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CONTINUOUS LOCAL RIGHT PSEUDOPROCESSES

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Local semi-dynamical systems introduced by HÁJEK [5] have been systematically studied within the framework of the theory of dynamical systems. The investigation of local semidynamical systems on an abstract set led Hájek to the introduction of an essentially more general concept for which he used the name "process" and whose properties were described in [6] and [7]. Further results concerning processes on a set without topological structure are found in [10], [11], [12]. The present paper deals with processes on a locally compact Hausdorff topological space. The notion of a continuous local right pseudoprocess is introduced, whose special case is e.g. Roxin's general control system [14]. An apparatus for the study of properties of solutions and orbits of continuous local right pseudoprocesses is developed and theorems on existence of solutions and their continuous dependence on a parameter are established.

1. FINITE TOPOLOGY AND CONTINUITY OF MULTIFUNCTIONS

1.1. Notation and terminology. Let X be an arbitrary set. By $A(X)$ or $P(X)$ we shall denote the set of all subsets or of all nonempty subsets of the set X, respectively. Let the set X be endowed with a topology $\mathscr{T}$ and let $O(X)$, $C(X)$, $K(X)$ denote the set of all nonempty open, closed, compact subsets of the space X, respectively. Let us define the maps

(1.1.1) $$u : A(X) \to A(P(X)) : u(G) = \{H \in P(X) \mid H \subset G\},$$

(1.1.2) $$l : A(X) \to A(P(X)) : l(G) = \{H \in P(X) \mid H \cap G \neq \emptyset\}$$

and let us denote

$$\mathscr{B}\mathscr{U}(X) = \{u(G) \mid G \in \mathscr{T}\}, \quad \mathscr{B}\mathscr{L}(X) = \{l(G) \mid G \in \mathscr{T}\},$$
$$\mathscr{B}\mathscr{S}(X) = \mathscr{B}\mathscr{U}(X) \cup \mathscr{B}\mathscr{L}(X).$$

1.2. Definition. Let X be a topological space and let $\mathscr{U}(X)$, $\mathscr{L}(X)$, $\mathscr{S}(X)$ be topologies in the set $P(X)$. The topology $\mathscr{U}(X)$, $\mathscr{L}(X)$, $\mathscr{S}(X)$ is called the *upper semi-finite topology*, the *lower semi-finite topology*, the *finite topology* in the set $P(X)$, respectively, iff $\mathscr{U}(X), \mathscr{L}(X), \mathscr{S}(X)$ is the coarsest topology in $P(X)$ such that $\mathscr{B}\mathscr{U}(X) \subset \mathscr{U}(X)$, $\mathscr{B}\mathscr{L}(X) \subset \mathscr{L}(X)$, $\mathscr{B}\mathscr{S}(X) \subset \mathscr{S}(X)$, respectively.

1.3. Remarks. 1.3.1. In what follows, the topological space $P(X)$ with the topology $\mathscr{U}(X)$ or $\mathscr{L}(X)$ or $\mathscr{S}(X)$ will be denoted by UX or LX or SX, respectively. In our investigations of properties of continuous local right pseudoprocesses we shall be concerned mainly with closed and compact subsets of the space X. Thus we shall work often in the subspaces $C(X)$ and $K(X)$ of the space $P(X)$. The topology on $C(X)$ or $K(X)$, induced by any one of the topologies $\mathscr{U}(X)$, $\mathscr{L}(X)$, $\mathscr{S}(X)$ will be denoted by the same symbol again, and for the topological spaces $(C(X), \mathscr{U}(X))$, $(C(X), \mathscr{L}(X))$, $(C(X), \mathscr{S}(X))$ and $(K(X), \mathscr{U}(X))$, $(K(X), \mathscr{L}(X))$, $(K(X), \mathscr{S}(X))$ we shall use the symbols UCX, LCX, SCX and UKX, LKX, SKX, respectively.

1.3.2. The topologies from Definition 1.2 were first introduced by Michael in his paper [9]. An exhaustive application of these topologies to the study of continuity of multifunctions was given among others by Ponomariev in [13].

1.3.3. It is easy to see that for each set $E \in P(X)$ the identities

$$u(E) = P(X) - l(X - E)\,, \quad l(E) = P(X) - u(X - E) \tag{1.3.1}$$

hold. Hence for each closed subset E of the space X the set $u(E)$ or $l(E)$ is closed in the space LX or UX, respectively. Thus the topologies $\mathscr{U}(X)$ and $\mathscr{L}(X)$ may be fully characterized by closed subsets of the space X as follows. The topology $\mathscr{U}(X)$ or $\mathscr{L}(X)$ is the coarsest topology in the set $P(X)$ such that for each closed subset $H \in C(X)$ the set $l(H)$ or $u(H)$ is closed in the space UX or LX, respectively. Thus, if a set $E \subset X$ is open and closed in X, then both the sets $u(E)$ and $l(E)$ are open and closed in SX.

1.4. Theorem. *Let X be a topological space, let I be an arbitrary index set and let the set*

$$\mathscr{M} = \{M_i \in K(X) \mid i \in I\}$$

be a compact subset of the space UCX. Then the set

$$M = \bigcup_{i \in I} M_i$$

is a compact subset of the space X.

Proof. Let $\mathscr{O}$ be an arbitrary open covering of the set M in the space X. For each $M_i \in \mathscr{M}$ let $\mathscr{O}_i$ be a finite open covering of M_i, chosen from $\mathscr{O}$. Let us denote

$$O_i = \bigcup_{G \in \mathscr{O}_i} G\,.$$

The system $\{u(O_i) \mid i \in I\}$ is an open covering of the compact set $\mathscr{M}$ in the space UCX, so that there exists a finite open covering, say $\{u(\mathscr{O}_1), u(\mathscr{O}_2), \ldots, u(\mathscr{O}_n)\}$, of the set $\mathscr{M}$. Since $M_i \subset O_i \in u(O_i)$, the system $\{O_1, O_2, \ldots, O_n\}$ is a finite open covering of the set M, chosen from the given covering $\mathscr{O}$. Hence, the set M is compact.

1.5. Theorem. *Let X be a topological space, let I be an arbitrary index set and let the set*

$$\mathcal{M} = \{M_i \in C(X) \mid i \in I\}$$

be a connected subset of the space SCX. Let at least one of the sets M_i, $i \in I$, be connected in the space X. Then the set

$$M = \bigcup_{i \in I} M_i$$

is a connected subset of the space X.

Proof. Let N be a proper subset of the set M, open and closed in M. We have to prove that $N = \emptyset$.

According to 1.3.1 the sets $u(N)$ and $u(M - N)$ are open and closed in SCM, so that the sets

$$(1.5.1) \qquad \mathcal{N}' = u(N) \cap \mathcal{M}, \quad \mathcal{N}'' = u(M - N) \cap \mathcal{M}$$

are open and closed in $\mathcal{M}$. Since $\mathcal{M}$ is connected, at least one of the sets (1.5.1) has to be empty. According to the assumption the set $\mathcal{M}$ contains at least one connected subset M_i, so that either $M_i \subset N$ or $M_i \subset M - N$. Hence either $\mathcal{N}' \neq \emptyset$ or $\mathcal{N}'' \neq \emptyset$. Let us suppose that $\mathcal{N}' \neq \emptyset$. Then, taking into account the assumption of connectedness of $\mathcal{M}$, we obtain the identity $\mathcal{M} = \mathcal{N}'$, and hence $M = N$, contradicting the assumption that N is a proper subset of M. Thus $\mathcal{N}' = \emptyset$ and it must be $\mathcal{N}'' \neq \emptyset$, so that $\mathcal{N}'' = \mathcal{M}$, i.e. $(M - N) \cap M = M$. Hence, with regard to the assumption $N \subset M$, one obtains $N = \emptyset$.

1.6. Convergence in a topological space. Let X be a topological space. By a *(generalized) sequence* of points or of subsets of the space X we mean a map defined on some directed set with values in the set X or in the set $P(X)$, respectively. Sequences of points x_i or of subsets H_i of the space X with indices i form a directed set I (with a partial ordering $\prec$) will be denoted by $(x_i, i \in I)$ or $(H_i, i \in I)$, respectively. If there is no danger of confusion of the index sets I, we shall use the simpler notation (x_i) and (H_i). Let us recall the terminology concerning the convergence of sequences.

Let I, J be directed sets and let $(x_i, i \in I)$ $(i_j, j \in J)$ be sequences of points of the sets X and I, respectively, such that $j, j' \in J, j \prec j'$ implies $i_j \prec i_{j'}$. Then the sequence $(x_{i_j}, j \in J)$ is called a *subsequence* of the sequence $(x_i, i \in I)$. A sequence $(x_i, i \in I)$ is said to *converge* to a point $x \in X$ (which is written as $x_i \to x$ for $i \in I$) iff for each neighbourhood U of x there exists an index $i \in I$ such that $x_i \in U$ holds for all $j \in I$, $i \prec j$. The point x is then called the *limit* of the sequence (x_i). This fact is written as $x = \lim_{i \in I} x_i$, or simply $x = \lim x_i$. A point $x \in X$ is called a *cluster point* of a sequence (x_i) iff there exists a subsequence of the sequence (x_i) converging to the point x. Analogous terminology and notation will be used for sequences of subsets of X.

