Pavel Doktor; Milan Kučera

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PERTURBATIONS OF VARIATIONAL INEQUALITIES AND RATE OF CONVERGENCE OF SOLUTIONS

PAVEL DOKTOR, MILAN KUČERA, Praha

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INTRODUCTION

Let H be a Hilbert space with an inner product $(\cdot, \cdot)$ and with the corresponding norm $\|\cdot\|$. We shall consider two closed convex sets K_1, K_2 in H and two (in general nonlinear) operators $A_1, A_2 : H \to H$. We shall study the connection between solutions u_1, u_2 of the following two variational inequalities:

(I_n) $u \in K_n$,

(II_n) $(A_n u, v - u) \geqq (f_n, v - u)$ for all $v \in K_n$

$(n = 1, 2)$, where $f_1, f_2 \in H$ are given. More precisely, we shall estimate the value $\|u_1 - u_2\|$ in terms of $\|f_1 - f_2\|$, the "distance between the sets K_1, K_2" and the "distance between the operators A_1, A_2" (see Section 2, Theorem 2.1). Further, we can consider a sequence $\{K_n\}$ of closed convex sets, a sequence $\{A_n\}$ of operators and a sequence $\{f_n\}$ of right-hand sides converging in a certain sense to a closed convex set K_0, to an operator A_0 and to $f_0 \in H$, respectively. Convergence of the sequence of solutions of the corresponding variational inequalities (I_n), (II_n) to a solution of the variational inequality (I_0), (II_0) (without an estimate of the rate of convergence) has been proved under various assumptions in a number of papers (see for example U. Mosco [3], [4]). As a consequence of the above mentioned Theorem 2.1, we obtain under certain special assumptions an estimate for the rate of convergence of solutions in terms of the rate of convergence of K_n, A_n, f_n (see Remark 2.6). Concrete examples are given in Section 3.

1. NOTATION, GENERAL REMARKS

If K is a closed convex set in the Hilbert space H, then we shall denote by P_K the projection onto K, i.e., $P_K u$ for an arbitrary $u \in H$ is the unique element of K satisfying the condition

$$\|u - P_K u\| = \inf_{v \in K} \|u - v\|$$

(see [1]).

Remark 1.1. It is well-known (and easy to see) that $P_K u$ is the unique element of K satisfying the condition

$$(u - P_K u, v - P_K u) \leqq 0 \quad \text{for all} \quad v \in K$$

(see [1]).

Remark 1.2. The projection onto a closed convex set is a Lipschitzian mapping:

$$\|P_K u - P_K v\| \leqq \|u - v\| \quad \text{for all} \quad u, v \in H . \tag{1.1}$$

Remark 1.3. Let γ be an arbitrary positive number. Then $u \in H$ is a solution of the variational inequality

$$u \in K , \tag{I}$$

$$(Au, v - u) \geqq (f, v - u) \quad \text{for all} \quad v \in K \tag{II}$$

if and only if

$$u = P_K(u - \gamma(Au - f)) \tag{1.2}$$

(see [1]). Indeed, it follows from Remark 1.1 that (1.2) is equivalent to (I) and

$$(u - \gamma(Au - f) - u, v - u) \leqq 0 \quad \text{for all} \quad v \in K , \tag{II'}$$

which is equivalent to (II).

Lemma 1.1. (see [1]). *Let $A : H \to H$ be an operator satisfying the assumptions*

$$(Au - Av, u - v) \geqq M\|u - v\|^2 \quad \textit{for all} \quad u, v \in H , \tag{1.3}$$

$$\|Au - Av\| \leqq L\|u - v\| \quad \textit{for all} \quad u, v \in H \tag{1.4}$$

where $M \leqq L$ are positive constants. Let $f \in H$ and $\gamma \in (0, 2M/L^2)$. Then the operator T defined by

$$Tu = P_K(u - \gamma(Au - f))$$

is a contraction. Namely, we have

$$\|Tu - Tv\| \leqq L'\|u - v\| \quad \textit{for all} \quad u, v \in H ,$$

where $L' = \sqrt{(1 - 2\gamma M + \gamma^2 L^2)} \in \langle 0, 1)$.

Remark 1.4. It follows from Lemma 1.1, Remark 1.3 and from the well-known Banach contraction principle that under the assumptions (1.3), (1.4) the problem (I), (II) has precisely one solution and this solution can be obtained by the usual iterative method as a fixed point of the operator T.

