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WEAK ISOMORPHISMS OF ABELIAN LATTICE ORDERED GROUPS

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The notions of weak homomorphism and weak isomorphism of general algebras have been introduced by GOETZ and MARCZEWSKI (cf. [3], [7], [8]). The concept of weak isomorphism of general algebras has been contained implicitly in MALCEV's papers [5], [6]; CSÁKÁNY [1] denotes this concept as equivalence of algebras.

Several authors investigated weak homomorphisms and weak isomorphisms of concrete types of algebraic structures (for references, cf. e.g., GŁAZEK and MICHALSKI [2]).

In this note it will be shown that if φ is a weak isomorphism of an abelian lattice ordered group $\mathfrak{G}$ onto a lattice ordered group $\mathfrak{G}_1$, then 1) φ is an isomorphism with respect to the group operation, and 2) φ is either an isomorphism or a dual isomorphism with respect to the partial order.

We recall some relevant basic notions concerning weak isomorphisms.

Let $\mathfrak{A} = (A; F)$ be a general algebra with the underlying set A and with the system F of fundamental operations. Let i, n be positive integers, $i \leqq n$. We define an n-ary operation a_i^n on the set A by putting $a_i^n(x_1, \ldots, x_n) = x_i$ for each n-tuple $x_1, \ldots, x_n$ of elements of A. We denote by $\mathscr{P}(\mathfrak{A})$ the least set of operations on the set A such that:

(i) $F \subseteq \mathscr{P}(\mathfrak{A})$ and $a_i^n \in \mathscr{P}(\mathfrak{A})$ for any positive integers i, n with $i \leqq n$;

(ii) $\mathscr{P}(\mathfrak{A})$ is closed with respect to superpositions.

The system $\mathscr{P}(\mathfrak{A})$ will be called *the system of all polynomials of the algebra* $\mathfrak{A}$.

Let $\mathfrak{A} = (A, F)$ and $\mathfrak{A}_1 = (A_1, F_1)$ be general algebras and let φ be a one-to-one mapping of A onto A_1. For each n-ary operation $f \in F$ and n-tuple $y_1, \ldots, y_n \in A_1$ we define

$$f^*(y_1, \ldots, y_n) = \varphi(f(\varphi^{-1}(y_1), \ldots, \varphi^{-1}(y_n))) .$$

Similarly, for each n-ary operation $f_1 \in F_1$ and each n-tuple $x_1, \ldots, x_n \in A$ we put

$$f^*(x_1, \ldots, x_n) = \varphi^{-1}(f_1(\varphi(x_1), \ldots, \varphi(x_n))) .$$

The mapping φ is called *a weak isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{A}_1$, if $f^* \in \mathscr{P}(\mathfrak{A}_1)$ and $f_1^* \in \mathscr{P}(\mathfrak{A})$ for each $f \in F$ and each $f_1 \in F_1$.

Without loss of generality we can assume that $A \cap A_1 = \emptyset$. In this case we can identify the elements x and $\varphi(x)$ for each $x \in A$. Thus we can suppose that the algebras $\mathfrak{A}$ and $\mathfrak{A}_1$ have the same underlying set and that the identity mapping is a weak isomorphism of the algebra $\mathfrak{A}$ onto $\mathfrak{A}_1$. Hence $f = f^* \in \mathscr{P}(\mathfrak{A}_1)$ and $f_1 = f_1^* \in \mathscr{P}(\mathfrak{A})$ for each $f \in F$ and each $f_1 \in F_1$.

