we have $b = c$. Since $\mathscr{S}$ is $\mathscr{D}(X)$ – projective and $U \in \mathscr{D}(X)$, thus there is $a \in X_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$ (see 4.3.7). Then $\xi_{U,x}(z) = r(x)$ for all $x \in U$. The proof is thereby complete.

4.3.9. Theorem. *Given a normal and T_1 presheaf $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ (see 2.1.1A) from* TOP, *assume that the following conditions are fulfilled:*

S1: *$\mathscr{S}$ fulfils the condition (2) or (2′) from* 4.2.5, ZP *and* UCP,

S2: *there is an open cover $\mathscr{V}$ of X such that every $V \in \mathscr{V}$ has* ZP,

S3: *the relation $\simeq$ from* 4.3.4 *is an equivalence in P,*

S4: *every $x \in X$ has a well ordered countable filter base $\langle Ax \leqq \rangle$ of its open* nbds *whose ordinal type is ω_0, such that ϱ_{UU+1} maps $\mathscr{X}_U$ onto a τ_{U+1} – closed set $\varrho_{UU+1}(X_U)$ homeomorphically for all $U \in Ax$.*

S5: *if M is a component of X, $x, y \in M$, $W \in Ax$, $W \subset M$, then $\mathscr{X}_W = \varprojlim \mathscr{S}_{B(W,y)}$ ($B(W, y) = \{U \in \mathscr{B}(X) \mid U$ connected, $W \subset U$, $y \in U\}$ – see* P3 *from* 4.3.6).

Then there is a topology t in P such that the canonical maps $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ are homeomorphisms and $A_U \subset \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$. Moreover, if every $V \in \mathscr{V}$ is a normal topological space, X has countable local character and $\mathscr{S}$ is $\mathscr{D}(X)$ – projective (see 4.3.7B), *then $\Gamma(U, t) = A_U$ for all $U \in \mathscr{D}(X)$. Especially, $\Gamma(U, t) = A_U$ for all $U \in \mathscr{B}(X)$ if $\mathscr{S}$ is projective, if every $V \in \mathscr{V}$ is normal and if X has countable local character.*

Proof. We set $\mathscr{E} = \{F_U = C(\mathscr{X}_U \to Q) \mid U \in \mathscr{B}(X)\}$ (Q is the compact unit interval). We show that the conditions of 4.3.6 are fulfilled. Clearly, P1 – P3 of 4.3.6 hold. As $\mathscr{S}$ has ZP, so F1 from 4.3.6 holds. It follows from the normality of $\mathscr{S}$, S4 and 4.3.3C that F2 from 4.3.6 holds while F3 holds as Ax is countable. Clearly, $\mathscr{E}$ fulfils F4, F5. Thus the statement of 4.3.6 holds, hence every $x \in X$ has an open nbd M of x such that for every $\alpha \in P_M$ there is an M – function for α with respect to $\simeq$ (M is the component of X with $x \in M$; M is open as X is locally connected for $\mathscr{S}$ has UCP). Given $x \in X$, $\alpha \in I_x$, we set $S(\alpha) = \{(K, f) \mid K$ is an open nbd of x such that there is a compoment M of X provided $K \subset M$, f is an M – function for $\alpha\}$. As F_U are closed under maximum and minimum of two functions, we get by 4.3.6B that the assumption 1b of 4.3.8 holds. In order to show that the conditions 1c, 3b of 4.3.8 hold, we prove the following statement T:

"Given $x \in X$, $\alpha \in I_x$, $\varepsilon > 0$, $\delta \geqq 0$ and a function g on I_x such that $\{g_V = g \circ \xi_{V,x} \mid V \in Ax\}$ is a thread through $\mathscr{E}_{Ax}$, then for the component M of X with $x \in M$ there is an M – function f for α with respect to $\simeq$ so that $f \geqq \max(g, \delta)$ on $M_\varepsilon = \{\gamma \in I_x \mid |g(\gamma) - g(\alpha)| \geqq \varepsilon\}$ and that $\{f \circ \xi_{V,x} \mid V \in Ax\}$ is a thread through $\mathscr{E}_{Ax}$."

Indeed, there is $W \in Ax$ and $a \in X_W$ with $\xi_{W,x}(a) = \alpha$. We may assume W to be the smallest element of $\langle Ax \leqq \rangle$ and put $a_V = \varrho_{WV}(a)$ for $V \in Ax$. By 4.3.3C, we can by induction construct a thread $\mathscr{F} = \{f_V \mid V \in Ax\}$ through $\mathscr{E}_{Ax}$ so that f_V is an a_V – function in $\mathscr{X}_V$ and that $f_V \geqq \max(g_V, \delta)$ on $\{b \in X_V \mid |g_V(b) - g_V(a_V)| \geqq \varepsilon\}$.