For the sake of completeness, we present

Proof of Lemma 1.1. Using (1.1), (1.3), (1.4) we obtain

$$\|Tu - Tv\|^2 = \|P_K(u - \gamma(Au - f)) - P_K(v - \gamma(Av - f))\|^2 \leqq$$
$$\leqq \|u - \gamma(Au - f) - v + \gamma(Av - f)\|^2 =$$

$$= (u - v, u - v) - 2\gamma(Au - Av, u - v) + \gamma^2(Av - Au, Av - Au) \leqq$$
$$\leqq (1 - 2\gamma M + \gamma^2 L^2) \|u - v\|^2 .$$

It is $1 - 2\gamma M + \gamma^2 L^2 \in \langle 0, 1)$ for $\gamma \in (0, 2M/L^2)$.

2. PERTURBATION OF THE VARIATIONAL INEQUALITY. RATE OF CONVERGENCE OF THE APPROXIMATIVE SOLUTION

In this section, we shall establish an estimate of the norm of the difference of solutions u_1, u_2 of the problems (I_i), (II_i), $i = 1, 2$. To this end, let us first define the expressions which characterize the "distance" between two closed convex sets and between two operators.

Let K_1, K_2 be closed convex nonempty sets in H. For each $r > 0$ such that $\{x \in K_i; \|x\| \leqq r\} \neq \emptyset$ $(i = 1, 2)$, we define

$$S(r; K_1, K_2) = \sup_{\substack{v \in K_1 \\ \|v\| \leqq r}} \inf_{u \in K_2} \|u - v\| ,$$

$$\sigma(r; K_1, K_2) = \max (S(r; K_1, K_2), S(r; K_2, K_1)) .$$

For each $r > 0$, we set

$$\varrho(r; K_1, K_2) = \sup_{\substack{u \in H \\ \|u\| \leqq r}} \|P_{K_1} u - P_{K_2} u\| .$$

(If no misunderstanding can occur we shall not specify the convex sets writing briefly $\sigma(r; K_1, K_2) = \sigma(r)$ e.t.c..)

Remark 2.1. The expression $\sigma(r)$ is the so-called local gap (or opening) of the sets K_1, K_2 (see [4]). Given a sequence of convex sets $\{K_n\}_{n=1}^{\infty}$ we can define the convergence $K_n \to K$ by means of the conditions

(CK) $$\lim_{n \to \infty} \varrho(r; K, K_n) = 0 \quad \forall r > 0$$

or

(CK') $$\exists r_0 \geqq 0 \quad \forall r > r_0 \lim_{n \to \infty} \sigma(r; K, K_n) = 0$$

which are equivalent (see Remark 2.2 and Lemma 2.1). The condition (CK') ensures that K_n tend to K in the following sense:

(M1) to each $u \in K$ there exist $u_n \in K_n$, $n = 1, 2, \ldots$, such that $u_n \to u$ *);

(M2) if $u_n \in K_{l_n}$ where l_n is an increasing sequence of indices and $u_n \rightharpoonup u$, then $u \in K$ *).

The conditions (M1), (M2) were used by U. Mosco [3] in the proof of convergence of the corresponding solutions (without estimates for the rate of convergence).

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*) By $\to$ and $\rightharpoonup$ we denote the strong convergence and the weak convergence in H, respectively.

Remark 2.2. We shall establish an estimate of $\|u_1 - u_2\|$ in terms of the expression ϱ. However, it is usually difficult to calculate this expression directly, while it is often possible to evaluate the expression σ (cf. also Section 3). The following lemma describes the relation (in general nonlinear) between the expressions ϱ, σ and hence between the conditions (CK), (CK′):

Lemma 2.1. *Let K_1, K_2 be closed convex nonempty sets in H, and let us denote $d_i = \operatorname{dist}(\theta, K_i)$ $(i = 1, 2)$, $d = \max(d_1, d_2)$.*) Then*

$$\sigma(r) \leqq \varrho(r) \leqq \sqrt{((8r + 4d)\,\sigma(r + d) + \sigma^2(r + d))} \tag{E}$$

for each $r > d$.