Let $(H_i, i \in I)$ be a sequence of subsets of the space X. Let us recall the definitions of the *upper limit* (denoted by $\limsup_{i \in I} H_i$) and the *lower limit* (denoted by $\liminf_{i \in I} H_i$) of the sequence (H_i).

A point $x \in X$ is said to belong to $\limsup_{i \in I} H_i$ iff for each neighbourhood V of x and for each index $i \in I$ there exists $j \in I$ such that $i \prec j$ and $V \cap H_j \neq \emptyset$.

A point $x \in X$ is said to belong to $\liminf_{i \in I} H_i$ iff for each neighbourhood V of x there exists an index $i \in I$ such that $V \cap H_j \neq \emptyset$ holds for each $j \in I$ satisfying $i \prec j$.

Clearly a point $x \in X$ belongs to $\limsup_{i \in I} H_i$ or $\liminf_{i \in I} H_i$ iff there exists a sequence $(x_i, i \in I)$ of points of the space X such that $x_i \in H_i$ for $i \in I$ and x is a cluster point or limit of the sequence (x_i), respectively.

In what follows, we shall often be concerned with convergence in a uniform space. This is due to the fact that each compact topological space X is uniformizable. The unique uniformity $\mathcal{U}$ on X, inducing on X the given topology, consists of all relations in X containing the identical relation 1_X on X. A base of this uniformity may be constructed using finite open coverings of the space X. That is why this uniformity is often called the *uniformity of finite open coverings* (see e.g. [2], chap. II, § 4).

Let $(x_i, i \in I)$ be a sequence of points of a uniform space X with a uniformity $\mathcal{U}$. The sequence $(x_i, i \in I)$ is called a *Cauchy sequence* iff for each vicinity $V \in \mathcal{U}$ there exists an index $i \in I$ such that $(x_j. x_k) \in V$ holds for each couple of indices $j, k \in I$, $i \prec j$, $i \prec k$. A uniform space X is said to be *complete* iff each Cauchy sequence of points of the space X has a limit in X. Recall that each compact uniform space with the uniformity of finite open coverings is complete.

Remember that each sequence of subsets of a topological space X may be interpreted as a sequence of points of anyone of the topological spaces UX or LX or SX. In such a way we obtain other three types of convergence of sequences of subsets which will be dealt with in what follows.

1.7. Definition. Let X be a topological space. Let a sequence $(H_i, i \in I)$ with $H_i \in C(X)$ for $i \in I$ and a set $H \in C(X)$ be given. The sequence $(H_i, i \in I)$ is said to *U-converge* or *L-converge* or *S-converge* to the set H in the space X (which will be denoted by $H_i \to^U H$ for $i \in I$ or $H_i \to^L H$ for $i \in I$ or $H_i \to^S H$ for $i \in I$, respectively), iff the sequence $(H_i, i \in I)$ of points of the set $C(X)$ converges to the point $H \in C(X)$ in the topological space UCX or LCX or SCX, respectively. The set H is then called the *U-limit* or the *L-limit* or the *S-limit* of the sequence $(H_i, i \in I)$ in the space X, which is written as $H \in U - \lim H_i$ for $i \in I$ or $H \in L - \lim H_i$ for $i \in I$ or $H \in S -$ $- \lim H_i$ for $i \in I$, respectively.

1.8. Lemma. *Let a sequence $(H_i, i \in I)$ with $H_i \in C(X)$ for $i \in I$ and a set $H \in C(X)$ be given. Then the following assertions hold.*

1.8.1. *$H \in U - \lim H_i$ for $i \in I$ iff for each neighbourhood U of H in X there exists an $i \in I$ such that $H_j \subset U$ holds for each $j \in I$, $i \prec j$.*

1.8.2. *$H \in L - \lim H_i$ for $i \in I$ iff for each open set $U \in O(X)$ with $H \cap U \neq \emptyset$ there exists an $i \in I$ such that $H_j \cap U \neq \emptyset$ holds for each $j \in I$, $i \prec j$.*

1.8.3. *$H \in S - \lim H_i$ for $i \in I$ iff for each open sets $U, V \in O(X)$ satisfying $H \subset U$, $H \cap V \neq \emptyset$ there exists an $i \in I$ such that $H_j \subset U$, $H_j \cap V \neq \emptyset$ holds for each $j \in I$, $i \prec j$.*

1.8.4. *$H \in S - \lim H_i$ for $i \in I$ iff $H \in U - \lim H_i$ for $i \in I$ and $H \in L - \lim H_i$ for $i \in I$.*

1.9. Remark. It is well known that in a Hausdorff topological space each sequence has at most one limit. For sequences of subsets this assertion does not hold. The assertion 1.8.1 implies immediately that $H \in U - \lim H_i$ for $i \in I$, $H \subset F \in C(X)$ implies $F \in U - \lim H_i$ for $i \in I$. Analogously, 1.8.2 implies that $H \in L - \lim H_i$ for $i \in I$, $F \in C(X)$, $F \subset H$ implies $F \in L - \lim H_i$ for $i \in I$.

1.10. Theorem. *Let a sequence $(H_i, i \in I)$ with $H_i \in C(X)$ for $i \in I$ be given. Then $H = \liminf_{i \in I} H_i \in L - \lim H_i$ for $i \in I$, and if $H' \in L - \lim H_i$ for $i \in I$ then $H' \subset H$.*

Proof. First, let us prove that $H \in L - \lim H_i$ for $i \in I$. Recall that $x \in H$ iff there exists a sequence of points $x_i \in H_i$ for $i \in I$ converging to the point x. Take an arbitrary open set $U \in O(X)$ with $U \cap H \neq \emptyset$ and an arbitrary point $x \in U \cap H$. Then there must exist a sequence of points $x_i \in H_i$ converging to the point x, i.e. there exists an $i \in I$ such that $x_j \in U \cap H_j$ holds for each $j \in I$ with $i \prec j$. Hence, according to 1.8.2, the assertion $H \in L - \lim H_i$ for $i \in I$ follows.

Now let us prove that $H' \in L - \lim H_i$ for $i \in I$ implies $H' \subset H$. Take an arbitrary point $x \in H'$ and assume that there exists no sequence of points $x_i \in H_i$ converging to the point x. Then for some open neighbourhood U of x and for each $k \in I$ there exists $i_k \in I$ with $k \prec i_k$ such that $U \cap H_{i_k} = \emptyset$. On the other hand, $H' \in L - \lim H_i$ for $i \in I$ and $U \cap H' \neq \emptyset$ imply $U \cap H_i \neq \emptyset$ for all sufficiently large $i \in I$ which is a contradiction. Thus $H' \subset H$.

1.11. Lemma. *Let X be a locally compact Hausdorff space and let a sequence $(H_i, i \in I)$ with $H_i \in C(X)$ for $i \in I$ be given. If $H' \in U - \lim H_i$ for $i \in I$, then $\limsup_{i \in I} H_i \subset H'$.*

Proof. Take an arbitrary point $x \in H = \limsup_{i \in I} H_i$. Recall that the point x is a cluster point of a certain sequence (x_i) with $x_i \in H_i$. Assume that $x \notin H'$. Since the space X is regular, there exist open sets $V, W \in O(X)$ such that $x \in V$, $H' \subset W$ and $V \cap W = \emptyset$. On the other hand, since x is a cluster point of the sequence (x_i), there must exist arbitrarily large indices $j \in I$ such that $x_j \in V \cap H_j$. The assumption $H' \in$ $\in U - \lim H_i$ for $i \in I$ implies $H_j \subset W$ for sufficiently large indices j. Hence $x_j \in$ $\in V \cap H_j \subset V \cap W = \emptyset$, which is a contradiction. Thus $x \in H'$, which yields $H \subset H'$.

1.13. Corollary. *Let a sequence* $(H_i, i \in I)$ *of nonempty closed subsets of a locally compact Hausdorff space be given. Then for each two sets* $H, F \in C(X)$ *satisfying* $H \in L - \lim H_i$ *for* $i \in I$, $F \in U - \lim H_i$ *for* $i \in I$ *it holds*

$$H \subset \liminf_{i \in I} H_i \subset \limsup_{i \in I} H_i \subset F . \tag{1.13.1}$$

1.14. Corollary. *Let X be a locally compact Hausdorff space, let* $(H_i, i \in I)$ *be a sequence with* $H_i \in C(X)$ *and let* $H, H', F, F' \in C(X)$. *Then the following assertions hold.*

If $H \in L - \lim H_i$ *for* $i \in I$, $H - H' \neq \emptyset$, *then* $H' \notin U - \lim H_i$ *for* $i \in I$.

If $F \in U - \lim H_i$ *for* $i \in I$, $F' - F \neq \emptyset$, *then* $F' \notin L - \lim H_i$ *for* $i \in I$.

1.15. Corollary. *In a locally compact Hausdorff space each sequence* $(H_i, i \in I)$ *of nonempty closed subsets has at most one limit. If there exists* $H \in S - \lim H_i$ *for* $i \in I$, *then*

$$H = \limsup_{i \in I} H_i = \liminf_{i \in I} H_i .$$

1.16. Multifunctions. The main objects of our investigation are certain kinds of multifunctions, called continuous local right pseudoprocesses. That is why we shall recall now several notions and assertions concerning multifunctions.