Then $f_1 = \varprojlim \mathscr{F}$ is an α – function on $\mathscr{I}_x$ and $f_1 \geqq \max(g, \delta)$ on M_ε. By 4.3.6A, there is an M – function f for α with $f_x = f_1$ as desired.

We show that the condition 1c of 4.3.8 holds. Given a component M of X, $x \in M$, $\alpha \in I_x$, $(K, g) \in S(\alpha)$, $y \in K$, $\beta \in I_y$, $\varepsilon > 0$, $\delta \geqq 0$, then $g_1 = g/I_y$ is a function on I_y such that $\{g_V = g_1 \circ \xi_{V,y} \mid V \in A_y\}$ is a thread through $\mathscr{E}_{Ay}$. By the statement T, there is $f \in S(\beta)$ with $f \geqq \max(g, \delta)$ on $\{\gamma \in I_y \mid |g(\gamma) - g(\beta)| \geqq \varepsilon\}$. By 4.3.6A, $f \geqq \max(g, \delta)$ on $\{\gamma \in P_M \mid |g(\gamma) - g(\beta)| \geqq \varepsilon\}$ as desired. By S3, S5 and 4.3.6C, we get that 1d of 4.3.8 is fulfilled with $K = M$ (M is connected).

By 2.2.2A and 4.2.5, there is a topology $t_x^\#$ in every stalk I_x which is projectively defined by a set $Dx \subset C^*(\mathscr{I}_x \to R)$, so that all the $p_U : \mathscr{X}_U \to (A_U, b_U(st_x^\#))$ are homeomorphisms and that $\{f \circ \xi_{U,x} \mid U \in Ax\}$ is a thread through $\mathscr{E}_{Ax}$ for every $f \in Dx$. We show that the condition 3b of 4.3.8 holds. Let $\alpha \in I_x$ and let a $t_x^\#$ – nbd L of α be given. As $t_x^\#$ is projectively made by Dx, we may assume that there is $g \in Dx$ with $L = \{\gamma \in I_x \mid |g(\gamma) - g(\alpha)| < \varepsilon\}$. Then $\mathscr{F} = \{g_V = g \circ \xi_{V,x} \mid V \in Ax\}$ is a thread through $\mathscr{E}$. We set $\delta = \varepsilon$. By the statement T and 4.3.6A, there is $(M, f) \in S(\alpha)$ with $f \geqq \max(g, \delta) \geqq \delta = \varepsilon$ on $I_x - L$, so $\{\gamma \in I_x \mid f(\gamma) < \varepsilon\} \subset L$ as desired. From the assumption S1 and from 4.1.17B it follows that the canonical maps $p'_U : H_U \to A'_U$ which belong to the $\mathscr{E}$ – hull ($\mathscr{E}_1^\beta$ – hull) of $\mathscr{S}$ by $\mathscr{E}$ (by $\mathscr{E}_1 = \{\mathscr{A}_{X_U}(F_U) \mid U \in \mathscr{B}(X)\}$ – see 2.2.5A) are 1–1. By 4.1.5 and 4.1.1E, $\mathscr{S}$ fulfils COND from 4.1.1E. Thus the conditions of 3.3.8 are fulfilled which proves the theorem.

The foregoing theorem can be easily strengthened in the sense of 4.3.7.

4.3.10. Definition. Let a presheaf $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \, |\varrho_{UV}| \, X\}$ from TOP and a nonempty set $\mathscr{D}(X) \subset \mathscr{B}(X)$ be given. $\mathscr{S}$ is called *topologically* $\mathscr{D}(X)$ – *projective* if for every $U \in \mathscr{D}(X)$ and any open cover $\mathscr{V}$ of U, provided $V \cap W \in \mathscr{V}$, if $V, W \in \mathscr{V}$ we have $\mathscr{X}_U = \varprojlim \mathscr{S}_{\mathscr{V}}$ ($\mathscr{S}_{\mathscr{V}} = \{\mathscr{X}_U | \varrho_{UV}| \langle \mathscr{V} \leqq \rangle\}$ – see 4.1.1A). If $\mathscr{D}(X) = \mathscr{B}(X)$ then $\mathscr{S}$ is called *topologically projective.* (The same definition can be given in terms of CLOS, UNIF, ...)

4.3.11. Remark. The following assertions are equivalent:

1) $\mathscr{S}$ is topologically $\mathscr{D}(X)$ – projective,
2) $\mathscr{S}$ is $\mathscr{D}(X)$ – projective (see 4.3.7B), fulfils COND from 4.1.1E, and the topology τ_U is projectively defined by the set of maps $\mathscr{Z} = \{\varrho_{UV} : X_U \to \mathscr{X}_V \mid V \in \mathscr{V}\}$ for every $U \in \mathscr{D}(X)$ and any open cover $\mathscr{V}$ of U.