Proof. (i) If $v \in K_1$, then $P_{K_1}v = v$ and therefore

$$\inf_{u \in K_2} \|u - v\| = \|P_{K_2}v - v\| = \|P_{K_2}v - P_{K_1}v\| .$$

Thus we have

$$S(r; K_1, K_2) = \sup_{\substack{v \in K_1 \\ \|v\| \leqq r}} \inf_{u \in K_2} \|u - v\| =$$

$$= \sup_{\substack{v \in K_1 \\ \|v\| \leqq r}} \|P_{K_2}v - P_{K_1}v\| \leqq \sup_{\|v\| \leqq r} \|P_{K_2}v - P_{K_1}v\| = \varrho(r)$$

for an arbitrary $r > d$; analogously for $S(r; K_2, K_1)$ and the first inequality of (E) is proved.

(ii) Secondly, let $u \in H$ be an arbitrary point, $\|u\| \leqq r$ and let us denote $u_1 = P_{K_1}u$, $u_2 = P_{K_2}u$. We have

$$\|u_i\| \leqq \|P_{K_i}(\theta)\| + \|P_{K_i}(u) - P_{K_i}(\theta)\| \leqq d + r\,, \quad i = 1, 2 \tag{2.1}$$

in virtue of Remark 1.2. This together with the definition of σ implies that

$$\operatorname{dist}(u_2, K_1) \leqq \sigma(r + d)\,. \tag{2.2}$$

It follows from Remark 1.1 that the set K_1 lies in the half-space $H_1 = \{w; (u - u_1, w - u_1) \leqq 0\}$ and (2.2) yields that

$$u_2 \in H_2 = H_1 + \frac{u - u_1}{\|u - u_1\|}\,\sigma\,,$$

where we write σ instead of $\sigma(r + d)$ (see Fig. 2.1). It follows from the definition of σ that $B(u_1, \sigma) \cap K_2 \neq \emptyset$, where $B(z, k)$ denotes the closed ball with the center z and with the radius k. Thus $u_2 \in B(u, R + \sigma)$, where $R = \|u - u_1\|$. Hence we have $\varrho(r) \leqq \sup \{\|w - u_1\|;\ w \in B(u, R + \sigma) \cap H_2\} =^{\text{def}} q$. Easy calculation by methods of the plane geometry yields $q = \sqrt{(4R\sigma + \sigma^2)}$. We have $R = \|u - u_1\| \leqq \|u\| + \|u_1\| \leqq 2r + d$ and this implies (E).

*) By θ we denote the origin in H.

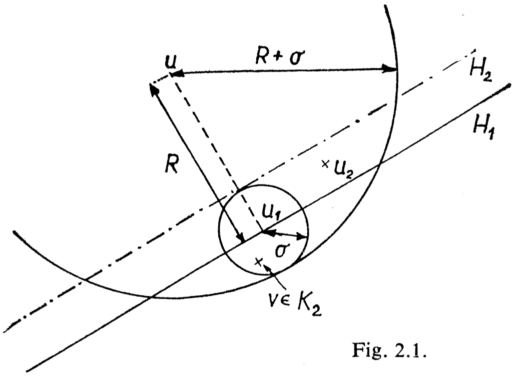

Fig. 2.1.

Remark 2.3. It is easy to see from the proof of Lemma 2.1 that the following more precise estimate holds for each $r > 0$:

$$(\mathrm{E}') \qquad \varrho(r) \leqq \sqrt{((8r + 4d_2)\,\sigma(r + d_1) + \sigma^2(r + d_1))}\,.$$

Particularly, if one of the sets K_i contains the origin, then $\varrho(r)$ is estimated in terms of $\sigma(r)$ instead of $\sigma(r + d)$.

Remark 2.4. Let us discuss the case of a sequence $\{K_n\}_{n=1}^{\infty}$. It is easy to see that if (CK') is valid, then $\operatorname{dist}(K_i, \theta) \leqq D$ for $i = 1, 2, \ldots$ and for some D. Thus, for each $r > 0$ we have $\varrho(r, K, K_n) \leqq 2(r + D) < \infty$ and hence there exists $C(r)$ such that

$$\varrho(r; K, K_n) \leqq C(r)\,\{\sigma(r + D; K, K_n)\}^{\alpha}, \quad \alpha = \tfrac{1}{2}\,.$$

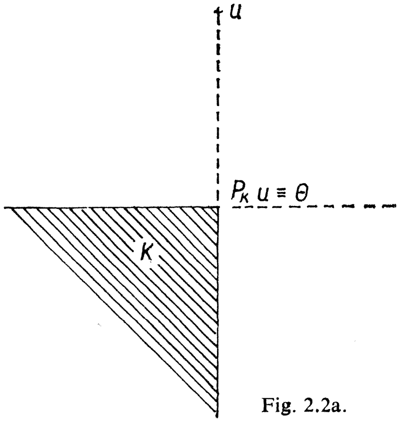

Fig. 2.2a.

