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In [3] the intersection graph of a tree was defined. The intersection graph of a tree T is an undirected graph whose vertices are in a one-to-one correspondence with all proper subtrees of T and in which two vertices are adjacent if and only if the corresponding subtrees have a non-empty intersection.

Analogously a join graph of a tree can be defined. The join graph $J(T)$ of a tree T is a graph whose vertices are in a one-to-one correspondence with all proper subtrees of T and in which two vertices are adjacent if and only if the join of the corresponding subtrees is not equal to T. (The join of two subtrees of T is the least subtree of T which contains both these subtrees.) A graph consisting only of one vertex is also considered a tree.

This definition was formulated in this form in order that the join graph might be a dual concept to the intersection graph. Nevertheless, in the following we shall study the complement $\bar{J}(T)$ of $J(T)$, i.e. the graph in which two vertices are adjacent if and only if the join of the corresponding subtrees is equal to T.

We shall define some auxiliary concepts.

By $IG(n)$ we shall denote the intersection graph of the family of all non-empty proper subsets of a set with n elements.

If G is an undirected graph without loops and multiple edges, let $\sim$ be a binary relation on the vertex set of G such that $a \sim b$ if and only if $\Gamma(a) = \Gamma(b)$. (By the symbol $\Gamma(x)$ we denote the set of all vertices of G which are adjacent to x.) Evidently this relation is an equivalence; we shall call it the *adjacency equivalence* on G. If $a \sim b$, $a \neq b$, then a and b are not adjacent in G; otherwise we should have $b \in \Gamma(a)$ which would imply $b \in \Gamma(b)$, i.e. the existence of a loop. If $a \sim a'$, $b \sim b'$, then evidently a is adjacent to b if and only if a' is adjacent to b'.

If we identify all adjacency-equivalent pairs of vertices of G, we obtain a graph $A(G)$ which will be called *the adjacency reduct of* G. Evidently there exists a discrete homomorphism [2] of G onto $A(G)$.

Lemma 1. *Let M be a set with n elements. Let G be a graph whose vertices are in a one-to-one correspondence with all non-empty proper subsets of M and in which two vertices are adjacent if and only if the union of the corresponding sets is equal to M. Then G is isomorphic to the complement of $IG(n)$.*

Proof. Let φ be a mapping of the vertex set of G onto the vertex set of the complement of $IG(n)$ such that if x is a vertex of G corresponding to the subset X of M, then $\varphi(x)$ is the vertex of $IG(n)$ corresponding to the set $M - X$; this mapping is evidently a bijection. By De Morgan's formula we have $X \cup Y = M$ if and only if $(M - X) \cap (M - Y) = \emptyset$. Therefore vertices $\varphi(x)$, $\varphi(y)$ are adjacent in the complement of $IG(n)$ if and only if x, y are adjacent in G and φ is an isomorphism.

Theorem 1. *Let T be a finite tree with $n \geqq 3$ vertices and with k terminal vertices, let $J(T)$ be its join graph, let $\bar{J}(T)$ be the complement of $J(T)$. Then the adjacency reduct of $\bar{J}(T)$ is isomorphic to the complement of $IG(k)$ with one isolated vertex added.*

Proof. Let K be the set of all terminal vertices of T. If $L \subset K$, then we denote by $\mathcal{T}(L)$ the family of all subtrees of T which contain all vertices of L and no vertex of $K - L$. Now let T_1, T_2 be two proper subtrees of T, let L_1, L_2 be such subsets of K that $T_1 \in \mathcal{T}(L_1)$, $T_2 \in \mathcal{T}(L_2)$. If $L_1 \cup L_2 = K$, then the join of T_1 and T_2 contains all terminal vertices of T and this is possible if and only if this join is equal to T. If $L_1 \cup L_2 \neq K$, let $x \in K - (L_1 \cup L_2)$. The vertex set of the join of L_1 and L_2 consists of all vertices of L_1 and L_2 and eventually also of all inner vertices of a path in T connecting a vertex of K_1 with a vertex of L_2. The vertex x belongs neither to L_1 nor to L_2 and, being a terminal vertex of T, it cannot be an inner vertex of a path in T. Therefore the join of T_1 and T_2 does not contain x and is not equal to T. We see that the vertices of $\bar{J}(T)$ corresponding to T_1 and T_2 are adjacent if and only if $L_1 \cup L_2 = K$. Evidently if $T_1 \in \mathcal{T}(L_1)$, $T_2 \in \mathcal{T}(L_2)$, then $T_1 \sim T_2$ (as vertices of $\bar{J}(T)$) if and only if $L_1 = L_2$. The adjacency reduct of $\bar{J}(T)$ is isomorphic to the graph whose vertices are all proper subsets of K and in which two vertices are adjacent if and only if their union is K. By Lemma 1 the vertices corresponding to non-empty proper subsets of K form a subgraph of this graph isomorphic to the complement of $G(k)$. The vertex corresponding to the empty set is isolated.

