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MINIMAL PRIME SUBGROUPS OF LATTICE-ORDERED GROUPS

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We investigate the structure and properties of l-groups which are determined by the minimal prime subgroups of their homomorphic images.

In [15] MARTINEZ introduced the concept of a *torsion class* of l-groups as a class $\mathbf{T}$ of l-groups that is closed with respect to

1) convex l-subgroups,
2) joins of families of convex l-subgroups, and
3) l-homomorphic images.

One example is the class $\mathbf{N}$ of all l-groups with each convex l-subgroup normal.

If P is a class of l-groups then $\Delta(P)$ will denote the class of l-groups for which each l-homomorphic image belongs to P. A subclass P of $\mathbf{N}$ is a *pretorsion* class if it satisfies (1) and (2). If P is a pretorsion class then $\Delta(P)$ is a torsion class. In section 2 we investigate six pretorsion classes determined by the minimal prime subgroups. Two interesting examples are

P_4: Each minimal prime is a polar.

P_6: Each minimal prime M is closed and G/M is discrete.

Then

$\Delta(P_4) \cap \mathbf{N}$ is the class of normal lex-sums of 0-groups, and

$\Delta(P_6) \cap \mathbf{N}$ is the class of l-groups such that each l-homomorphic image is compactly generated.

The fact that these are torsion classes is rather surprising.

In section 3 classes Q determined by minimal primes that are not pretorsion, and such that $\Delta(Q) \cap \mathbf{N}$ is a torsion class are investigated. For example

$$Q_1 : G = 0 \text{ or } G \text{ has a minimal prime that is closed.}$$

Then $\Delta(Q_1) \cap \mathbf{N}$ is the torsion class of all l-groups for which each l-homomorphic image is completely distributive.

Section 1 consists essentially of two mapping theorems about convex l-subgroups that are interesting in their own right and crucial in proving the later theory.

We review some of the notation and theory that is needed. Proofs for most of these results may be found in [5] and [6].

The set $\mathscr{C}(G)$ of all the convex l-subgroups of an l-group G is a complete algebraic Brouwerian lattice. $C \in \mathscr{C}(G)$ is *closed* if $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq C$ and $a = \bigvee a_\lambda$ exists implies $a \in C$. If $A \in \mathscr{C}(G)$ then $A' = \{g \in G \mid |g| \wedge |a| = 0 \text{ for each } a \in A\}$ is a closed convex l-subgroup called the *polar* of A. The set of all polars forms a complete Boolean algebra. If $g \in G$ then $G(g) = \{x \in G \mid |x| \leqq n|g| \text{ for some } n > 0\}$ is the principal convex l-subgroup generated by g and $g'' = G(g)''$ is the *principal polar* determined by g. The elements $G(g)$ are the compact elements in the lattice $\mathscr{C}(G)$.

$M \in \mathscr{C}(G)$ is called *regular* if it is maximal without containing some element $g \in G$. Then we say that M is a *value* of g, and M or g is called *special* if M is the only value of g. G is called *finite valued* if each element in G has only a finite number of values.

$M \in \mathscr{C}(G)$ is called *prime* if the convex l-subgroups that contain it form a chain or equivalently if $a \wedge b = 0$ implies $a \in M$ or $b \in M$. Each regular subgroup is prime and each convex l-subgroup is the intersection of regular subgroups.

Let $\Gamma = \Gamma(G)$ denote the set of all regular subgroups G_γ and for each G_γ let G^γ be the convex l-subgroup that covers it. If $G_\gamma \lhd G^\gamma$ then G^γ/G_γ is an archimedean o-group. If $G_\gamma \lhd G^\gamma$ for each $\gamma \in \Gamma(G)$ then G is *normal valued.* Now Γ is a root system (i.e., a po set such that the elements that exceed a fixed element form a chain). G_γ is *essential* if there exists an element $y \in G$ with all of its values contained in G_γ. A subset of Γ is *plenary* if it is a dual ideal of Γ with zero intersection. Each plenary subset must contain the essential elements and if Γ admits a minimal plenary subset then it consists of the essential elements and so is unique.