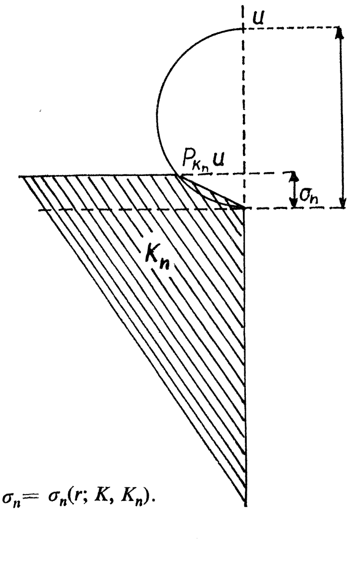

Fig. 2.2b. $\sigma_n = \sigma_n(r; K, K_n)$.

A simple example illustrated by Fig. 2.2a, b shows that this estimate is not true with $\alpha > \frac{1}{2}$.

Now, let us consider operators $A_1, A_2 : H \to H$. The following assumptions will be used:

(M) $(A_n u - A_n v, u - v) \geqq M\|u - v\|^2$ for all $u, v \in H$, $n = 1, 2$, where $M > 0$ (monotonicity);

(B) $B(r) = \sup_{\substack{n=1,2 \\ \|u\| \leqq r}} \|A_n u\|$ is a finite number for each $r > 0$ (boundedness);

(L) $\|A_n u - A_n v\| \leqq L\|u - v\|$ for all $u, v \in H$, $n = 1, 2$, where $L > 0$ (Lipschitz property).

For each $r > 0$, let us denote

$$a(r) = a(r; A_1, A_2) = \sup_{\|u\| \leqq r} \|A_1 u - A_2 u\| .$$

Remark 2.5. If $\{A_n\}$ is a sequence of operators, then the following convergence condition can be considered:

$$\text{(CA)} \qquad \lim_{n \to \infty} a(r; A, A_n) = 0 \quad \text{for each} \quad r > 0 .$$

This condition is stronger than the assumptions about the convergence of operators studied by U. Mosco [3].

Theorem 2.1. *Let K_1, K_2 be closed convex sets in H and let $A_1, A_2 : H \to H$ be operators satisfying the conditions* (M), (B), (L). *Let us suppose that $f_1, f_2 \in H$. Let us denote by u_n, $n = 1, 2$ the unique solutions of* (I_n), (II_n).*) *Let us choose $\gamma \in (0, 2M/L^2)$. Then*

$$\|u_n\| \leqq U , \quad n = 1, 2 ; \tag{2.3}$$

$$\|u_1 - u_2\| \leqq \frac{1}{1 - L'} [\varrho(U + \gamma B(U) + \gamma F) + \gamma\|f_1 - f_2\| + \gamma a(U)] , \tag{2.4}$$

where

$$L' = \sqrt{(1 - 2\gamma M + \gamma^2 L^2)} \in \langle 0, 1) ,$$

$$U = \frac{1}{M} [F + B(d)] + d,$$

$$F = \max (\|f_1\|, \|f_2\|) , \quad d = \max_{n=1,2} (\operatorname{dist} (K_n, \theta)) .$$

Proof. Choose $v_n \in K_n$ such that $\|v_n\| \leqq d$, $n = 1, 2$. The conditions (M), (II_n), (B) imply that

$$M\|u_n - v_n\|^2 \leqq (A_n u_n - A_n v_n, u_n - v_n) \leqq (f_n - A_n v_n, u_n - v_n) \leqq$$
$$\leqq [F + B(d)] \|u_n - v_n\|$$

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*) The existence and unicity of the solution of (I), (II) is well-known under more general assumptions (for example, see [2]). In our special case it follows directly from Remark 1.4.