Let X, Y be topological spaces. A *multifunction* $f : X \to Y$ is any relation f between X and Y. Let D_f denote the *domain* of the multifunction f, i.e. $D_f = \{x \in X \mid f(x) \neq \emptyset\}$. A value $f(x)$ of a multifunction $f : X \to Y$ at a point $x \in D_f$ is a nonempty subset of the space Y. A multifunction $f : X \to Y$ is said to be *point closed* or *point compact* or *point connected*, iff for each point $x \in D_f$ the set $f(x)$ is closed or compact or connected in the space Y, respectively. In what follows we assume all multifunctions to be point closed and to satisfy $D_f = X$.

For a given multifunction $f : X \to Y$ and given sets $A \subset X$, $B \subset Y$ we denote

$$f(A) = \{y \in Y \mid y \in f(x) \text{ for some } x \in A\} = \bigcup_{x \in A} f(x)$$
$$f^-(B) = \{x \in X \mid f(x) \cap B \neq \emptyset\} , \quad f_-(B) = \{x \in X \mid f(x) \subset B\} .$$

The set $f(A)$ is called the *image* of the set A, the sets $f^-(B)$ and $f_-(B)$ are called the *upper* and the *lower inverse images* of B, respectively. It is easy to see that for each multifunction $f : X \to Y$ and for each $A \subset X$, $B \subset Y$ the identities

$$X - f_-(B) = f^-(Y - B) , \quad X - f^-(B) = f_-(Y - B) \tag{1.16.1}$$

hold.

According to our assumption, each multifunction is point closed. Thus each multifunction $f : X \to Y$ can be assigned a map

$$f^\vee : X \to C(Y) : f^\vee(x) = f(x) \quad \text{for each} \quad x \in X .$$

The map $f^\vee$ is said to be generated by the multifunction f.

1.17. Definition. Let topological spaces X, Y, a multifunction $f : X \to Y$ and a point $a \in X$ be given. The multifunction f is said to be *upper semi-continuous* or *lower semi-continuous* or *continuous at the point a* iff the map $f^\vee$ generated by the multifunction f is continuous at the point a with respect to the upper semi-finite or lower semi-finite of finite topology on the space $C(X)$, respectively.

The multifunction f is said to be *upper semi-continuous* or *lower semi-continuous* or *continuous* iff it is upper semi-continuous or lower semi-continuous or continuous at each point $a \in X$.

1.18. Remark. In the preceding definition the continuity problem for multifunctions $f : X \to Y$ was reduced to the continuity problem for maps $f^\vee : X \to C(Y)$ with a convenient topology on the set $C(Y)$. Recall that a map $g : P \to Q$ is continuous at a point $a \in P$ (with respect to given topologies on the sets P and Q) iff for each element U of a subbase of a neighbourhood system of the point $g(a)$ there exists a neighbourhood V of the point a such that $g(V) \subset U$. In the following lemma this characterization of continuity of maps will be applied to the maps $f^\vee : X \to UCY$, $f^\vee : X \to LCY$ and $f^\vee : X \to SCY$ in order to obtain a very useful description of notions introduced in Definition 1.17 in terms of topologies on the sets X and Y.

1.19. Lemma. *Let topological spaces X, Y, a multiplication $f : X \to Y$ and a point $a \in X$ be given. The following assertions hold.*

1.19.1. *The multifunction f is upper semi-continuous at the point a iff for each open set $U \in O(Y)$ with $f(a) \subset U$ there exists a neighbourhood V of the point a such that $f(x) \subset U$ for each point $x \in V$.*

1.19.2. *The multifunction f is lower semi-continuous at the point a iff for each open set $U \in O(Y)$ with $f(a) \cap U \neq \emptyset$ there exists a neighbourhood V of the point a such that $f(x) \cap U \neq \emptyset$ for each point $x \in V$.*

1.19.3. *The multifunction f is continuous at the point a iff for each two open sets $U, W \in O(Y)$ with $f(a) \subset U$, $f(a) \cap W \neq \emptyset$ there exists a neighbourhood V of the point a such that $f(x) \subset U$ and $f(x) \cap W \neq \emptyset$ for each point $x \in V$.*

1.19.4. *The multifunction f is continuous at the point a iff it is both upper and lower semi-continuous at the point a.*

1.20. Theorem. *A multifunction $f : X \to Y$ is upper semi-continuous iff for each open set $G \in O(Y)$ the lower inverse image $f_-(G)$ is an open set in X.*

Proof. Let $G \in O(Y)$. Then the set $u(G)$ is open in UCX and the identity $f_-(G) = \{x \in X \mid f(x) \subset G\} = \{x \in X \mid f^\vee(x) \in u(G)\} = (f^\vee)^{-1}(u(G))$ holds. Hence the assertion follows immediately.

1.21. Theorem. *A multifunction $f : X \to Y$ is upper semi-continuous iff for each closed set $H \in C(Y)$ the upper inverse image $f^-(H)$ is a closed set in X.*

Proof. See Theorem 1.20 and the identities (1.16.1).

1.22. Theorem. *A multifunction $f: X \to Y$ is upper semi-continuous at a point $a \in X$ iff $f(a) \in U - \lim f(x_i)$ for $i \in I$ holds for each sequence $(x_i, i \in I)$ in X converging to the point a.*

Proof. Let us suppose that $x_i \to a$ implies $f(x_i) \to^U f(a)$ and that the multifunction f is not upper semi-continuous at the point a. Then there exists a neighbourhood U of the set $f(a)$ such that $f(V) \not\subset U$ for each neighbourhood V of the point a. Now we can construct such a sequence (x_V) in X that $x_V \in V$ for each neighbourhood V of a, $x_V \to a$ and $f(x_V) \not\subset U$. Thus $x_V \to a$ does not imply $f(x_V) \to^U f(a)$, which contradicts the assumption.

To prove the converse implication, assume that the multifunction f is upper semi-continuous at the point a and that there is given a sequence $(x_i, i \in I)$ in X converging to the point a. Let U be an arbitrary neighbourhood of the set $f(a)$. Then there exists a neighbourhood V of the point a such that $f(V) \subset U$. For the neighbourhood V there exists an index $i \in I$ such that for each $j \in I$ with $i \prec j$ it is $x_j \in V$, and thus $f(x_j) \subset U$. Hence $f(x_i) \to^U f(a)$, which was to be proved.

1.23. Theorem. *A multifunction $f: X \to Y$ is lower semi-continuous iff for each open set $G \in O(Y)$ the upper inverse image $f^-(G)$ is an open set in X.*

Proof. For each $G \in O(Y)$ the set $l(G)$ is open in LCY, so that the assertion follows from the continuity of the map $f^\vee : X \to LCY$ and the identity

$$f^-(G) = \{x \in X \mid f(x) \cap G \neq \emptyset\} = \{x \in X \mid f^\vee(x) \in l(g)\} = (f^\vee)^{-1}(l(G)).$$

1.24. Theorem. *A multifunction $f: X \to Y$ is lower semi-continuous iff for each closed set $H \in C(Y)$ the lower inverse image $f_-(H)$ is a closed set in X.*

Proof. See Theorem 1.23 and the identities (1.16.1).

1.25. Theorem. *A multifunction $f: X \to Y$ is lower semi-continuous at a point $a \in X$ iff $f(a) \in L - \lim f(x_i)$ for $i \in I$ holds for each sequence $(x_i, i \in I)$ in X converging to the point a.*

Proof. Suppose that $x_i \to a$ implies $f(x_i) \to^L f(a)$ and that f is not lower semi-continuous at the point a. Then there exists an open set $U \in O(Y)$ such that $f(a) \cap \cap U \neq \emptyset$ and in each neighbourhood V of the point a there exists a point x_V such that $f(x_V) \cap U = \emptyset$. Clearly, the sequence (x_V) may be chosen so as to converge to the point a. Thus $x_V \to a$ does not imply $f(x_V) \to^L f(a)$, which contradicts the assumption.

Suppose now that the multifunction f is lower semi-continuous at the point a, let $(x_i, i \in I)$ be a sequence in x converging to the point a, and let U be an arbitrary open set in Y such that $f(a) \cap U \neq \emptyset$ holds for each point $x \in V$. To the neighbourhood V there exists an index $i \in I$ such that $x_j \in V$ holds for each $j \in I$, $i \prec j$. Thus $f(x_j) \cap U \neq \emptyset$. Hence, according to 1.8.2, $f(x_i) \to^L f(a)$, which was to be proved.

1.26. Theorem. *Let a multifunction $f : X \to Y$ be upper semi-continuous and point compact. If the space X is compact and the space Y is T_1, then the space $f(X)$ is also compact.*

Proof. The map $f^\vee : X \to UCY$ generated by the multifunction f is continuous, so that the set

$$f^\vee(X) = \{f^\vee(x) \mid x \in X\} = \{f(x) \mid x \in X\}$$

is compact in the space UCX. Thus the assumptions of Theorem 1.4 are fulfilled, hence the set $f(x) = \bigcup\{f(x) \mid x \in X\}$ is compact.

1.27. Theorem. *Let a multifunction $f : X \to Y$ be continuous. Let the space X be connected and let there exist a point $x \in X$ such that the set $f(x)$ is connected. Then the space $f(X)$ is also connected.*

Proof. The map $f^\vee : X \to SCY$ is continuous, so that $f^\vee(X)$ is a connected subset of the space SCX. Now apply Theorem 1.5.