If Γ is an arbitrary root system and T_γ is an archimedean o-group for each $\gamma \in \Gamma$, then $V(\Gamma, T_\gamma)$ will denote the subgroup of the direct product of the T_γ, where the support of each element satisfies the ACC. Define $v \in V$ to be positive if each maximal component is positive in the respective T_γ. Then V is an abelian l-group and the main result in [5] asserts that if G is a divisible abelian l-group then there is an l-isomorphism of G into $V(\Gamma(G), G^\gamma/G_\gamma)$. Let $\Sigma(\Gamma, T_\gamma)$ be the direct sum of the T_γ. Then Σ is a finite valued l-subgroup of V.

Finally if $\{S_\gamma \mid \gamma \in \Delta\}$ is a set of o-groups then $\Pi S_\gamma(\Sigma S_\gamma)$ will denote the cardinal product (sum) of the S_γ. If Δ is finite then we shall denote the cardinal sum by $S_1 \boxplus \ldots \boxplus S_n$.

1. THE MAPPING THEOREMS FOR CONVEX l-SUBGROUPS

Let $S \neq 0$ be a convex l-subgroup of an l-group G and let

$\mathscr{I}$ = all the prime subgroups of G that do not contain S, and
$\mathscr{S}$ = all the proper prime subgroups of S.

For each $M \in \mathscr{I}$ let $M\sigma = M \cap S$.

Theorem 1.1. *For an l-group G we have the following properties*

1) σ *is a one to one inclusion preserving map of* $\mathscr{I}$ *onto* $\mathscr{S}$. *If* $N \in \mathscr{S}$ *and* $0 < s \in \in S \setminus N$ *then* $N\sigma^{-1} = \{x \in G \mid |x| \wedge s \in N\}$.
2) $\mathscr{N}_G(M) \cap S = \mathscr{N}_S(M \cap S)$ *and* $M \lhd [M \cup S]$ *iff* $M \cap S \lhd S$, *where* $\mathscr{N}_G(M)$ *is the normalizer of* M *in* G.
3) M *is regular in* G *iff* $M\sigma$ *is regular in* S, *and in this case* $(M\sigma)^* = M^* \cap S$ *and* $M \lhd M^*$ *iff* $M\sigma \lhd (M\sigma)^*$, *where* $M^*((M\sigma)^*)$ *is the convex l-subgroup that covers* $M(M\sigma)$ *in* $G(S)$. *Moreover, if* $M \lhd M^*$ *then* $M^*/M \cong (M\sigma)^*/M\sigma$.
4) *If* $S = G(g)$ *then* M *is a value of* g *in* G *iff* $M\sigma$ *is a maximal convex l-subgroup of* $G(g)$ *and in this case* $M \lhd M^*$ *iff* $M\sigma \lhd G(g) = (M\sigma)^*$. *In particular, if* M *is the only value of* g *then* $M \lhd M^*$.
5) M *is closed iff* $M\sigma$ *is closed.*
6) M *is a polar iff* $M\sigma$ *is a polar and in this case both are minimal primes.*
7) M *is special iff* $M\sigma$ *is special.*
8) M *is essential iff* $M\sigma$ *is essential.*
9) *If* $M \lhd G$ *then* G/M *is discrete iff* $S/M\sigma$ *is discrete.*
10) *If* Δ *is a plenary subset of* $\Gamma(G)$ *then*

$$\Delta(S) = \{S \cap G_\gamma \mid \gamma \in \Delta \text{ and } G_\gamma \not\supseteq S\}$$

is a plenary subset of $\Gamma(S)$. *If* Δ *is minimal then so is* $\Delta(S)$.