which yields (2.3). With respect to Remark 1.3, we have

$$(2.5)\quad \|u_1 - u_2\| = \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_2}(u_2 + \gamma(f_2 - A_2u_2))\| \leqq$$
$$\leqq \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_2}(u_1 + \gamma(f_1 - A_1u_1))\| +$$
$$+ \|P_{K_2}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_2}(u_2 + \gamma(f_1 - A_1u_2))\| +$$
$$+ \|P_{K_2}(u_2 + \gamma(f_1 - A_1u_2)) - P_{K_2}(u_2 + \gamma(f_2 - A_2u_2))\| .$$

Using (B) and (2.3) we obtain

$$(2.6)\qquad \|u_n + \gamma(f_n - A_nu_n)\| \leqq U + \gamma F + \gamma\, B(U)\,, \quad n = 1, 2$$

and therefore

$$(2.7)\qquad \|P_{K_1}(u_1 + \gamma(f_1 - A_1u_1)) - P_{K_2}(u_1 + \gamma(f_1 - A_1u_1))\| \leqq$$
$$\leqq \varrho(U + \gamma F + \gamma\, B(U))\,.$$

Further, Lemma 1.1 implies that

$$(2.8)\quad \|P_{K_2}(u_1 + \gamma(f_1 - A_1u_1) - P_{K_2}(u_2 + \gamma(f_1 - A_1u_2))\| \leqq L'\|u_1 - u_2\|\,.$$

Remark 1.2 implies that

$$(2.9)\qquad \|P_{K_2}(u_2 + \gamma(f_1 - A_1u_2)) - P_{K_2}(u_2 + \gamma(f_2 - A_2u_2))\| \leqq$$
$$\leqq \gamma(\|f_1 - f_2\| + a(U))\,.$$

Putting (2.7)–(2.9) into (2.5) we obtain (2.4).

Remark 2.6. Let us consider closed convex sets K, K_n in H ($n = 1, 2, \ldots$) satisfying the condition (CK). Further, let $A, A_n : H \to H$ ($n = 1, 2, \ldots$) be operators satisfying the assumptions (M), (L) (with some positive M, L independent of n), (CA) and

$$(\tilde{\mathrm{B}})\qquad \tilde{B}(r) = \sup_{\substack{\|u\| \leqq r \\ n = 1,2,\ldots}} \|A_nu\| \text{ is a finite number for each } r > 0.$$

Suppose that $f, f_n \in H$, $f_n \to f$. Denote by u and u_n the unique solutions of the problems (I), (II) and (I_n), (II_n), respectively.

Theorem 2.1 ensures that $u_n \to u$ and it gives an estimate of the rate of this convergence. If we set $\varrho_n(r) = \varrho(r; K, K_n)$, $a_n(r) = a(r; A, A_n)$, then

$$\|u - u_n\| \leqq \frac{1}{1 - L'}\,[\varrho_n(U + \gamma\,\tilde{B}(U) + \gamma\tilde{F}) + \gamma\|f - f_n\| + \gamma\, a_n(U)]\,,$$

where $\gamma \in (0, 2M/L^2)$ is arbitrary,

$$L' = 1 - 2\gamma M + \gamma^2L^2\,,$$
$$\tilde{F} = \sup_{n=1,2,\ldots} \|f_n\|\,, \quad d = \sup_{n=1,2,\ldots} \operatorname{dist}(K_n, \theta)\,,$$
$$U = \frac{1}{M}[\tilde{F} + \tilde{B}(d)] + d\,.$$

Further,

$$\|u_n\| \leqq U .$$

Let us remark that the convergence of solutions without an estimate of its rate is proved in [3] in a more general situation.

3. EXAMPLES

In this section, we shall explain two easy applications of Theorem 2.1. For the sake of simplicity, we shall choose the simplest fixed operator A $(= A_1 = A_2)$ in Example 3.1 and give the estimate of the difference between the solutions in terms of the distance between the sets K_1, K_2 only. On the other hand, a simple fixed set K $(= K_1 = K_2)$ will be considered in Example 3.2, where the estimate of the difference between the solutions in terms of the distance between the operators A_1, A_2 will be given. It will be clear that both examples can be generalized and combined.