1.28. Theorem. *Let a multifunction $f : X \to Y$ be upper semi-continuous and point connected. Let the space X be connected. Then the space $f(X)$ is also connected.*

Proof. Suppose that $f(X) = Y$ and that the space Y is not connected. Then there exist two non empty disjoint open sets A, B such that $Y = A \cup B$. According to Theorem 1.20 the sets $f_-(A)$ and $f_-(B)$ are open. Since $f(x) \subset A \cup B$ and $f(X)$ is connected, it is either $f(X) \subset A$ or $f(X) \subset B$. Hence $X = f_-(A) \cup f_-(B), f_-(A) \cap \cap f_-(B) = \emptyset$ and $f_-(A) \neq \emptyset \neq f_-(B)$, which contradicts the assumption of connectedness of the space X.

2. CONTINUOUS LOCAL RIGHT PSEUDOPROCESSES

2.1. Notation. In what follows, P denotes a locally compact Hausdorff space, R the 1-dimensional Euclidean space of all reals. Let

$$D = \{(v, u, x) \in R \times R \times P \mid u \leqq v\} .$$

The main object of our investigation will be a multifunction p defined on a subset D_p of the set D such that for each point $(v, u, x) \in D_p$ the set $p(v, u, x)$ is a non-empty subset of the space P. In the study of properties of the multifunction p a very important role will be played by the set

$$G_p = \{(t, u, x) \in D_p \mid \bigcup_{v \in [u,t]} \{t\} \times \{v\} \times p(v, u, x) \subset D_p\} .$$

The set $R \times R \times P$ is supposed to be endowed with the topology of the cartesian product and the sets D, D_p and G_p are topologized with the topologies induced by the topology of the space $R \times R \times P$. Thus all the topologies involved are locally compact and Hausdorff.

2.2. Definition. A multifunction $p : R \times R \times P \to P$ is called a *local right pseudoprocess* (shortly written as an *lr-pseudoprocess*) in the space P iff it satisfies the following three axioms:

(E) *The local existence axiom*: Each point $(u, x) \in R \times P$ reals $e(u, x) \in (0, +\infty]$ and $g(u, x) \in (0, e(u, x)]$ are assigned such that

$$(v, u, x) \in D_p \quad \text{iff} \quad u \leqq v < u + e(u, x)$$

and

$$(v, u, x) \in G_p \quad \text{iff} \quad u \leqq v < u + g(u, x) .$$

(I) *The initial value axiom*: $p(u, u, x) = \{x\}$ for each point $(u, x) \in R \times P$.

(O) *The domain openness axiom*: The sets D_p and G_p are open in the space D.

2.3. Remarks. 2.3.1. To each *lr*-pseudoprocess p in P there corresponds an abstract right pseudoprocess $\mathfrak{p}$ in P over R in the sense of Definition 2.1 in [11]. These two pseudoprocesses are evidently related in the following way:

$$y \in p(v, u, x) \quad \text{iff} \quad (y, v)\, \mathfrak{p}(x, u) .$$

2.3.2. The initial value axiom implies immediately $p(D_p) = P$. Thus the set P is fully determined by the multifunction p itself.

2.3.3. The local existence axiom may be formulated in the following manner: There exist functions

$$e, g : R \times P \to (0, +\infty] \tag{2.3.1}$$

defined on the whole space $R \times P$ such that

$$D_p = \bigcup_{(u,x) \in R \times P} [u, u + e(u, x)) \times \{(u, x)\} , \tag{2.3.2}$$

$$G_p = \bigcup_{(u,x) \in R \times P} [u, u + g(u, x)) \times \{(u, x)\} . \tag{2.3.3}$$

Clearly

$$e(u, x) = \sup \{t - u \in R \mid (t, u, x) \in D_p\} ,$$
$$g(u, x) = \sup \{t - u \in R \mid (t, u, x) \in G_p\} ,$$

where the supremum is taken over the extended real line $R \cup \{-\infty, +\infty\}$.

2.3.4. If

$$D_p = D , \tag{2.3.4}$$

then also $G_p = D$ so that $g(u, x) = e(u, x) = +\infty$ for all $(u, x) \in R \times P$. In this case all axioms of an *lr*-pseudoprocess are fulfilled trivially. An *lr*-pseudoprocess satisfying (2.3.4) is said to be *global* (shortly written as *gr-pseudoprocess*).

2.4. Example. Let us consider a one-parametric Cauchy problem

$$\dot{y} = (y + a^2 t)^2 , \quad y(u) = x , \quad a > 0 . \tag{2.4.1}$$

For each $a > 0$, $u, x \in R$ the Cauchy problem (2.4.1) has exactly one maximal solution

$$(2.4.2) \qquad \varphi(t; u, x, a) = -a^2 t + a \operatorname{tg}\left[at - au + \operatorname{arctg}\left(\frac{x}{a} + au\right)\right],$$

defined for all $t \in R$ satisfying

$$u - \frac{\pi}{2a} - \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a} + au\right) < t < u + \frac{\pi}{2a} - \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a} + au\right).$$

Denote by

$$(2.4.3) \qquad e(u, x, a) = \frac{\pi}{2a} - \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a} + au\right)$$

the escape time of the solution (2.4.2).

Take any compact subset I of the interval $(0, +\infty)$ and define a lr-pseudoprocess p as follows:

$$(2.4.4) \qquad p(t, u, x) = \{\varphi(t; u, x, a) \mid a \in I\}$$
$$\text{for each } (t, u, x) \in R \times R \times R, \quad u \leqq t.$$

It may be easily verified that the local existence axiom is fulfilled for

$$(2.4.5) \qquad e(u, x) = \max\{e(u. x, a) \mid a \in I\},$$
$$g(u, x) = \min\{e(u, x, a) \mid a \in I\}.$$

2.5. Lemma. *Given a lr-pseudoprocess p, then the set D_p or G_p is open in the space D iff the function e or g from* (2.3.1), *respectively, is lower semi-continuous.*

2.6. Remark. Recall that the functions e and g are lower semi-continuous iff

$$(2.6.1) \qquad e(u, x) \leqq \liminf (v_i, y_i), \quad g(u, x) \leqq \liminf g(v_i, y_i)$$

holds for each point $(u, x) \in R \times P$ and for each (generalized) sequence (v_i, y_i) in $R \times P$ converging to the point (u, x). From (2.6.1) one easily obtains that for any compact sets $I \subset R$, $K \subset P$ the reals

$$(2.6.2) \qquad e(I, K) = \inf\{e(u, x) \mid (u, x) \in I \times K\}$$

and

$$(2.6.3) \qquad g(I, K) = \inf\{g(u, x) \mid (u, x) \in I \times K\}$$

are positive. Thus

$$(2.6.4) \qquad [u, u + e(I, K)) \times \{u\} \times K \subset D_p,$$
$$[u, u + g(I, K)) \times \{u\} \times K \subset G_p$$

holds for each compact set $I \times K \subset R \times P$ and for each $u \in I$.

In what follows, the symbol $p(t, v, p(v, u, x)$ denotes the union of all sets $p(t, v, z)$ with $z \in p(v, u, x)$ such that $(t, v, z) \in D_p$. Thus the statement "$p(t, v, p(v, u, x))$ is defined" asserts that $(v, u, x) \in D_p$ and $(t, v, z) \in D_p$ for a certain point $z \in p(v, u, x)$. In the special cases $t = v$ or $v = u$ this reduces to the assertion $(t, u, x) \in D_p$. Similarly, the statement "the given inclusion holds whenever the left hand side is defined" means that if the left hand side of the inclusion is defined, then the right hand side is defined as well and the inclusion holds.

2.7. Definition. An lr-pseudoprocess p is said to be *compositive* or *transitive* iff the following axiom (C) or (T) respectively, is fulfilled.

(C) *The compositivity axiom*: The inclusion

$$p(t, u, x) \subset p(t, v, p(v, u, x)) \tag{2.7.1}$$

holds whenever $v \in [u, t]$ and the left hand side is defined.

(T) *The transitivity axiom*: The inclusion

$$p(t, v, p(v, u, x)) \subset p(t, u, x) \tag{2.7.2}$$

holds whenever the left hand side is defined.

An lr-pseudoprocess p is called a *local right process* (shortly written as an *lr-process*) iff it satisfies the following axiom.

(C) *The semi-group axiom*: The identity

$$p(t, u, x) = p(t, v, p(v, u, x)) \tag{2.7.3}$$

holds whenever one of its sides is defined.

2.8. Remarks. 2.8.1. Recall that the semi-group axiom is satisfied iff both the axioms (C) and (T) are satisfied simultaneously. In other words a lr-pseudoprocess is an lr-process iff it is both compositive and transitive.

2.8.2. In the case of a gr-pseudoprocess the assumptions of the axioms (C), (T) and (G) reduce to the requirement that (2.7.1), (2.7.2) and (2.7.3), respectively, takes place for all $x \in P$ and all $u, v, t \in R$ satisfying $u \leqq v \leqq t$.