We see that every further information about T from $J(T)$ can be obtained only from the numbers of vertices of the classes of the adjacency equivalence.

Lemma 2. *Let the graph $IG(n)$ be given for some n. Then for each vertex x of $IG(n)$, we can determine the cardinality of the set to which x corresponds.*

Proof. The vertices corresponding to one-element subsets of the set M (from the definition of $IG(n)$) form an independent set in $IG(n)$ of the cardinality n. No other family of n pairwise disjoint non-empty subsets of M can exist. Therefore a vertex x of $IG(n)$ corresponds to a one-element subset of M if and only if it belongs to the (unique) independent set M_0 of the greatest cardinality. If $2 \leqq m \leqq n - 1$, a vertex $y \notin M_0$ corresponds to a set of the cardinality m if and only if it is adjacent to exactly m vertices of M_0 (i.e., the corresponding set has non-empty intersections with exactly m one-element sets).

Theorem 2. *Let T be a finite tree, let $J(T)$ be its join graph. Given $J(T)$, we can reconstruct T up to isomorphism.*

Proof. We construct the adjacency reduct of $\bar{J}(T)$ and its complement; by Theorem 1 it is isomorphic to $IG(k)$, therefore we can determine the number k of terminal vertices of T. By Theorem 1 and Lemma 2 we can determine the classes of the adjacency equivalence in $J(T)$ which correspond to families $\mathcal{T}(L)$ for L of the cardinality 1; to these families we assign vertices $t_1, \ldots, t_k$ in a one-to-one way. These vertices $t_1, \ldots, t_k$ are the terminal vertices of T; the cardinality of the class corresponding to t_i for $i = 1, \ldots, k$ is the number of the subtrees of T which contain t_i and no other terminal vertex of T. Thus we take $K = \{t_1, \ldots, t_k\}$. If L is a subset of K with $l \geqq 2$ vertices, then $\mathcal{T}(L)$ is the class of all vertices of $\bar{J}(T)$ which are adjacent to all vertices from the classes corresponding to vertices of L and not adjacent to vertices from the classes corresponding to vertices of $K - L$. Therefore for each proper subset L of K we can determine the number $\mu(L)$ of subtrees of T which contain all vertices of L and no vertex of $K - L$.

If T is a snake (a tree consisting of one simple path), we have $|K| = 2$, therefore $K = \{t_1, t_2\}$ and there are three proper subsets of K, namely $\{t_1\}$, $\{t_2\}$ and $\emptyset$. Then $\bar{J}(T)$ consists of a complete bipartite graph with some isolated vertices added (by Theorem 1). Therefore we can conclude that T is a snake; otherwise $\bar{J}(T)$ would contain a triangle. We determine $\mu(\{t_1\})$ and $\mu(\{t_2\})$; evidently $\mu(\{t_1\}) = \mu(\{t_2\}) = n - 1$, where n is the number of vertices of T. By its number of vertices a snake is determined up to isomorphism.

If T is not a snake, we proceed by induction with respect to the number n of vertices of T. For $n = 2$ and $n = 3$ the tree is always a snake and for this case the assertion was proved. Let $n_0 \geqq 4$. Suppose that the assertion is true for $n = n_0 - 1$ and prove it for $n = n_0$.

Let T be a tree with n_0 vertices, let its join graph $J(T)$ be given. For T let $K = \{t_1, \ldots, t_k\}$. Let T' be the tree obtained from T by deleting t_k; this is a tree with $n_0 - 1$ vertices. Let t' be the vertex adjacent to t_k in T. Distinguish two cases:

(i) The vertex t' is a terminal vertex of T'.

(ii) The vertex t' is not a terminal vertex of T'.