In the whole section, Ω is a given domain in $\mathbb{R}^N$ with a lipschitzian boundary.

Example 3.1. Denote $H = W_2^1(\Omega)$ (the well-known Sobolev space). Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in H$ be given functions satisfying the conditions

$$\psi_1 - \varphi_1 \geqq \delta , \tag{3.1}$$

$$\|\varphi_2 - \varphi_1\| \leqq \varepsilon , \quad \|\psi_2 - \psi_1\| \leqq \varepsilon , \tag{3.2}$$

$$\varphi_n \leqq 0 \leqq \psi_n , \quad n = 1, 2 , \tag{3.3}$$

where ε, δ are constants such that

$$0 < \varepsilon \leqq \frac{\delta}{4} . \tag{3.4}$$

(We write $v \leqq u$ for the functions $v, u \in H$ if and only $v(x) \leqq u(x)$ for almost all $x \in \Omega$ etc..) The assumption (3.3) is not necessary and it is considered for the sake of simplicity only. This assumption ensure that $d = 0$ in Theorem 2.1 and that (E) holds for all $r > 0$ in Lemma 2.1. Therefore the estimate of $\|u_1 - u_2\|$ will be simpler in this case.

Let us consider convex closed sets

$$K_n = \{u \in H;\ \varphi_n \leqq u \leqq \psi_n\} \tag{3.5}$$

$(n = 1, 2)$ and an operator $A : H \to H$ defined by

$$(Au, v) = \int_\Omega \left[\sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + uv \right] \mathrm{d}x \quad \text{for all} \quad u, v \in H . \tag{3.6}$$

We shall show that if u_n is the solution of the problem (I_n), (II_n) $(n = 1, 2)$ with K_n from (3.5), $A_1 = A_2 = A$ from (3.6) and with some $f_1 = f_2 = f \in H$, then

$$\|u_1 - u_2\| \leqq \left[24\|f\| \frac{4\varepsilon}{\delta}(3\|f\| + \tfrac{1}{2}\|\varphi_1 + \psi_1\|) + \right.$$
$$\left. + \frac{16\varepsilon^2}{\delta^2}(3\|f\| + \tfrac{1}{2}\|\varphi_1 + \psi_1\|)^2\right]^{1/2} .$$

It is clear the assumptions (M), (B), (L) are fulfilled with

(3.7) $$M = L = 1 , \quad B(r) = r$$

and we can choose

(3.8) $$\gamma = 1 , \quad L' = 0$$

in Theorem 2.1. The assumption (3.3) implies

(3.9) $$d = 0 , \quad U \doteq \|f\| .$$

Now, we want to estimate $\sigma(r; K_1, K_2)$. Denote $\xi = \frac{1}{2}(\psi_1 + \varphi_1)$. Let $u \in K_1$ be an arbitrary function such that $\|u\| \leqq r$. Set

$$u_k = k(u - \xi) + \xi = ku + (1 - k)\,\xi$$

for each $k \in \langle 0, 1 \rangle$. It follows from (3.1), (3.5) that

$$u_k \geqq ku + (1 - k)\,\tfrac{1}{2}(2\varphi_1 + \delta) \geqq \varphi_1 + \frac{1 - k}{2}\,\delta ,$$

$$u_k \leqq ku + (1 - k)\,\tfrac{1}{2}(2\psi_1 - \delta) \leqq \psi_1 - \frac{1 - k}{2}\,\delta .$$

If we set $k = 1 - 2\varepsilon/\delta$, we obtain

$$\varphi_1 + \varepsilon \leqq u_k \leqq \varphi_1 - \varepsilon$$

and this together with (3.2), (3.5) implies $u_k \in K_2$. Further,

$$\sup_{\substack{u \in K_1 \\ \|u\| \leqq r}} \|u - u_k\| = \sup_{\substack{u \in K_1 \\ \|u\| \leqq r}} (1 - k)\,\|u - \xi\| \leqq \frac{2\varepsilon}{\delta}(r + \|\xi\|)$$

and hence

$$S(r; K_1, K_2) \leqq \frac{2\varepsilon}{\delta}(r + \|\xi\|) .$$

On the other hand, let $u \in K_2$, $\|u\| \leqq r$. It follows from (3.1), (3.2), (3.4) that

$$u_k \geqq ku + (1 - k)\,\tfrac{1}{2}(2\varphi_1 + \delta) \geqq$$
$$\geqq k\varphi_2 + (1 - k)\,\tfrac{1}{2}(2\varphi_2 - 2\varepsilon + \delta) \geqq \varphi_2 + \frac{1 - k}{4}\,\delta ,$$

$$u_k \leqq ku + (1-k)\tfrac{1}{2}(2\psi_1 - \delta) \leqq$$

$$\leqq k\psi_2 + (1-k)\tfrac{1}{2}(2\psi_2 + 2\varepsilon - \delta) \leqq \psi_2 - \frac{1-k}{4}\delta .$$