2.8.3. If an lr-pseudoprocess p satisfies the axiom (C) or (T) or (G) and $(t, u, x) \in G_p$, then (2.7.1) or (2.7.2) or (2.7.3), respectively, holds with

$$p(t, v, p(v, u, x)) = \bigcup_{z \in p(v,u,x)} p(t, v, z)\,.$$

2.9. Example. Let us investigate an lr-pseudoprocess p defined in Example 2.4 with $I = \{\frac{1}{2}, 1\}$. According to (2.4.4)

$$p(t, u, x) = \{\varphi(t; u, x, \tfrac{1}{2}),\ \varphi(t; u, x, 1)\}\,, \tag{2.9.1}$$

where

$$\varphi(t; u, x, \tfrac{1}{2}) = -\tfrac{1}{4}t + \tfrac{1}{2}\operatorname{tg}\left(\tfrac{1}{2}t - \tfrac{1}{2}u + \operatorname{arctg}\frac{4x + u}{2}\right),$$

$$\varphi(t; u, x, 1) = - t + \operatorname{tg}(t - u + \operatorname{arctg}(x + u)) .$$

According to (2.4.5)

$$e(u, x) = \max\{e(u, x. \tfrac{1}{2}), e(u, x, 1)\} ,$$

$$g(u, x) = \min\{e(u, x, \tfrac{1}{2}), e(u, x, 1)\}$$

with

$$e(u, x, a) = \frac{\pi}{2a} - \frac{1}{a}\operatorname{arctg}\frac{x + a^2u}{a} .$$

Since the function $\varphi(.; u, x, a)$ is a solution of the Cauchy problem (2.4.1), the identity

(2.9.2) $$\varphi(t, u, x, a) = \varphi(t, v, \varphi(v, u, x, a), a)$$

holds for all $u \leqq v \leqq t < u + e(u, x, a)$. Using (2.9.2), one may easily verify that

(2.9.3) $$p(t, u, x) \subset p(t, v, p(v, u, x))$$

whenever $(t, u, x) \in D_p$ and $v \in [u, t]$.

2.10. Lemma. *Let p be an lr-pseudoprocess. Then the following assertions hold.*

2.10.1. *Let p be compositive. If $(v, u, x) \in D_p$, then*

(2.10.1) $$u + e(u, x) \leqq v + e(v, y)$$

holds for a certain point $y \in p(v, u, x)$.

2.10.2. *Let p be transitive. If $(v, u, x) \in D_p$, then*

(2.10.2) $$u + e(u, x) \geqq v + e(v, y)$$

holds for each point $y \in p(v, u, x)$.

2.10.3. *Let p be an lr-process. If $(v, u, x) \in D_p$, then*

(2.10.3) $$u + e(u, x) = v + e(v, y)$$

holds for a certain point $y \in p(v, u, x)$.

2.11. Lemma. *Let an lr-pseudoprocess p be transitive and let $(u, x) \in R \times P$ be such that $e(u, x) < +\infty$. Then*

(2.11.1) $$\limsup_{t_i \to u + e(u,x)} p(t_i, u, x) = \emptyset$$

holds for each (generalized) sequence of reals $t_i \in [u, u + e(u, x))$ converging to $u + e(u, x)$.

Proof. Suppose that (2.11.1) does not hold. Then to each $z \in \limsup p(t_i, u, x)$ there exists a subsequence (z_{j_i}) of a sequence of points $z_i \in p(t_i, u, x)$ converging to the point z. According to (2.10.2),

$$t_{j_i} + e(t_{j_i}, z_{j_i}) \leqq u + e(u, x)$$

holds for each i. This together with the local existence axiom and (2.6.1) implies

$$u + e(u, x) < u + e(u, x) + e(u + e(u, x), z) \leqq$$
$$\leqq \liminf_{(t_{j_i}, z_{j_i}) \to (u + e(u,x), z)} (t_{j_i} + e(t_{j_i}, z_{j_i})) \leqq$$
$$\leqq \liminf_{(t_{j_i}, z_{j_i}) \to (u + e(u,x), z)} (u + e(u, x)) = u + e(u, x),$$

which is a contradiction.

2.13. Corollary. *Let p be a transitive lr-pseudoprocess.*

(i) *If $(u, x) \in R \times P$ with $e(u, x) < +\infty$, then*

$$\liminf_{t_i \to u + e(u,x)} p(t_i, u, x) = \emptyset$$

holds for each (generalized) sequence (t_i) in $[u, u + e(u, x))$ converging to $u + e(u, x)$.

(ii) *If the space P is compact, then $e(u, x) = +\infty$ holds for all $(u, x) \in R \times P$. Thus, each transitive lr-pseudoprocess in a compact space is global.*

(iii) *If $(u, x) \in R \times P$ with $e(u, x) < +\infty$, then to each compact subset $B \subset P$ there exists $v \in [u, e(u, x))$ such that $p(t, u, x) \not\subset B$ for all $t \in (v, u + e(u, x))$.*

2.14. Definition. An *lr*-pseudoprocess p is said to be *continuous* iff it satisfies the following two axioms.

(S) *The semi-continuity axiom*: The multifunction p is upper semi-continuous.

(K) *The point closedness axiom*: The multifunction p is point closed.

2.15. Remarks. 2.15.1. For a continuous *lr*-pseudoprocess we shall use the shorter notation a *clr-pseudoprocess*; analogously a *clr-process*, a *cgr-pseudoprocess* and a *cgr-process* will stand for a continuous *lr*-process, a continuous *gr*-pseudoprocess and a continuous *gr*-process, respectively.

2.15.2. Let us define a map

(2.15.1) $$k : R \times P \to (0, +\infty] : k(u, x) =$$
$$= \sup \{t \in R \mid p(u + v, u, x) \text{ is compact for all } v \in [0, t]\} .$$

We shall prove that $k(u, x) > 0$ for each point $(u, x) \in R \times P$. Let (u, x) be an arbitrary point of $R \times P$. The multifunction p is upper semi-continuous, hence to each compact neighbourhood U of the set $\{x\} = p(u, u, x)$ there exists a neighbourhood V of the point (u, u, x) in D_p such that $p(v', u', x') \subset U$ holds for each point $(v', u', x') \in V$. Since the multifunction p is point closed and the set U is compact, $p(v', u', x')$ is compact for each point $(v', u', x') \in V$. Hence $k(u, x) > 0$ follows.

As the space P is locally compact and Hausdorff, the same reasoning yields the following assertion. If the set $p(t, u, x)$ is compact, then there exists a neighbourhood V of the point (t, u, x) in the space D_p such that $p(t', u', x')$ is compact for each point $(t', u', x') \in V$. Hence the set

$$K_p = \bigcup_{(u,x) \in R \times P} [u, u + k(u, x)) \times \{u\} \times \{x\} \tag{2.15.2}$$

is open in D_p, which implies that the map (2.15.1) is lower semi-continuous. Thus for any compact sets $I \subset R$ and $K \subset P$, the real

$$k(I, K) = \inf \{k(u, x) \mid (u, x) \in I \times K\} \tag{2.15.3}$$

is positive and

$$[u, u + k(I, K)) \times \{u\} \times K \subset K_p \tag{2.15.4}$$

holds for each $u \in I$.

2.15.3. For each point $(u, x) \in R \times P$ and each real $v \in [u, e(u, x))$, the set $p([u, v], u, x)$ will be called an *orbit* of the *lr*-pseudoprocess p through the point (u, x). The set $p([u, e(u, x)), u, x)$ will be called a *maximal orbit* of the *lr*-pseudoprocess p through the point (u, x).

2.15.4. Since the space P is locally compact and Hausdorff by our assumptions, 2.15.2 and Theorem 1.26 imply that each orbit $p([u, v], u, x)$ with $v \in [u, u + k(u, x))$ is compact.

2.15.5. Suppose that at least one of the following two conditions a), b) is satisfied.

a) The set $p(v, u, x)$ is connected for each real $v \in [u, e(u, x))$.

b) The multifunction $p(\cdot, u, x) : [u, e(u, x)) \to P$ is lower semi-continuous.

Then each orbit $p([u, v], u, x)$ of the *clr*-pseudoprocess p is connected.

Indeed, the set $[u, v] \times \{u\} \times \{x\}$ is a connected subset of D_p, the set $p(u, u, x) = \{x\}$ is connected, so that Theorem 1.28 or Theorem 1.27 may be applied in the case a) or b), respectively.

2.15.6. Let $v \in [u, u + k(u, x))$ and let one of the conditions a), b) in 2.15.5 be satisfied on the interval $[u, v]$. Then the orbit $p([u, v], u, x)$ is a continuum.

2.16. Lemma. *Let a clr-pseudoprocess p be given and let $(v, u, x) \in D_p$ with $v \in [u, k(u, x))$. Then*

$$\emptyset \neq \limsup_{t_i \to v} p(t_i, u, x) \subset p(v, u, x) \tag{2.16.1}$$

holds for each sequence of reals $t_i \in [u, v]$ converging to the real v.

Proof. Each sequence of points $z_i \in p(t_i, u, x)$ is contained in the compact set $p([u, v], u, x)$, so that it has a cluster point. This point belongs to $\limsup p(t_i, u, x)$, hence this set is non-empty. The inclusion in (2.16.1) follows from the semi-continuity axiom, Theorem 1.22 and Lemma 1.11.

2.17. Lemma. *Let p be a clr-pseudoprocess. Let sequences of points $(t_i, u_i, x_i) \in D_p$ and $y_i \in p(t_i, u_i, x_i)$ converge to points $(t, u, x) \in D_p$ and $y \in P$, respectively. Then $y \in p(t, u, x)$.*

Proof. The semi-continuity axiom, Theorem 1.22 and Lemma 1.11 imply

$$z \in \limsup p(t_i, u_i, x_i) \subset p(t, u, x) \in U - \lim p(t_i, u_i, x_i) .$$

2.18. Lemma. *Let a clr-pseudoprocess p and compact sets $I \subset R$, $K \subset P$ be given. Then to each neighbourhood U of the set K there exists a real $d > 0$ such that*

$$p([u, u + d], u, x) \subset U \tag{2.18.1}$$

holds for each point $(u, x) \in I \times K$.