Now let us investigate the case when $\mathfrak{A} = \mathfrak{G} = (G; +, -, \wedge, \vee)$ and $\mathfrak{A}_1 = \mathfrak{G}_1 = (G; +_1, -_1, \wedge_1, \vee_1)$ are lattice ordered groups. The positive cone and the negative cone of $\mathfrak{G}$ will be denoted by G^+ and G^-, respectively. The symbols G_1^+ and G_1^- have the analogous meaning with respect to $\mathfrak{G}_1$. The relation of the partial order in $\mathfrak{G}$ or in $\mathfrak{G}_1$ will be denoted by $\leqq$ and $\leqq_1$, respectively. If $a \in G$, $a_1 = a_2 = \ldots = a_n = a$, then we denote $a_1 + a_2 + \ldots + a_n = na$, $a_1 +_1 \ldots +_1 a_n = n \circ a$. The following result has been established in [4]:

(*) Suppose that (i) $\wedge, \vee \in \mathscr{P}(\mathfrak{G}_1)$, $\wedge_1, \vee_1 \in \mathscr{P}(\mathfrak{G})$, and (ii) the neutral element of $\mathfrak{G}$ coincides with the neutral element of $\mathfrak{G}_1$. Then we have either

(1) $$G^+ = G_1^+ \quad \text{and} \quad G^- = G_1^- \,,$$

or

(2) $$G^+ = G_1^- \quad \text{and} \quad G^- = G_1^+ \,.$$

In what follows we assume that the identity is a weak isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}_1$. Further we suppose that $\mathfrak{G}$ is abelian. The case card $G = 1$ being trivial we assume that card $G > 1$. From the basic algebraic rules valid for lattice ordered groups it follows that each binary operation belonging to $\mathscr{P}(\mathfrak{G})$ with variables x_1, x_2 can be expressed in the form

(3) $$\bigwedge_{i \in I} \bigvee_{j \in J} (m_{ij}x_1 + n_{ij}x_2) \,,$$

where I, J are nonempty finite sets and n_{ij}, m_{ij} are integers for each $i \in I$, $j \in J$.

Lemma 1. *The neutral element of $\mathfrak{G}$ coincides with the neutral element of $\mathfrak{G}_1$.*

Proof. Let 0 be the neutral element of $\mathfrak{G}$. Then $0 +_1 0$ can be expressed in the form (3) with $x_1 = x_2 = 0$. Hence $0 +_1 0 = 0$ and thus 0 is the neutral element in $\mathfrak{G}_1$ as well.

From (*) and from Lemma 1 we obtain:

Corollary 1. *Either the relations* (1) *or the relations* (2) *are fulfilled.*

The following two assertions are immediate consequences of the fact that the identity mapping is a weak isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}_1$.

Lemma 2. *Let $A \subseteq G$. If A is closed with respect to all fundamental operations of $\mathfrak{G}$, then A is closed with respect to all fundamental operations of $\mathfrak{G}_1$, and conversely.*

Lemma 3. *Let R be a congruence relation of $\mathfrak{G}$. Then R is a congruence relation of $\mathfrak{G}_1$, and conversely.*

From Lemmas 1 and 3 we obtain:

Corollary 2. *Let $A \subseteq G$. If A is an l-ideal of $\mathfrak{G}$, then A is an l-ideal of $\mathfrak{G}_1$, and conversely.*

We denote by N, N^+ and N_0 the set of all positive integers, the set of all non-negative integers and the set of all integers, respectively.

Lemma 4. *Let $0 < t \in G$. Then $nt = n \circ t$ is valid for each positive integer n.*

Proof. Suppose that the condition (1) is valid (in the case when (2) holds we can use the dual argument). Denote $A = \{nt\}_{n \in N_0}$. Then A is the least l-subgroup of $\mathfrak{G}$ containing the element t. Hence according to Lemma 2, A is also the least l-sugroup of $\mathfrak{G}_1$ containing the element t, thus $A = \{n \circ t\}_{n \in N_0}$. This together with (1) implies

$$\{nt\}_{n \in N} = \{n \circ t\}_{n \in N} . \tag{4}$$

Suppose that $x_1 +_1 x_2$ is expressed by (3) for each $x_1, x_2 \in G$. Consider the system S of all planes $z = m_{ij}x + n_{ij}y$ in the three-dimensional euclidean space with coordinates x, y, z. Let P be the set of all points $P(x, y, z)$ with $x > 0$, $y > 0$, having the property that $P(x, y, z)$ belongs to the intersection of two distinct planes of the system S. Then either $P = \emptyset$ or there exists $P_0(x_0, y_0, z_0) \in P$ such that $y_0 x_0^{-1} \leqq yx^{-1}$ for each $P(x, y, z) \in P$. In the first case we put $M = N^+ \times N^+$; in the second we denote $M = \{(m, n) \in N \times N^+ : nm^{-1} \leqq y_0 x_0^{-1}\} \cup \{(0, 0)\}$.