In the case (ii) there exists a proper subset L of $K - \{t_k\}$ such that the least tree from $\mathcal{T}(L)$ contains t' and thus all trees from $\mathcal{T}(L)$ contain t'. Then there is a one-to-one correspondence between $\mathcal{T}(L)$ and $\mathcal{T}(L \cup \{t_k\})$; each tree from $\mathcal{T}(L)$ is obtained from a tree from $\mathcal{T}(L \cup \{t_k\})$ by deleting the vertex t_k and the edge $t't_k$ and each tree from $\mathcal{T}(L \cup \{t_k\})$ is obtained from a tree from $\mathcal{T}(L)$ by adding this vertex and this edge. Therefore $\mu(L) = \mu(L \cup \{t_k\})$. In the case (i), to each proper subset L of $K - \{t_k\}$ there exist trees from $\mathscr{T}(L)$ which contain t' and their number is equal to $\mu(L \cup \{t_k\})$, but there are also trees from $\mathscr{T}(L)$ which do not contain t'; therefore $\mu(L) > \mu(L \cup \{t_k\})$ for each $L \subset K - \{t_k\}$. We see that we are able to recognize whether (i) or (ii) occurs.

Consider the case (i). In the tree T' we determine the classes $\mathscr{T}'(L)$ and numbers $\mu'(L)$ analogous to $\mathscr{T}(L)$ and $\mu(L)$ for all proper subsets L of $K' = (K \cup \{t'\}) - \{t_k\}$. If L is a proper subset of K' and $t' \notin L$, then a tree from $\mathscr{T}(L)$ belongs to $\mathscr{T}'(L)$ if and only if it does not contain t'. As was shown above, the number of trees from $\mathscr{T}(L)$ containing t' is equal to $\mu(L \cup \{t_k\})$. Therefore $\mu'(L) = \mu(L) - \mu(L \cup \{t_k\})$. If $t' \in L$, then we can prove analogously as above that there is a one-to-one correspondence between $\mathscr{T}'(L)$ and $\mathscr{T}((L - \{t'\}) \cup \{t_k\})$ for $L \neq \{t'\}$, therefore in this case $\mu'(L) = \mu((L - \{t'\}) \cup \{t_k\})$. For $L = \{t'\}$ there is such a correspondence between $\mathscr{T}'(L)$ and the set obtained from $\mathscr{T}((L - \{t'\}) \cup \{t_k\})$ by deleting the one-vertex tree consisting of t_k; thus $\mu'(\{t'\}) = \mu(\{t_k\}) - 1$. Hence we can transform $J(T)$ to $J(T')$. By the induction assumption the tree T' can be reconstructed from $J(T')$. In the reconstructed tree T' we find the vertex t' and add the vertex t_k and the edge $t't_k$ to T' to obtain T.

Now consider the case (ii). We put $K' = K - \{t_k\}$. If L is a proper subset of K', then each tree from $\mathscr{T}(L)$ is also in T' and thus $\mu'(L) = \mu(L)$ for each such L. We reconstruct the tree T'. Now it is more difficult to find t' in T', because t' is not a terminal vertex of T'. Let u_1, u_2 be two elements of K'. If t' lies between u_1 and u_2, then each tree from $\mathscr{T}(\{u_1, u_2\})$ contains t' and analogously to the above considerations we can prove $\mu(\{u_1, u_2\}) = \mu(\{u_1, u_2, t_k\})$. In the opposite case $\mu(\{u_1, u_2\}) > \mu(\{u_1, u_2, t_k\})$. Therefore we can determine for any two vertices of K' whether t' lies between them or not. If there exist three vertices u_1, u_2, u_3 of K' such that t' lies between any two of them, then t' is uniquely determined [1]. If not, then there is an equivalence on K' such that two vertices are in this equivalence if and only if t' does not lie between them and this equivalence has exactly two classes L_1, L_2. Let T_1 (or T_2) be the least tree from $\mathscr{T}(L_1)$ (or $\mathscr{T}(L_2)$, respectively). The trees T_1, T_2 are disjoint and there exists a path P in T' connecting a vertex v_1 of T_1 with a vertex v_2 of T_2 whose inner vertices belong neither to T_1 nor to T_2. One of those inner vertices is t'. If d is the distance between v_1 and t', then $\mathscr{T}(L_1)$ contains exactly d trees not containing t' and $\mu(L_1 \cup \{t_k\})$ trees containing t'. This yields $d = \mu(L_1) - \mu(L_1 \cup \{t_k\})$ and t' is determined. We add the vertex t_k and the edge $t't_k$ to T' and T is reconstructed.

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