Proof. Let U be neighbourhood of the set K. Then to each point $(u, x) \in I \times K$ there exist an open neighbourhood $V(u, x)$ of the point (u, x) and a real $d(u, x)$ such that

$$p([v, v + d(u, x)], v, y) \subset U \tag{2.18.2}$$

holds for each point $(v, y) \in V(u, x)$. The system of sets

$$\{V(u, x) \mid (u, x) \in I \times K\} \tag{2.18.3}$$

forms an open covering of the compact set $I \times K$. Take any finite open subcovering $\{V(u_i, x_i) \mid i = 1, 2, \ldots, n\}$ of the covering (2.18.3) and show that (2.18.1) is satisfied with

$$d = \min \{d(u_1, x_1), d(u_2, x_2), \ldots, d(u_n, x_n), e(I, K)\} , \tag{2.18.4}$$

where $e(I, K)$ is the real assigned to I, K according to (2.6.2).

To each point $(u, x) \in I \times K$ there exists an index i, $1 \leqq i \leqq n$ such that $(u, x) \in \in V(u_i, x_i)$. Hence, using (2.18.2) one obtains

$$p([u, u + d], u, x) \subset p([u, u + d(u_i, x_i)], u, x) \subset U .$$

2.19. Lemma. *Let p be a clr-pseudoprocess in P, K a compact subset of the space P and $\mathcal{U}$ the uniformity of finite open coverings of K. Then to each $U \in \mathcal{U}$ and to each compact subset $I \subset R$ there exists a real $d > 0$ such that*

$$p([u, u + d], u, x) \subset U(x) \tag{2.19.1}$$

holds for each $(u, x) \in I \times K$, where $U(x) = \{y \in P \mid (y, x) \in U\}$.

Proof. Take an arbitrary $U \in \mathcal{U}$ and a compact set $I \subset R$. Let $V \in \mathcal{U}$ be an open symmetric vicinity such that $V^2 \subset U$. To each point $(u, x) \in I \times K$ there exist an open neighbourhood $W(u, x)$ of the point (u, x) and a positive real $d(u, x)$ such that

$$(2.19.2) \qquad p([v, v + d(u, x)], v, y) \subset V(x)$$

holds for each point $(v, y) \in W(u, x)$. The system of sets

$$(2.19.3) \qquad \{W(u, x) \mid (u, x) \in I \times K\}$$

is an open covering of the compact set $I \times K$. Let $\{W(u_i, x_i) \mid i = 1, 2, \ldots, n\}$ be any finite subcovering of (2.19.3). Let a real d be chosen in the same way as in (2.18.4) and let $(u, x) \in I \times K$ be arbitrary. Then $(u, x) \in W(u_i, x_i)$ for some i, $1 \leqq i \leqq n$. Hence, applying (2.19.2) one obtains

$$(2.19.4) \qquad p([u, u + d], u, x) \subset p([u, u + d(u_i, x_i)], u, x) \subset V(x_i)\,.$$

Since $p(u, u, x) = \{x\} \subset V(x_i)$ and V is symmetric, $x_i \in V(x)$. Hence $V(x_i) \subset \subset V^2(x) \subset U(x)$, which together with (2.19.4) gives (2.19.1).

2.20. Theorem. *Let a clr-process p and a point $(u, x) \in R \times P$ be given. Denote $w = \min\{k(u, x), g(u, x)\}$. Then the multifunction*

$$(2.20.1) \qquad p(\cdot, u, x) : [u, w) \to P$$

is lower semi-continuous.

Proof. According to Theorem 1.23 we have to prove that for each open set $G \subset P$ the set

$$T = \{t \in [u, w) \mid p(t, u, x) \cap G \neq \emptyset\}$$

is open in $[u, w)$. Let t_0 be an arbitrary point of the set T.

First we shall suppose that $u < t_0$. Take an arbitrary point $z \in p(t_0, u, x) \cap G$ and an arbitrary real $u' \in (t_0, w)$. Since the set $p([u, u'], u, x)$ is compact, the set $p([u, u'], u, x) - G = A$ is compact as well. Thus there exists an open neighbourhood U of the set A such that

$$(2.20.2) \qquad z \notin U\,.$$

According to Lemma 2.18, to the neighbourhood U of the set A there exists a real $d > 0$ such that

$$(2.20.3) \qquad p([v, v + d], v, y) \subset U \quad \text{for all} \quad v \in [u, u'], \quad y \in A\,.$$

The real d may be supposed to satisfy $d < t_0 - u$. Let us show that $(t_0 - d, t_0) \subset T$. Take an arbitrary $t \in (t_0 - d, t_0)$. Since $z \in p(t_0, u, x) \subset p(t_0, t, p(t, u, x))$, there exists a point $z' \in p(t, u, x)$ such that $z \in p(t_0, t, z')$. Let us prove that

$$(2.20.4) \qquad z' \in p(t, u, x) \cap G\,.$$

Suppose that $z' \notin G$. Then $z' \in A$, so that $z \in p(t_0, t, z') \subset U$ according to (2.20.3), which contradicts (2.20.2). Thus (2.20.4) holds. Hence $t \in T$ and since t was an arbitrary point of the interval $(t_0 - d, t_0)$, the whole interval $(t_0 - d, t_0)$ has to be contained in the set T. Thus the first part of the proof is complete.

Now suppose that $u \leqq t_0$ and that the real $d > 0$ was chosen so as to satisfy $d < w - t_0$ and $p([t_0, t_0 + d], t_0, z) \subset G$. Then $p(t, u, x) \cap G = p(t, t_0, p(t_0, u, x)) \cap G \supset p(t, t_0, z) \cap G = p(t, t_0, z) \neq \emptyset$ holds for all $t \in [t_0, t_0 + d]$. Hence $[t_0, t_0 + d) \subset T$, which completes the proof.

2.21. Theorem. *Let p be a clr-process. Then for each point $(u, x) \in R \times P$ and for each real $v \in [u, w)$ with $w = \min\{k(u, x), g(u, x)\}$ the set $p([u, v], u, x)$ is a continuum.*

Proof. Apply Theorem 2.20 and Remark 2.15.6.

2.22. Lemma. *Let p be a clr-pseudoprocess. Then to each point $(u, x) \in R \times P$ there exist a real $k > 0$ and a compact neighbourhood K of the point x such that $(t, w, z) \in G_p$ holds for each point $(w, z) \in [u - k, u + k] \times K$ and each $t \in \in [w, w + k]$.*

Proof. Let (u, x) be an arbitrary point of $R \times P$ and let U be an arbitrary compact neighbourhood of the point x. Then there exist a compact neighbourhood V of the point x and compact neighbourhoods I, J of the point u in R such that $p(t, v, y) \subset U$ holds for each $(t, v, y) \in (J \times I \times V) \cap D_p = H$. The set H is compact in D_p, the sets $p(t, v, y)$ being closed subsets of the compact set U are compact for each $(t, v, y) \in H$ and the multifunction p is upper semi-continuous, so that the set

$$K = \bigcup_{(t,v,y) \in H} p(t, v, y)$$

is compact by Theorem 1.26. Since $(u, u, y) \in H$ for each point $y \in V$ implies $V \subset K$, the set K is a compact neighbourhood of the point x. Let $g(I, K)$ be the real from (2.6.3) assigned to the neighbourhood K of the point x and the neighbourhood I of the real u. Then $(t, w, z) \in G_p$ holds for all $w \in I$, $z \in K$ and $t \in [w, w + g(I, K))$. One may easily verify that the assertion of the lemma is satisfied with the set K and the real $k = g(I, K)/3$.

2.23. Lemma. *Let p be a clr-pseudoprocess. Then to each point $(u, x) \in R \times P$ there exist a real $k > 0$ and a compact neighbourhood K of the point x such that $(t, w, z) \in K_p$ holds for each point $(w, z) \in [u - k, u + k] \times K$ and each $t \in \in [w, u + k]$.*

Proof. The reasoning is the same as in the proof of Lemma 2.22 with $g(u, x)$ and G_p replaced by $k(u, x)$ and K_p, respectively.

3.1. Remark. In the paper [11] the concept of a solution of a right pseudoprocess was introduced.This concept will be recalled and its properties will be investigated in the case of continuous local right pseudoprocesses in a locally compact Hausdorff space P.

3.2. Definition. Let p be an lr-pseudoprocess in P and let $s : R \to P$ be a map. The map s is called a *solution* of the lr-pseudoprocess p iff it has the following two properties:

(i) The domain D_s of the map s is a non-empty interval in R.

(ii) If $u, v \in D_s$, $u \leqq v$, then

(3.2.1) $$s(v) \in p(v, u, s(u)) .$$

3.3. Remarks. In what follows the set of all solutions of a given lr-pseudoprocess will be denoted by S.

3.3.1. If $s \in S$ and $J \subset D_s$ is an interval, then $s|_J \in S$.

3.3.2. Let I be any index set and let $s_i \in S$ for $i \in I$ be such that $\bigcap s_i$ is a map and $D_{\cap s_i}$ is a non-empty interval. Then $\bigcap s_i \in S$.