It we set $k = 1 - 4\varepsilon/\delta$, we obtain

$$\varphi_2 + \varepsilon \leqq u_k \leqq \psi_2 - \varepsilon$$

and this together with (3.2), (3.5) implies $u_k \in K_1$. Hence we have

$$\sup_{\substack{u \in K_2 \\ \|u\| \leqq r}} \|u - u_k\| = \sup_{\substack{u \in K_2 \\ \|u\| \leqq r}} (1-k)\|u - \xi\| \leqq \frac{4\varepsilon}{\delta}(r + \|\xi\|)$$

which yields

$$S(r; K_2, K_1) \leqq \frac{4\varepsilon}{\delta}(r + \|\xi\|) .$$

On the whole, we have

$$\sigma(r; K_1, K_2) \leqq \frac{4\varepsilon}{\delta}(r + \|\xi\|) . \tag{3.10}$$

Using (E) from Lemma 2.1, (3.9) and (3.10), we obtain

$$\varrho(r; K_1, K_2) \leqq \sqrt{\left(8r\frac{4\varepsilon}{\delta}(r + \|\xi\|) + \frac{16\varepsilon^2}{\delta^2}(r + \|\xi\|)^2\right)} . \tag{3.11}$$

Putting (3.7), (3.8), (3.9), (3.11) into (2.4), we obtain the estimate announced above.

Example 3.2. Let us denote by $H = \mathring{W}_2^1(\Omega)$ the subspace of $W_2^1(\Omega)$ of functions with zero traces on the boundary of Ω and introduce the inner product on H by

$$(u, v) = \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x \quad \text{for all} \quad u, v \in H .$$

Let g_1, g_2 be two continuous functions defined on $\langle 0, \infty)$ which have the first derivative on $(0, \infty)$, and satisfy the following conditions:

$$M \leqq g_n(t) \leqq B \quad \text{for all} \quad t \in \langle 0, \infty) , \quad n = 1, 2 , \tag{3.12}$$

$$M \leqq \frac{\mathrm{d}}{\mathrm{d}t}(g_n(t)\,t) = g_n(t) + g_n'(t)\,t \leqq L, \quad t \in (0, \infty) , \quad n = 1, 2 , \tag{3.13}$$

$$|g_1(t) - g_2(t)| \leqq \varepsilon \quad \text{for all} \quad t \in \langle 0, \infty) , \tag{3.14}$$

where M, B, L are positive constants. We shall consider operators $A_1, A_2;\ H \to H$ defined by

$$(A_n u, v) = \int_\Omega g_n(|\mathrm{grad}\, u|) \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x \tag{3.15}$$

and a closed convex set

(3.16) $$K = \{u \in H;\, u \geqq 0\}\,.$$

We shall show that if u_n is the solution of the problem (I_n), (II_n) $(n = 1, 2)$ with A_n from (3.15), $K_1 = K_2 = K$ from (3.16) and with some $f_1, f_2 \in H$, then

$$\|u_1 - u_2\| \leqq \frac{\gamma}{1 - \sqrt{(1 - 2\gamma M + \gamma^2 L^2)}} \left(\|f_1 - f_2\| + \frac{\varepsilon}{M} \max\left(\|f_1\|, \|f_2\|\right)\right)$$

for an arbitrary $\gamma \in (0, 2M/L^2)$.