From the definition of the set M it follows that there exists a plane $z = m_{i_0 j_0}x + n_{i_0 j_0}y \in S$ having the property

$$mt +_1 nt = m_{i_0 j_0}(mt) + n_{i_0 j_0}(nt) = (m_{i_0 j_0}m + n_{i_0 j_0}n)\, t \tag{5}$$

for each $(m, n) \in M$.

Let $m \in N$, $n = 0$. According to Lemma 1 we have $mt +_1 0\, t = mt$, and hence (5) yields

$$m_{i_0 j_0} = 1 . \tag{6}$$

There exists $m \in N$ with $(m, 1) \in M$; let m_0 be the least positive integer with this property. If $m > m_0$, then $(m, 1)$ also belongs to M.

Clearly $n_{i_0 j_0} \neq 0$. Assume that $n_{i_0 j_0} < 0$. Then $0 < m_0 t +_1 t = (m_0 + n_{i_0 j_0})\, t$, hence $m_0 > -n_{i_0 j_0}$. For each $i \in N$ with $i \leqq m_0$ we have (cf. (4))

$$it +_1 t = k_i t\,, \quad k_i > 0\,;$$

put $k = \max k_i$ $(i = 1, 2, \ldots, m_0 - 1)$. We can easily verify that all elements $m_0 t +_1 n \circ t$ $(n = 1, 2, 3, \ldots)$ belong to the set $\{t, 2t, \ldots, kt\}$. On the other hand, the set $\{m_0 t +_1 n \circ t\}_{n \in N}$ is infinite and so we arrived at a contradiction. Hence $n_{i_0 j_0} > 0$.

Assume that $n_{i_0 j_0} > 1$. Let $m \geqq m_0$. By calculating $mt +_1 t$, $(mt +_1 t) +_1 t, \ldots$ we obtain that

$$mt +_1 n \circ t = (m + n_{i_0 j_0} n)\, t$$

for each $n \in N$. From this and from $n_{i_0 j_0} > 1$ it follows that the set

$$\{nt\}_{n\in N} \setminus \{mt +_1 n \circ t\}_{n\in N} = B$$

is infinite. Now (4) implies

$$B = \{n \circ t\}_{n\in N} \setminus \{mt +_1 n \circ t\}_{n\in N}$$

and this set has only a finite number of elements, which is a contradiction. Therefore $n_{i_0 j_0} = 1$. In view of (5) and (6) we obtain

$$mt +_1 nt = (m + n)\, t \tag{7}$$

for each $(m, n) \in M$.

Let m_0 be as above. According to (4) there exists $m_0' \in N$ with $m_0 t = m_0' \circ t$. Thus (7) implies

$$(m_0' + 1) \circ t = m_0' \circ t +_1 t = (m_0 + 1)\, t\,,$$

and by induction we obtain

$$(m_0' + n) \circ t = (m_0 + n)\, t \tag{8}$$

for each $n \in N$.

Let a positive integer $p > 1$ be given. For each $i \in \{0, 1, 2, \ldots, p - 1\}$ we denote $A_i = \{(m_0 + i + np)\, t\}_{n\in N^+}$. Further, for each $x \in G$ we denote by $A(x)$ the l-subgroup of $\mathfrak{G}$ generated by the element x. Then $x = pt$ satisfies the following condition:

(α) There exists $i \in \{0, 1, 2, \ldots, p - 1\}$ such that $A_i \subseteq A(x)$ and $A_j \cap A(x) = \emptyset$ for each $j \in \{0, 1, 2, \ldots, p - 1\}$ with $j \neq i$.