Proof. (1) through (4) with the exception of $M^*/M \cong (M\sigma)^*/M\sigma$ are proved in [10] and the essential ideas for the proofs are contained in [6]. Now

$$\frac{(M\sigma)^*}{M\sigma} = \frac{M^* \cap S}{M \cap S} = \frac{M^* \cap S}{M \cap M^* \cap S} \cong \frac{M + (M^* \cap S)}{M} \subseteq \frac{M^*}{M}.$$

Consider $0 < t \in M^*$ and pick $0 < s \in (M^* \cap S) \setminus M$. Then since M^*/M is an archimedean o-group, $M + ns > M + t$ for some integer $n > 0$ so $M + t = M + + ns \wedge t$ and hence $M^* \subseteq M + (M^* \cap S)$.

Now let $N = M\sigma = M \cap S$, where $M \in \mathscr{I}$.

5) Suppose that M is closed, $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq N^+$ and $a = \bigvee_S a_\lambda$. Then since S is a convex l-subgroup, $a = \bigwedge_G a_\lambda$ so $a \in M \cap S = N$ and hence N is closed.

Conversely suppose that N is closed, $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq M^+$ and $a = \bigvee_G a_\lambda$. Pick $0 < s \in S$; then $s \wedge a = \bigvee_G (s \wedge a_\lambda)$ and since $s \wedge a \in S$ we have $s \wedge a = = \bigvee_S (s \wedge a_\lambda)$. But each $s \wedge a_\lambda \in S \cap M = N$ and so since N is closed $s \wedge a \in N$ for all $0 < s \in S$. Now suppose (by way of contradiction) that $a \notin M$ and pick $0 < s \in S \setminus M$. Then since M is prime $s \wedge a \notin M$, a contradiction. Therefore M is closed.

6) If $M = A'$ then M is a minimal prime and $M = a'$ where a is basic [10]. Now if $a'' \cap S = 0$ then $S \subseteq a' \subseteq M$, a contradiction, so pick $0 < x \in a'' \cap S$. Then $x'' = a''$ and x is basic in S. Now $x \notin N$ a minimal prime in S so N must be the polar of x in S.

Conversely if N is a polar then N is a polar in S of a basic element k. Clearly k is also basic in G and since $k \notin M$ we have $k' \subseteq M$ but since N is minimal so is M and so since k' is prime $k' = M$.

7) If M is special and the only value of $0 < g \in G$ then N is regular so pick $0 < s \in \in S$ with value N. Then M is the only value of $g \wedge s$ in S so N is special.

Conversely suppose that N is special and $0 < s \in S$ with N as its only value. Then M is the only value of s in G.

8) If M is essential then there exists $0 < g \in G$ all of whose values are contained in M. Now N is regular and hence a value of some $0 < s \in S$. Since $g \wedge s \in S$ and all the values of $g \wedge s$ are contained in M, all of the values of $g \wedge s$ in S are contained in N.

Conversely suppose that N is essential. Then there exists $0 < s \in S$ such that all the values of s in S are contained in N. Then all the values of s in G are contained in M.

9) G/M is discrete iff $M^*/M \cong Z$ and $M^*/M \cong (M\sigma)^*/M\sigma$.

10) A straightforward argument shows that $\Delta(S)$ is a plenary subset of $\Gamma(S)$. If Δ is minimal then each $G_\gamma \in \Delta$ is essential so each $G_\gamma \cap S$ is essential and hence $\Delta(S)$ is minimal.

Proposition 1.2. *For a minimal prime subgroup M of an l-group G the following are equivalent.*

1) *M is regular and a polar.*
2) *$M = g'$ where $G(g)$ is an archimedean o-group.*
3) *M is regular and closed and $M \lhd M^*$.*
4) *M is essential.*
5) *M is special.*

Proof. $(1 \to 2)$ $M = a'$ where $0 < a$ is basic. Pick $0 < x \in M^* \setminus M$. Then $g = x \wedge a \in M^* \setminus M$. Thus $M^* \supseteq g' \boxplus G(g) \supset g' = M$ and since M^* covers M we have $M^* = g' \boxplus G(g)$ and $G(g) \cong M^*/M$ an archimedean o-group.