First, we shall show that A_1, A_2 satisfy the assumptions of Theorem 2.1. If $r = [r_1, \ldots, r_N]$, $s = [s_1, \ldots, s_N] \in \mathbb{R}^N$ are arbitrary, then we can write (omitting the index of g for a moment) $\sum_{i=1}^{N} (g(|r|)\, r_i - g(|s|)\, s_i)(r_i - s_i) = F(1) \setminus F(0)$, where $F(t) = \sum_{i=1}^{N} g(|s + t(r - s)|)(s_i + t(r_i - s_i))(r_i - s_i)$. There exist $\tau \in (0, 1)$ and $\theta = s + \tau(r - s)$ such that

$$F(1) - F(0) = g(|\theta|)\, |r - s|^2 + \frac{g'(|\theta|)}{|\theta|} \sum_{i=1}^{N} (r_i - s_i)\, \theta_i \sum_{j=1}^{N} (r_j - s_j)\, \theta_j\,.$$

If $g'(|\theta|) \geqq 0$, then $F(1) - F(0) \geqq g(|\theta|)\, |r - s|^2$; if, conversely, $g'(|\theta|) < 0$, we use the Cauchy inequality which yields

$$F(1) - F(0) \geqq g(|\theta|)\, |r - s|^2 + \frac{g'(|\theta|)}{|\theta|}\, |r - s|^2\, |\theta|^2\,;$$

hence we have in both cases

$$\sum_{i=1}^{N} [g(|r|)\, r_i - g(|s|)\, s_i](r_i - s_i) \geqq M \sum (r_i - s_i)^2\,.$$

This implies that

$$(A_n u - A_n v,\, u - v) \geqq M \|u - v\|^2\,,$$

i.e. the condition (M) is fulfilled.

To obtain the condition (B), we conclude from the relations

$$\|A_n u\| = \sup_{\|v\| \leqq 1} |(A_n u, v)| = \sup_{\|v\| \leqq 1} \left| \int_\Omega g_n(|\mathrm{grad}\, u|) \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right| \leqq$$

$$\leqq \sup_{\|v\| \leqq 1} B \int_\Omega \left| \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right| \leqq B \sup_{\|v\| \leqq 1} \|u\|\, \|v\|$$

that (B) is fulfilled with

(3.17) $$B(r) = B\,.\,r\,.$$

Now, if $r, s, t \in \mathbb{R}^N$, then we have (supposing $g_n(|r|) - g_n(|s|) \geqq 0$)

$$\sum_{i=1}^{N} [g_n(|r|)\, r_i - g_n(|s|)\, s_i]\, t_i =$$

$$= \sum_{i=1}^{N} \{[g_n(|r|) - g_n(|s|)]\, r_i t_i + g_n(|s|)\,(r_i - s_i)\, t_i\} \leqq$$

$$\leqq [g_n(|r|) - g_n(|s|)]\, |r|\, |t| + g_n(|s|)\, |r - s|\, |t| \leqq$$

$$\leqq [g_n(|r|)\, |r| - g_n(|s|)\, |s|]\, |t| \leqq L|r - s|\, |t|\,;$$

analogously as above we obtain

$$\|A_n u - A_n v\| \leqq L\|u - v\|\,,$$

i.e. the condition (L) is fulfilled. Further, it is easy to see that

$$a(r; A_1, A_2) = \sup_{\substack{\|u\| \leqq r \\ \|w\| \leqq 1}} \left| \int_\Omega \sum_{i=1}^{N} \left[g_1(|\text{grad}\, u|) \frac{\partial u}{\partial x_i} - g_2(|\text{grad}\, u|) \frac{\partial u}{\partial x_i} \right] \frac{\partial w}{\partial x_i}\, dx \right| \leqq \tag{3.18}$$

$$\leqq \varepsilon \sup_{\substack{\|u\| \leqq r \\ \|w\| \leqq 1}} \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i}\, dx = r\,.\,\varepsilon\,.$$

Obviously, we have $\varrho(r) = 0$ for all r because $K_1 = K_2$. The assumptions (M), (B), (L) are fulfilled with the constants M, L from (3.12), (3.13) and putting (3.17), (3.18), into (2.4) we obtain the estimate mentioned above.

Remark 3.1. Evidently, we could consider sequences of sets K_n, of operators A_n and right hand sides f_n ($n = 1, 2, \ldots$) converging to K, A, f and we could give an estimate of the rate of convergence in Examples 3.1, 3.2 (cf. Remark 2.6).

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Authors' addresses: P. Doktor, 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK); M. Kučera, 115 67 Praha 1, Žitná 25, ČSSR (MÚ ČSAV).