Proof. 1 ⇒ 2: Let an open cover $\mathscr{V}$ of $U \in \mathscr{D}(X)$ and a $\mathscr{V}$ – smooth family $\mathscr{H} = \{a_V \in X_V \mid V \in \mathscr{V}\}$ be given (see 4.2.1). As we can add all the finite intersections of the sets from $\mathscr{V}$ to $\mathscr{V}$ and adapt $\mathscr{H}$ accordingly, we may assume $V \cap W \in \mathscr{V}$ if $V, W \in \mathscr{V}$. Setting $h_V(\mathscr{H}) = a_V$, we get a family between $\{\mathscr{H}\}$ and $\mathscr{S}_{\mathscr{V}}$ (see 0.6), thus there is a unique map $f : \{\mathscr{H}\} \to \mathscr{X}_U$ so that $\varrho_{UV} \circ f = h_V$ for all $V \in \mathscr{V}$. Thus for $a = f(\mathscr{H}) \in X_U$ we have $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$, hence $\mathscr{S}$ is $\mathscr{D}(X)$ – projective.

If $a, b \in X_U$, $a_V = \varrho_{UV}(a) = \varrho_{UV}(b) = b_V$ for all $V \in \mathscr{V}$, then for $\mathscr{H} = \{a_V \mid V \in \mathscr{V}\}$ we have $\mathscr{H} = \{b_V \mid V \in \mathscr{V}\}$. As f is the unique map with $\varrho_{UV} \circ f(\mathscr{H}) = a_V$ we have $f(\mathscr{H}) = a$, so $a = b$. Thus $\mathscr{S}$ fulfils COND. Clearly, τ_U is projectively defined by $\mathscr{X}$.

$2 \Rightarrow 1$: Given an open cover $\mathscr{V}$ of $U \in \mathscr{D}(X)$, a topological space $\mathscr{R} = (R, t)$ and. a family $\{f_V : \mathscr{R} \to \mathscr{H}_V \mid V \in \mathscr{V}\}$ between $\mathscr{R}$ and $\mathscr{S}_{\mathscr{V}}$, we may assume $V \cap W \in \mathscr{V}$ if $V, W \in \mathscr{V}$ as above. If $x \in R$ then $\{f_V(x) = a_V \mid V \in \mathscr{V}\}$ is $\mathscr{V}$ – smooth, so there is $a \in X_U$ with $\varrho_{UV}(a) = a_V$ for all $V \in \mathscr{V}$. We set $f(x) = a$. Then $f : \mathscr{R} \to \mathscr{X}_U$ is continuous (for $\varrho_{UV} \circ f = f_V$ are), and from COND it follows that f is unique. By 0.6, $\mathscr{X}_U = \varprojlim \mathscr{S}_{\mathscr{V}}$.

4.3.12. Remark. Given a topologically $\mathscr{D}(X)$ – projective presheaf $\mathscr{S} = \{\mathscr{X}_U = (X_U, \tau_U) \,|\varrho_{UV}|\, X\}$ from CLOS, suppose that for every $x \in X$ and any open nbd V of x there is $D \in \mathscr{D}(X)$ with $x \in D \subset V$. Let us have an open cover $\mathscr{V}$ of X such that for every $W \in \mathscr{V}$ there is a closure (topology) t_W in P_W (see 4.3.4B; clearly, P_W is the covering space of the presheaf $\mathscr{S}_W = \{\mathscr{X}_U|\varrho_{UV}|\, W\}$). Then there is a closure (topology) t in P with the following properties:

A. If for any $W \in \mathscr{V}$ the canonical maps $p_U : \mathscr{X}_U \to (A_U, b_U(t_W))$ are homeomorphisms for all $U \in \mathscr{D}(X)$ with $U \subset W$, then $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ are homeomorphisms for all $U \in \mathscr{D}(X)$. Especially, if $\mathscr{S}$ is topologically projective, then $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ are homeomorphisms for all $U \in \mathscr{B}(X)$.

B. If for any $W \in \mathscr{V}$ we have $A_U = \Gamma(U, t_W)$ for all $U \in \mathscr{D}(X)$ with $U \subset W$, then $A_U = \Gamma(U, t)$ for all $U \in \mathscr{D}(X)$. Especially, if $\mathscr{S}$ is topologically projective, then $A_U = \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$.

Proof. A: If $x \in X$, we put $st\, x = \{W \in \mathscr{V} \mid x \in W\}$. If $\alpha \in I_x$ and $F \subset st\, x$ is finite, and if N_W is a t_W – nbd of α for $W \in F$, we set $E_F = \bigcap\{N_W \mid W \in F\}$, $\mathscr{B}\alpha = \{E_F \mid F \subset st\, x,\ F \text{ finite}\}$. Then $\mathscr{B}\alpha$ is a filter base round α. These define a closure t in P; if all the t_W are topologies then so is t as well.