According to Lemma 2, $A(x)$ is also the l-subgroup of $\mathfrak{G}_1$ generated by x. Put $A_i' = \{(m_0' + i + np) \circ t\}_{n\in N^+}$. From (8) it follows that $A_i = A_i'$ for each $i \in \{0, 1, 2, \ldots, p - 1\}$. Thus from ($\alpha$) we infer that the following condition is fulfilled:

(α_1) There exists $i \in \{0, 1, 2, \ldots, p - 1\}$ such that $A_i' \subseteq A(x)$ and $A_j' \cap A(x) = \emptyset$ for each $j \in \{0, 1, 2, \ldots, p - 1\}$ with $j \neq i$.

Moreover, from (4) we get that there is $p' \in N$ with $x = p' \circ t$. From the definition of A_i' we obtain that the following assertion is valid:

(β) Let q be a positive integer. Suppose that there exists $i \in \{0, 1, 2, \ldots, p - 1\}$ such that $A_i \subseteq A(q \circ t)$ and $A_j \cap A(q \circ t) = \emptyset$ for each $j \in \{0, 1, 2, \ldots, p - 1\}$, $j \neq i$. Then $q = p$.

(In fact, from $A_i \subseteq A(q \circ t)$ it follows that there is a positive integer m with $p = mq$; from $A_j \cap A(q \circ t) = \emptyset$ we get $m = 1$.)

From (α_1) and (β) we conclude $p' = p$. Hence $pt = p \circ t$ is valid for each positive integer p.

Lemma 4'. *Let $0 < t \in G$. Then $nt = n \circ t$ is valid for each integer n.*

Proof. In view of Lemma 4 it suffices to verify that $-t = -_1 t$. Further, we can suppose that (1) holds (in the case (2) the proof would be analogous). The set $M =$

$= \{nt\}_{n \in N_0}$ is the l-subgroup of G generated by t. Hence this set is also the l-subgroup of G_1 generated by t. From (1) it follows that $t >_1 0$, whence $-_1 t <_1 0$, thus there exists a positive integer m such that $-_1 t = -mt$. The l-subgroup of G_1 generated by $-_1 t$ coincides with M. Hence the l-subgroup of G generated by $-mt$ coincides with M. Thus $m = 1$.

Corollary. *Let $t \in G$ be such that either $t \geqq 0$ or $t \leqq 0$. Then $nt = n \circ t$ is valid for each $n \in N_0$.*

For each $x \in G$ we denote, as usual, $|x| = (x \vee 0) - (x \wedge 0)$. We have $x = (x \vee 0) + (x \wedge 0)$. If m_1, m_1', m_2, m_2' are integers and $k_1 = \max\{m_1, m_2\}$, $k_2 = \min(m_1 . m_2)$, $l_1 = \max(m_1', m_2')$, $l_2 = \min(m_1', m_2')$, then

(γ_1) $(m_1(x \vee 0) + m_1'(x \wedge 0)) \vee (m_2(x \vee 0) + m_2'(x \wedge 0)) = k_1(x \vee 0) + l_2(x \wedge 0)$,

(γ_2) $(m_1(x \vee 0) + m_1'(x \wedge 0)) \wedge (m_2(x \vee 0) + m_2'(x \wedge 0)) = k_2(x \vee 0) + l_1(x \wedge 0)$.

(This is an easy consequence of the fact that $x \vee 0$ and $x \wedge 0$ are disjoint, i.e., $(x \vee 0) \wedge (-(x \wedge 0)) = 0$.)

In the following lemma we assume that for all $x_1, x_2 \in G$, $x_1 +_1 x_2$ is given by the expression (3).

Lemma 5. *Let r be a positive integer such that $r > 2|n_{ij}|$ is valid for each $i \in I$ and each $j \in J$. Let $x, y \in G$, $x \geqq r|y|$. Then $x +_1 y = x + y$.*

Proof. We have
$$x +_1 y = \bigwedge_{i \in I} \bigvee_{j \in J} (m_{ij}x + n_{ij}y) .$$
Denote $m_{ij}x + n_{ij}y = t_{ij}$. Let $i, i_1 \in I$, $j, j_1 \in J$.