$(2 \to 3)$ $M = g'$ is closed and since $g' \boxplus G(g)$ covers M we have M is regular and $M^* = g' \boxplus G(g)$ so $M \lhd M^*$.

$(3 \to 4)$ This follows from Theorem 3.11 in [3].

$(4 \to 5)$ There exists $0 < g \in G$ with all its values contained in M but since M is minimal it is the only value of G.

$(5 \to 1)$ We may assume that M is the unique value of $0 < g \in G$. Then by Theorem 3.6 in [6] $M = K \boxplus g'$ where K is the largest convex l-subgroup of $G(g)$, $G(g)$ is a lexicographic extension of K and $G(g)/K$ is an archimedean o-group. Suppose (by way of contradiction) that $0 < k \in K$. Then since M is a minimal prime there exists $0 < h \notin M$ since that $h \wedge k = 0$. If $h \notin M = G(g) \boxplus g'$ then by Lemma

6.2 in [4] $h > g > k$, a contradiction. If $h \in M^* \setminus M$ then $h = t + x$ where $0 < t \in G(g) \setminus K$ and $x \in g'$ but then $h \geqq t > k$, a contradiction. Therefore $K = 0$ so $M = g'$.

Question. Can (3) be replaced by (3') M is regular and closed?

If K is an l-ideal of an l-group G then the map σ of M onto M/K is an inclusion preserving bijection between the convex l-subgroups of G that contain K and the convex l-subgroups of G/K. Moreover, it is easy to see that M is prime (regular) iff $M\sigma$ is prime (regular) and if M is special so is $M\sigma$. If K is closed in G and M/K is closed in G/K then M is closed in G [11]. If $M = a'$ for $0 < a \in G$ then $M\sigma = (K + a)^*$ where $'$ (*) is the polar operation in $G(G/K)$. For if $0 < x \in G$ and $K + x \wedge K + a = K$ then $x \wedge a \in K \subseteq a'$ so $x \wedge a \in a' \cap a'' = 0$ and hence $a'/K \supseteq (K + a)^*$. Conversely if $0 < g \in a'$ then $K + g \wedge K + a = K$ so $a'/K \subseteq (K + a)^*$.

Now consider $0 < g \in G$ and suppose that $K = g' = G(g)' \lhd G$. Let $\mathcal{M}$ = the set of prime subgroups of G that contain g', and $\mathcal{N}$ = the set of prime subgroups of G/g'

Theorem 1.3. *σ induces an inclusion preserving bijection of $\mathcal{M}$ onto $\mathcal{N}$ and for $M \in \mathcal{M}$ we have:*

1) *M is regular iff $M\sigma$ is regular.*
2) *If M is special so is $M\sigma$.*
3) *If $M\sigma$ is closed so is M.*
4) *M is a minimal prime in G iff $M\sigma$ is a minimal prime in G/g'.*
5) *M is a polar iff $M\sigma$ is a polar.*
6) *If M is a minimal prime then M is special iff $M\sigma$ is special.*

Proof. 1, 2 and 3 follow from the above remarks. To prove the remainder we will use the fact that a prime subgroup M of G is minimal iff $0 < x \in M$ implies $x' \nsubseteq M$.

4) Clearly if M is a minimal prime in G then M/g' is a minimal prime. Conversely suppose that M/g' is minimal. If $g \in M$ then there exists $0 < x \notin M$ such that $g \wedge x \in g'$ so $g \wedge x \in G(g)' \cap G(g) = 0$, but the $x \in g' \subseteq M$, a contradiction. Now pick $0 < a \notin M$. Then there exists $0 < y \notin M$ such that $y \wedge a \in g'$. Thus $(g \wedge y) \wedge a = 0$ and $g \wedge y \notin M$ so M is minimal.