Let $t_x(t_W^x)$ be the closure or the topology induced in I_x by t (by t_W for $W \in st\, x$). By 4.2.1, the closure or the topology $b_U(t)$ is projectively defined by the canonical maps $\{\eta_{U,x} : A_U \to (I_x, t_x) \mid x \in U\}$ (see 4.1.1E). Thus $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ is continuous iff so is $\eta_{U,x} \circ p_U = \xi_{U,x} : \mathscr{X}_U \to (I_x, t_x)$ for every $x \in U$. It follows from the definition of t that t_x is projectively defined by the identities $\{i : I_x \to (I_x, t_W^x) \mid W \in st\, x\}$, so $\xi_{U,x} : \mathscr{X}_U \to (I_x, t_x)$ is continuous iff so is $\xi_{U,x} : \mathscr{X}_U \to (I_x, t_W^x)$ for all $W \in st\, x$. If $W \in st\, x$, $D \in \mathscr{D}(X)$, $x \in D \subset V$ then by the definition of $b_D(t_W)$ and 4.1.4A, the canonical maps $\xi_{D,x} : \mathscr{X}_D \to (I_x, t_W^x)$ are continuous for all $x \in D$. If $U \in \mathscr{B}(X)$, $x \in U$, $W \in st\, x$ then there is $D \in \mathscr{D}(X)$ with $x \in D \subset U \cap W$, and $\xi_{U,x} = \xi_{D,x} \circ \varrho_{UD} : \mathscr{X}_D \to (I_x, t_W^x)$. As $\varrho_{UV} : \mathscr{X}_U \to \mathscr{X}_V$ and $\xi_{D,x} : \mathscr{X}_D \to (I_x, t_W^x)$ are continuous, thus $\xi_{U,x}$ is. Hence $p_U : \mathscr{X}_U \to (A_U, b_U(t))$ is continuous for all $U \in \mathscr{B}(X)$. If $U \in \mathscr{D}(X)$ then there is an open cover $\mathscr{C}$ of U with $\mathscr{C} \subset \mathscr{D}(X)$ such that for every $C \in \mathscr{C}$ there is $W \in \mathscr{V}$ with $C \subset W$. Then $p_C : \mathscr{X}_C \to (A_C, b_C(t_W))$ is a homeomorphism for every

$C \in \mathscr{C}$ and any $W \in \mathscr{V}$ with $C \subset W$. Let $W \in \mathscr{V}$, $C \in \mathscr{C}$, $C \subset W$. As $b_C(t)$ is finer than $b_C(t_W)$, so $p_C^{-1} = (A_C, b_C(t)) \to \mathscr{X}_C$ is continuous. The map $p_U^{-1} : (A_U, b_U(t)) \to \mathscr{X}_U$ is continuous iff so is $\varrho_{UC} \circ p_U^{-1} : (A_U, b_U(t)) \to \mathscr{X}_C$ for all $C \in \mathscr{C}$ as $\mathscr{S}$ is topologically $\mathscr{D}(X)$ – projective. But $\varrho_{UC} \circ p_U^{-1} = p_C^{-1} \circ \sigma_{UC}$, where $\sigma_{UC} = p_C \circ \varrho_{UC} \circ p_U^{-1} :$ $: (A_U, b_U(t)) \to (A_C, b_C(t))$. By 4.1.4A, σ_{UC} is continuous, hence so is p_U^{-1} as desired.

B. Let $U \in \mathscr{D}(X)$, $r \in \Gamma(U, t)$. Take the cover $\mathscr{C}$ of U from the proof of A. If $C \in \mathscr{C}$ and $W \in \mathscr{V}$ with $C \subset W$ then clearly $r/C \in \Gamma(C, t_W)$, so there is $a_C \in X_C$ with $\xi_{C,x}(a_C) = r(x)$ for all $x \in C$. As $\mathscr{S}$ fulfils COND (see 4.3.11), the family $\{a_C \mid C \in \mathscr{C}\}$ is $\mathscr{C}$ – smooth. As $\mathscr{S}$ is $\mathscr{D}(X)$ – projective, there is $a \in X_U$ with $\varrho_{UC}(a) = a_C$ for all $C \in \mathscr{C}$. Thus $r(x) = \xi_{U,x}(a)$ for all $x \in U$, so $\Gamma(U, t) \subset A_U$. The other inclusion follows easily from the definition of t. If $\mathscr{S}$ is topologically projective, we can set $\mathscr{D}(X) = \mathscr{B}(X)$ and the special statements follow.

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