First, suppose that $m_{ij} \neq m_{i_1j_1}$. We shall verify that in this case the elements t_{ij} and $t_{i_1j_1}$ are comparable in $\mathfrak{G}$. In fact, let $m_{ij} > m_{i_1j_1}$. Then
$$(m_{ij} - m_{i_1j_1})\, x \geqq x \geqq r|y| ,$$
$$(n_{i_1j_1} - n_{ij})\, y \leqq |(n_{i_1j_1} - n_{ij})\, y| =$$
$$= |n_{i_1}j_1 - n_{ij}|\, |y| \leqq (|n_{i_1j_1}| + |n_{ij}|)\, |y| < r|y| ,$$
whence $t_{ij} > t_{i_1j_1}$.

In the case $m_{ij} = m_{i_1j_1}$ we have according to (γ_1)
$$t_{ij} \vee t_{i_1j_1} = (m_{ij}x + n_{ij}y) \vee (m_{ij}x + n_{i_1j_1}y) =$$
$$= m_{ij}x + (n_{ij}y \vee n_{i_1j_1}y) = m_{ij}x + k(y \vee 0) + l(y \wedge 0) ,$$
where $k, l \in \{n_{ij}\}_{i \in I, j \in J}$. From this and from ($\gamma_1$), ($\gamma_2$) we infer that there are integers m, k, l with $m \in \{m_{ij}\}_{i \in I, j \in J}$, $k, l \in \{n_{ij}\}_{i \in I, j \in J}$ such that
$$x +_1 y = mx + k(y \vee 0) + l(y \wedge 0)$$
is valid whenever $x \geqq ry$.

If we put $y = 0$, $x > 0$, then we obtain $m = 1$. Thus
$$x \geqq r|y| \Rightarrow x +_1 y = x + k(y \vee 0) + l(y \wedge 0) .$$

Choose $y > 0$, $x \approx ry$. Then in view of Lemma 4,

$$(r + 1)\, y = (r + 1) \circ y = ry +_1 y = ry + ky\,,$$

whence $k = 1$. Further choose $y < 0$, $x = -ry$. According to Corollary of Lemma 4′ we obtain

$$(-r + 1)\, y = (-r + 1) \circ y = -ry +_1 y = -ry + ly = (-r + l)\, y\,,$$

thus $l = 1$, completing the proof.

Lemma 6. *Let $y_1, y_2 \in G$. Then $y_1 +_1 y_2 = y_1 + y_2$.*

Proof. Let r be as in Lemma 5. There exists $x \in G$ such that the relations

$$x \geqq r|y_1 + y_2|\,, \quad x \geqq r|y_1|\,, \quad x + y_1 \geqq r|y_1|$$

are valid. This and Lemma 5 yields

$$x +_1 (y_1 + y_2) = x + (y_1 + y_2)\,,$$
$$(x + y_1) +_1 y_2 = (x + y_1) + y_2\,,$$
$$x + y_1 = x +_1 y_1\,,$$

whence $y_1 + y_2 = y_1 +_1 y_2$.

Lemma 7. *Suppose that (1) is valid. Let $y_1, y_2 \in G$. Then $y_1 \leqq y_2$ if and only if $y_1 \leqq_1 y_2$.*

Proof. From $y_1 \leqq y_2$ it follows that there exists $z \in G^+$ with $y_1 + z = y_2$. According to Lemma 6 we have $y_1 +_1 z = y_2$, whence $y_1 \leqq_1 y_2$. Similarly, from $y_1 \leqq_1 y_2$ we infer that $y_1 \leqq y_2$.