5) Suppose $M/g' = (g' + a)^*$ with $0 < a \notin g'$. Then M/g' is a minimal prime and $g' + a$ is basic, so by the above argument $g \notin M$ and hence $g \wedge a \notin M$. It suffices to show that $g \wedge a$ is basic; for then $M \supseteq (g \wedge a)'$ a minimal prime and hence $M = (g \wedge a)'$. Suppose (by way of contradiction) that $g \wedge a > x, y > 0$ with $x \wedge y = 0$. Then $x, y \notin g'$ and

$$g' + a \geqq g' + g \wedge a \geqq g' + x\,, \quad g' + y > g'$$

and $g' + x \wedge g' + y = g'$, but this means that $g' + a$ is not basic, a contradiction.

6) Suppose that M is minimal and $M\sigma$ is special. Then $M\sigma$ is a minimal prime and special in G/g' so by Proposition 1.2 $M\sigma$ is regular and closed and $M\sigma \lhd K\sigma$ where $K\sigma$ is the convex l-subgroup of G that covers $M\sigma$. Now by (1) and (3) M is regular and closed, and $M \lhd K$ so by Proposition 1.2 M is special.

Let $\mathbf{N}$ be the class of all l-groups in which each convex l-subgroup is normal or equivalently each regular subgroup is normal. Thus $G \in \mathbf{N}$ iff $a, b \in G^+$ implies $G(a) = G(a^b) = G(a)^b$ where $a^b = -b + a + b$.

Proposition 1.4. *$\mathbf{N}$ is a torsion class.*

Proof. It is clear that $\mathbf{N}$ is closed with respect to l-homomorphic images and convex l-subgroups. Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a set of convex l-subgroups of an l-group G each of which belongs to $\mathbf{N}$ and let K be a convex l-subgroup of $\bigvee C_\lambda$. Then

$$K = K \cap (\bigvee C_\lambda) = \bigvee (K \cap C_\lambda) .$$

Pick $0 < a \in C_\alpha \cap K$ and $0 < b \in C_\beta$. In order to prove that $K \lhd \bigvee C_\lambda$ it suffices to show that $a^b \in C_\alpha \cap K$.

Now $a = a \wedge b + \bar{a}$ and $b = a \wedge b + \bar{b}$ with $\bar{a} \wedge \bar{b} = 0$ so $\bar{a}^{\bar{b}} = \bar{a} \in C_\alpha \cap K$ and $a \wedge b \in C_\alpha \cap K \cap C_\beta \lhd C_\beta$ so $(a \wedge b)^{\bar{b}} \in C_\alpha \cap K$. Finally $(a \wedge b)^{a \wedge b} = a \wedge b$ and $a^{a \wedge b} \in C_\alpha \cap K$ since $a, a \wedge b \in C_\alpha \cap K$.

Remark. $\mathbf{N}$ is not closed with respect to products so it is not a variety. For let $K = Q \times Z$ with $(q, m) + (p, n) = (2^n q + p, m + n)$ and (q, z) positive if $z > 0$ or $z = 0$ and $q \geqq 0$. Let

$$G = \prod_{i=1}^{\infty} K_i , \quad a = \begin{pmatrix} 0, 0, 0, \ldots \\ 1, 1, 1, \ldots \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1, 2, 3, \ldots \\ 0, 0, 0, \ldots \end{pmatrix} .$$

Then

$$a^b = \begin{pmatrix} 0, 0, 0, \ldots \\ 2, 4, 8, \ldots \end{pmatrix} \quad \text{so} \quad G(a) \neq G(a^b) .$$

2. PRETORSION CLASSES OF l-GROUPS

Throughout this section we shall only consider l-groups that belong to the torsion class $\mathbf{N}$. For a class P of l-groups let $\Delta(P)$ be the class of l-groups such that each l-homomorphic image belongs to P.

I. *If P is closed with respect to joins of convex l-subgroups then so is $\Delta(P)$.*

Proof. Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a set of convex l-subgroups of G each of which belongs to $\Delta(P)$ and let K be an l-ideal of $\bigvee C_\lambda$. Then

$$\frac{\bigvee C_\lambda}{K} = \bigvee \left(\frac{K + C_\lambda}{K} \right) \quad \text{and} \quad \frac{K + C_\lambda}{K} \cong \frac{C_\lambda}{C_\lambda \cap K} \in P \quad \text{for each} \quad \lambda \in \Lambda .$$

Moreover, each $(K + C_\lambda)/K$ is a convex l-subgroup of $(\bigvee C_\lambda)/K$ so $(\bigvee C_\lambda)/K \in P$ and hence $\bigvee C_\lambda \in \Delta(P)$.