3.3.3. Let an lr-pseudoprocess p be transitive, let $s_1, s_2 \in S$ be such that $D_{s_1} \cap \cap D_{s_2} \neq \emptyset$ and $s_1 \cup s_2$ is a map. Then $s_1 \cup s_2 \in S$.

3.3.4. Let an lr-pseudoprocess p be transitive, let I be any index set and let $s_i \in S$ for $i \in I$ be such that $s_j \cup s_k \in S$ whenever $j, k \in I$ and $D_{s_j} \cup D_{s_k}$ is an interval. Then $\bigcup s_i \in S$.

3.4. Definition. Let an lr-pseudoprocess p, a point $(u, x) \in R \times P$ and a solution s of p be given.

The solution s is called a *right* (*left*) *solution through the point* (u, x) iff $s(u) = x$ and $\min D_s = u$ ($\max D_s = u$).

The solution s is called a *maximal right* (*maximal left*) *solution through the point* (u, x) if it is a right (left) solution through the point (u, x) and $s = s'$ holds for each right (left) solution s' through the point (u, x) satisfying $s \subset s'$.

3.5. Remark. The set of all right (left) or maximal right (maximal left) solutions through the point (u, x) will be denoted by $S^+(u, x)$ ($S^-(u, x)$) or $C^+(u, x)$ ($C^-(u, x)$), respectively.

3.6. Theorem. *Let p be a transitive clr-pseudoprocess in P and let $s \in S^-(u, x)$ or $s \in S^+(u, x)$ be a continuous solution such that $D_s = (a, u]$ or $D_s = [u, b)$, respectively, with $-\infty < a < u < b < +\infty$. Then exactly one of the following two possibilities occurs.*

(i) *For each (generalized) sequence of reals* $t_i \in (a, u]$ *or* $t_i \in [u, b)$ *converging to the real* a *or* b*, respectively, the sequence* $(s(t_i))$ *has no cluster point. Then the solution* s *is maximal.*

(ii) *There exists a limit*

$$\lim_{t \to a+} s(t) \quad \text{or} \quad \lim_{t \to b-} s(t)\,.$$

Then the solution s *may be extended to the solution* s' *defined on the interval* $[a, u]$ *or* $[u, b]$*, respectively. Thus the solution* s *is not maximal.*

Proof. Take any continuous $s \in S^-(u, x)$ and suppose that there exists a sequence of reals $t_i \in (a, u]$ converging to the real a such that the sequence $(s(t_i))$ converges to a point $y \in P$. Define the map s'

$$s' : [a, u] \to P : s'(t) = \begin{cases} s(t) & \text{for} \quad t \in (a, u]\,, \\ y & \text{for} \quad t = a \end{cases} \tag{3.6.1}$$

and show that $s' \in S^-(u, x)$.

Clearly, $D_{s'} = [a, u]$ and $s'(v) \in p(v, w, s'(w))$ holds for all $v, w \in (a, u]$, $w \leqq v$. It remains to prove that

$$s'(v) \in p(v, a, s'(a)) \tag{3.6.2}$$

holds for all $v \in [a, u]$. Since (3.6.2) with $v = a$ is fulfilled trivially, $a < v$ may be assumed. Thus (3.6.1) implies

$$s'(v) = s(v) \in p(v, t_i, s(t_i)) = p(v, t_i, s'(t_i))$$

for all $t_i \in (a, v)$. Hence (3.6.2) holds for all $v \in [a, a + e(a, y)) \cap [a, u]$ in virtue of Lemma 2.17. Therefore it is sufficient to prove that $u < a + e(a, y)$.

Suppose that there exists $w \in [a, u]$ such that $w = a + e(a, y)$. Then

$$s'(w) = \lim_{v \to w-} s'(v) \in \limsup_{v \to (a + e(a,y))-} p(v, a, y) = \emptyset$$

by Lemma 2.11, which is a contradiction.

The proof for right solutions is similar.

3.7. Theorem. *Let a clr-pseudoprocess* p *be given. Then each solution* $s \in S^+(u, x)$ *with* $D_s \subset [u, k(u, x))$ *is continuous.*

Proof. Let w be an arbitrary point of D_s. Let us prove that s is continuous at the point w. Take $v \in D_s$ such that $w \in [u, v] \subset [u, k(u, x))$ and denote by s' the restriction of the solution s to the interval $[u, v]$. Let $\mathscr{U}$ denote the uniformity of finite open coverings of the compact set $K = p([u, v], u, s(u))$. We shall prove that the solution

$$s' : [u, v] \to (K, \mathscr{U})$$

is uniformly continuous.

Recall that $s(t) \in p(t, u, s(u)) \subset K$ holds for each $t \in [u, v]$. Take an arbitrary symmetric $U \in \mathscr{U}$. According to Lemma 2.19, to the vicinity U and to the compact interval $[u, v]$ there exists a real $d > 0$ such that $p(t, t', s(t')) \subset U(s(t'))$ holds for all $t, t' \in [u, v]$, $0 \leqq t - t' < d$.

Now,

$$s(t) \in p(t, t', s(t')) \subset U(s(t')) \quad \text{for all} \quad 0 \leqq t - t' < d \quad \text{in} \quad [u, v]$$

implies

$$(s(t), s(t')) \in U \quad \text{for all} \quad 0 \leqq t - t' < d \quad \text{in} \quad [u, v],$$

which was to be proved.

3.8. Lemma. *Let a clr-pseudoprocess p be given. Then to each continuous solution s with D_s compact there exist a real $k > 0$ and a compact set $K \subset P$ such that $p(t, u, s(u)) \subset K$ holds for each $u \in D_s$ and each $t \in [u, u + k] \cap D_s$.*

Proof. According to Lemmas 2.22 and 2.23, to each point $(u, s(u))$ with $u \in D_s$ there exist a positive real $k(u)$ and a compact neighbourhood $K(u)$ of the point $s(u)$ such that $(t, w, z) \in K_p \cap G_p$ holds for all $(w, z) \in [u - k(u), u + k(u)] \times K(u)$ and all $t \in [w, u + k(u)]$. Let $O(u)$ denote the interior of the set $[u - k(u), u + k(u)] \times K(u)$. Clearly, the system $\{O(u) \mid u \in D_s\}$ is an open covering of the compact set $H = \{(u, s(u)) \mid u \in D_s\}$. Now take any open subcovering $\{O(u_1), O(u_2), \ldots, O(u_n)\}$ and set

$$k = \min \{k(u_1), k(u_2), \ldots, k(u_n)\},$$

$$K = \bigcup \{p([w, u_i + k(u_i)], w, z) \mid (w, z) \in$$

$$\in [u_i - k(u_i), u_i + k(u_i)] \times K(u_i),\ i = 1, 2, \ldots, n\}.$$

3.9. Theorem. *Let a clr-pseudoprocess p and a solution s of p be given. Then s is continuous iff to each $v \in D_s$, $v \neq \inf D_s$ there exists a real $w \in D_s$ such that $w < v < k(w, s(w))$.*

Proof. Let s be continuous and $v \in D_s$, $v \neq \inf D_s$. Let $I \subset D_s$ be any compact neighbourhood of v and let s' denote the restriction of the solution s to the interval I. According to Lemma 3.8 there exist a real $k > 0$ and a compact set $K \subset P$ such that $p(t, u, s(u)) \subset K$ holds for each point $u \in I$ and each $t \in [u, u + k] \cap I$. Now, taking any $k' \in (0, \frac{1}{2}k)$ such that $[v - k', v + k'] \subset I$ it is easy to show that the assertion of the theorem is satisfied with $w = v - k'$.

Suppose now that the assertion of the theorem is satisfied and show that the solution s is continuous at each point $v \in D_s$. If $v = \inf D_s \in D_s$, then s is continuous at the point v by Theorem 3.7. If $v > \inf D_s$, then there exists a real $w \in D_s$ such that $w < v < k(w, s(w))$. Now Theorem 3.7 may be applied to the restriction of the solution s to the interval $[w, k(w, s(w)))$.

3.10. Theorem. *Let a clr-process p, a point $(v, u, x) \in K_p$ and a point $y \in p(v, u, x)$ be given. Then there exists a continuous solution $s \in S^+(u, x)$ such that $s(u) = x$, $s(v) = y$.*

Proof. Without loss of generality we may assume that $v - u = 1$. Denote

$$B_n = \left\{ t_n^k \mid t_n^k = u + \frac{k}{2^n}, \quad k = 0, 1, 2, \ldots, 2^n \right\}$$

for each positive integer $n \in N$. Recall that

$$t_n^k = u + \frac{2m}{2^n} = t_{n-1}^m \in B_{n-1} \quad \text{for all} \quad n > 1 \quad \text{and} \quad k = 2m\,,$$

and that the set

$$B = \bigcup_{n=1}^{\infty} B_n$$

is dense in the interval $[u, v]$. We shall define the points $s(t_n^k)$ for all $k, n \in \mathsf{N}$, $n > 0$, as follows:

For each $n \in \mathsf{N}$, $n > 0$, we define the set

$$G_n = \{s(t_n^k) \mid t_n^k \in B_n\}$$

by induction.