Now suppose that (2) is valid. Let $\precsim$ be the partial order on G that is dual to $\leqq_1$. By applying Lemma 7 to the lattice ordered groups G and $G_1' = (G; +_1, -_1, \precsim)$ we obtain:

Lemma 7′. *Suppose that (2) is valid. Let $y_1, y_2 \in G$. Then $y_1 \leqq y_2$ if and only if $y_1 \geqq_1 y_2$.*

Lemmas 6, 7 and 7′ imply:

Theorem. *Let $\mathfrak{G} = (G; +, -, \wedge, \vee)$ and $\mathfrak{G}_1 = (G; +_1, -_1, \wedge_1, \vee_1)$ be lattice ordered groups. Let $\leqq$ and $\leqq_1$ be the corresponding partial orders of $\mathfrak{G}$ and $\mathfrak{G}_1$, respectively. Suppose that $\mathfrak{G}$ is abelian and that the identity mappings is a weak isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}_1$. Then (i) the operations $+$ and $+_1$ on G coincide, and (ii) either $\leqq$ coincides with $\leqq_1$, or $\leqq$ is dual to $\leqq_1$.*

Corollary 2. *Let $\mathfrak{G} = (G; +, -, \wedge, \vee)$ and $\mathfrak{G}_1 = (G_1; +_1, -_1, \wedge_1, \vee_1)$ be lattice ordered groups. Assume that $\mathfrak{G}$ is abelian. Let φ be a weak isomorphism of $\mathfrak{G}$ onto $\mathfrak{G}_1$. Then (i) φ is an isomorphism of the group $(G; +)$ onto the group $(G_1; +_1)$, and (ii) φ is either an isomorphism or a dual isomorphism of the lattice $(G; \wedge, \vee)$ onto the lattice $(G_1; \wedge_1, \vee_1)$.*

Remark. It can be shown that the assertion of Lemma 4 remains valid without assuming the commutativity of the operation $+$. The question of the validity of Corollary 2 for a non-abelian lattice ordered group $\mathfrak{G}$ is open.

Let $\mathfrak{G} = (G; +, -, \wedge, \vee)$ and $\mathfrak{G}_1 = (G; +_1, -_1, \wedge_1, \vee_1)$ be lattice ordered groups with the same underlying set. Assume that $\mathfrak{G}_1$ is abelian. Let f be an $(n + m)$-ary polynomial belonging to $\mathscr{P}(\mathfrak{G}_1)$, $n \geqq 1$. Suppose that f can be expressed by using merely the operations $+_1$ and $-_1$ (i.e., without using the operations $\wedge_1$, $\vee_1$). Let $a_1, \ldots, a_m$ be fixed elements of G. Consider the n-ary operation

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, a_1, \ldots, a_m)$$

on G and let investigate the problem whether g can belong to $\mathscr{P}(\mathfrak{G})$.

Since $\mathfrak{G}_1$ is abelian, there exists a fixed element $b \in G$ and an n-ary operation $f_1 \in \in \mathscr{P}(\mathfrak{G}_1)$ such that

$$g(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) +_1 b\,; \tag{9}$$

the polynomial f_1 does not contain the lattice operation $\wedge_1$ and $\vee_1$.

Proposition. *Let $\mathfrak{G}$, $\mathfrak{G}_1, f, b$ be as above. Suppose that the zero element of $\mathfrak{G}$ coincides with the zero element of $\mathfrak{G}_1$ (this element will be denoted by 0). If $g \in \in \mathscr{P}(\mathfrak{G})$, then $b = 0$ (i.e., f does not depend on $a_1, \ldots, a_n$).*

Proof. Assume that $g \in \mathscr{P}(\mathfrak{G})$. Denote $h(x) = g(x, \ldots, x)$. Then $h \in \mathscr{P}(\mathfrak{G})$. Hence $h(x)$ can be expressed in the form

$$h(x) = \bigwedge_{i \in I} \bigvee_{j \in J} m_{ij} x\,, \tag{3'}$$

where I, J are finite sets and m_{ij} are integers. Thus $h(0) = 0$. From (9) we obtain $h(0) = f_1(0, \ldots, 0) +_1 b = 0 +_1 b = b$. Therefore $b = 0$.

Question. Does the above proposition remain valid without assuming that $\mathfrak{G}_1$ is abelian or without assuming that the zero element of $\mathfrak{G}$ coincides with the zero element of $\mathfrak{G}_1$?

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