For $n = 1$ we set $G_1 = \{s(u), s(u + \frac{1}{2}), s(v)\}$ with $s(u) = x$, $s(v) = y$ and $s(u + \frac{1}{2}) \in p(u + \frac{1}{2}, u, s(u))$ such that $s(v) \in p(v, u + \frac{1}{2}, s(u + \frac{1}{2}))$. This is possible since

$$s(v) = y \in p(v, u, x) = p(v, u, s(u)) = p(v, u + \tfrac{1}{2}, p(u + \tfrac{1}{2}, u, s(u)))\,.$$

Suppose now that for a given $n \in \mathsf{N}$ the set G_n is defined in such a way that $s(u) = x$, $s(v) = y$ and

$$s(t_n^k) \in p(t_n^k, t_n^r, s(t_n^r)) \quad \text{for all} \quad t_n^k, t_n^r \in B_n\,, \quad r \leqq k\,,$$

and define the set G_{n+1}. For $k = 2m$ set

$$s(t_{n+1}^k) = s(t_n^m) \in G_n$$

and for $k = 2m + 1$ take

$$s(t_{n+1}^k) \in p(t_{n+1}^k, t_{n+1}^{k-1}, s(t_{n+1}^{k-1}))$$

such that

$$s(t_{n+1}^{k+1}) \in p(t_{n+1}^{k+1}, t_{n+1}^k, s(t_{n+1}^k))$$

holds. This is possible since

$$s(t_{n+1}^{k+1}) = s(t_n^{m+1}) \in p(t_n^{m+1}, t_n^m, s(t_n^m)) = p(t_{n+1}^{k+1}, t_{n+1}^{k-1}, s(t_{n+1}^{k-1})) = \\ = p(t_{n+1}^{k+1}, t_{n+1}^k, p(t_{n+1}^k, t_{n+1}^{k-1}, s(t_{n+1}^{k-1})))\,.$$

Let us show that

$$s(t_{n+1}^k) \in p(t_{n+1}^k, t_{n+1}^r, s(t_{n+1}^r))$$

holds for all $t^r_{n+1}, t^k_{n+1} \in B_{n+1}$, $r < k$. It will be seen in what follows that it is possible to assume that none of the integers r, k is even. Denote $r = 2q + 1$, $k = 2m + 1$. Then, using the induction hypothesis on G_n, we obtain

$$s(t^k_{n+1}) \in p(t^k_{n+1}, t^{k-1}_{n+1}, s(t^{k-1}_{n+1})) = p(t^k_{n+1}, t^m_n, s(t^m_n)) \subset$$
$$\subset p(t^k_{n+1}, t^m_n, p(t^m_n, t^{q+1}_n, s(t^{q+1}_n))) = p(t^k_{n+1}, t^{r+1}_{n+1}, s(t^{r+1}_{n+1})) \subset$$
$$\subset p(t^k_{n+1}, t^{r+1}_{n+1}, p(t^{r+1}_{n+1}, t^r_{n+1}, s(t^r_{n+1}))) = p(t^k_{n+1}, t^r_{n+1}, s(t^r_{n+1}))\,.$$

Thus we have constructed a map $s : B \to P$ such that

$$s(u) = x\,, \quad s(v) = y\,, \quad s(t) \in p(t, t', s(t')) \quad \text{for all} \quad t, t' \in B\,, \quad t' < t\,.$$

In the same way as in the proof of Theorem 3.7 it may be proved that the map

$$s : B \subset R \to p([u, v], u, x)$$

is uniformly continuous. Since the map s maps the dense subset B of the compact metric space $[u, v]$ into a complete uniform Hausdorff space $p([u, v], u, x)$, (see 1.6), it may be uniquely extended to the uniformly continuous map of the whole interval $[u, v]$. Let the symbol s denote also this extended map and let us show that the map s is the solution the existence of which is asserted in the theorem.

Clearly $s(u) = x$, $s(v) = y$. Take any $t, t' \in [u, v]$, $t' < t$. Let (t_n), (t'_n) be sequences of reals in B such that $t_n > t'_n$ for all n, $t_n \to t$, $t'_n \to t'$ for $n \to +\infty$. The continuity of s implies $s(t_n) \to s(t)$, $s(t'_n) \to s(t')$ for $n \to +\infty$. As $s(t_n) \in p(t_n, t'_n, s(t'_n))$ for all n, Lemmas 2.16 and 2.17 may be applied. Hence

$$s(t) = \lim_{n \to +\infty} s(t_n) \in \limsup_{n \to +\infty} p(t_n, t'_n, s(t'_n)) \subset p(t, t', s(t'))\,.$$

Thus $s \in S^+(u, x)$.

3.11. Remark. The proof of Theorem 3.10 is similar to those of Budak [4] and Bronshtein [3]. Analogous theorems for generalized dynamical systems in metric spaces are proved in [1] and [14].

Recall that in terms of the paper [11] the preceding theorem asserts that each point compact *clr*-process is solution complete (see Definition 2.11 in [11]).

3.12. Corollary. *Let a clr-process p be given. Then* $S^+(u, x) \neq \emptyset$ *for each point* $(u, x) \in R \times P$.

3.13. Lemma. *Let a clr-pseudoprocess p, a real* $u \in R$ *and a compact set* $H \subset P$ *be given. Let T be a real such that* $0 < T < \inf\{k(u, y) \mid y \in H\}$ *and let M denote the set of all solutions* $s \in S$ *such that* $D_s = [u, T]$ *and* $s(u) \in H$. *Then the set M is equicontinuous.*

Proof. Let $\mathcal{U}$ denote the uniformity of finite open coverings of the compact set $K = p([u, T], u, H)$. We have to prove that to each vicinity $U \in \mathcal{U}$ and to each real

$v \in [u, T]$ there exists a real $d > 0$ such that

(3.13.1)
$$(s(t), s(v)) \in U \quad \text{for all} \quad t \in [v - d, v + d] \cap [u, T] \quad \text{and all} \quad s \in M .$$

Since the set of all open symmetric vicinities forms a base for the uniformity $\mathscr{U}$, it is sufficient to prove (3.13.1) only for U symmetric and open.

Let any symmetric open $U \in \mathscr{U}$ and any $v \in [u, T]$ be given.

First, suppose that $v \neq T$. According to Lemma 2.19, to the compact sets $K \subset \mathsf{P}$, $I = [u, T]$ and the vicinity U there exists a real $d' \in (0, T - v)$ such that $p([v, v + d'], v, s(v)) \subset U(s(v))$ holds for each $s \in M$. Thus $s(t) \in p(t, v, s(v)) \subset U(s(v))$ holds for all $u \leqq v \leqq t \leqq v + d' \leqq T$. Hence

$$(3.13.2) \qquad (s(t), s(v)) \in U \quad \text{for all} \quad t \in [v, v + d'] \quad \text{and all} \quad s \in M .$$

Now suppose that $v \neq u$. In a similar way as above, a real $d'' \in (0, v - u)$ may be found such that $p([t, v], t, s(t)) \subset U(s(t))$ holds for all $t \in [v - d'', v]$ and all $s \in M$. Thus $s(v) \in p(v, t, s(t)) \subset U(s(t))$ holds for all $t \in [v - d'', v]$ and all $s \in M$. Hence

$$(3.13.3) \qquad (s(v), s(t)) \in U \quad \text{for all} \quad t \in [v - d'', v] \quad \text{and all} \quad s \in M .$$

Now (3.13.2) and (3.13.3) imply (3.13.1) with $d = \min \{d', d''\}$.

3.14. Theorem. *Let a clr-pseudoprocess, a sequence of points $z_i \in \mathsf{P}$ converging to a point $z \in \mathsf{P}$ and a real $u \in \mathsf{R}$ be given. Let H denote the set of all members of the sequence (z_i) and its limit point z. Let T be a real such that $0 < T < \inf \{k(u, y) \mid y \in H\}$. Then to each sequence of solutions $s_n \in S^+(u, z_n)$ with $D_{s_n} = [u, T]$ there exists a continuous solution $s \in S^+(u, z)$ with $D_s = [u, T]$ and a subsequence (s_m) of the sequence (s_n) converging to the solution s uniformly on the interval $[u, T]$.*

Proof. Let K denote the compact subset $p([u, T], u, H)$ of the space P. Let $C([u, T], K)$ denote the space of all continuous maps of the interval $[O, T]$ into the space K and let the space $C)[u, T], K)$ be endowed with the topology of uniform convergence. Let M denote the set of all members of the sequence (s_n). According to Theorem 3.7 all solutions s_n are continuous, so that M is a subset of the space $C([u, T], K)$. The set M is equicontinuous by Lemma 3.13 and the closure of the set $\{s_n(v) \mid s_n \in M\}$ being a closed subset of the compact set K is compact for each $v \in [u, T]$. Thus Ascoli Theorem (see [8], chap. 7, Th. 17) may be applied. Hence the closure of the set M in the space $C([u, T], K)$ is compact, so that there exists a subsequence (s_m) of the sequence (s_n), converging to an element $s \in C([u, T], K)$. Let us show that the map s is the solution we are looking for.

Clearly $s(u) = z$ and $D_s = [u, T]$. Take arbitrary $t, t' \in D_s$, $t' \leqq t$. Then $s_m(t) \in p(t, t', s_m(t'))$ holds for all m. Hence, using Lemma 2.17 one obtains $s(t) \in p(t, t', s(t'))$ for all $t' \leqq t$ in D_s, which completes the proof.

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