II. *If P is closed with respect to convex l-subgroup so is $\Delta(P)$.*

Proof. Suppose that $G \in \Delta(P)$, C is a convex l-subgroup of G and K is an l-ideal of C. Then since $G \in \mathbf{N}$, $K \lhd G$ and hence $G/K \in P$. Therefore $C/K \in P$ and hence $C \in \Delta(P)$.

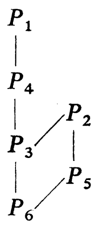
A class P of l-group that is closed with respect to convex l-subgroups and joins of convex l-subgroups will be called a *pre-torsion* class. We have shown that if P is a pre-torsion class then $\Delta(P)$ is a torsion class, but as we shall see the converse is not valid.

Note that if P and Q are classes of l-groups then $\Delta(P \cap Q) = \Delta(P) \cap \Delta(Q)$ and if P contains a torsion class $\mathbf{T}$ then $\mathbf{T} \subseteq \Delta(P) \subseteq P$.

We shall investigate the following subclasses of $\mathbf{N}$.

P_1: Each minimal prime subgroup is closed.
P_2: Each minimal prime subgroup is regular.
P_3: Each minimal prime subgroup is special or essential.
P_4: Each minimal prime subgroup is a polar.
P_5: For each minimal prime subgroup M, G/M is discrete.
P_6: Each minimal prime subgroup M is closed and G/M is discrete.

By Proposition 1.2 $P_1 \cap P_2 = P_3 \supset P_6 = P_1 \cap P_5$ so we have the following diagram of the inclusion relations between these classes.



Proposition 2.1. *Each P_i is a pretorsion class so $\Delta(P_i)$ is a torsion class for $i = 1, 2, \ldots, 6$.*

Proof. Let C be a convex l-subgroup of $G \in P_i$ and let N be a minimal prime in C. Then $N = M \cap C$ where M is a minimal prime in G. Thus M satisfies P_i and hence by Theorem 1.1 N satisfies P_i so $C \in P_i$.

Next let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a set of convex l-subgroups of G and suppose that each C_λ belongs to P_i. If N is a minimal prime in $\bigvee C_\lambda$ then $N = M \cap (\bigvee C_\lambda) = \bigvee(M \cap C_\lambda)$ where M is a minimal prime in G. Now $M \supset M \cap C_\lambda$ for some λ so $M \cap C_\lambda$ is a minimal prime in C_λ and hence satisfies P_i. But then by Theorem 1.1 M satisfies P_i and hence so does N. Therefore $\bigvee C_\lambda \in P_i$.

In [3] it is shown that each prime subgroup that contains a closed prime subgroup is itself closed and clearly the intersection of closed subgroups is closed. Thus condition P_1 is equivalent to the hypothesis that all the convex l-subgroups are closed.

Theorem 2.2 (Anderson) $\Delta(P_1) = \mathbf{Fv}$ *the torsion class of all finite valued l-groups or equivalently the class of all l-groups such that each regular subgroup is special.*

Proof. $G \in \mathbf{Fv}$ iff each regular subgroup is special [13] and hence each regular subgroup is closed. Thus $\mathbf{Fv} \subseteq P_1$ and since $\mathbf{Fv}$ is a torsion class $\mathbf{Fv} \subseteq \Delta(P_1)$.

Suppose (by way of contradiction) that $G \in \Delta(P_1)$ and $0 < g \in G$ has an infinite number of values. Let $\{M_\lambda | \lambda \in \Lambda\}$ be the values of g in $G(g) \in \Delta(P_1)$. Then there is a natural l-homomorphism

$$G(g) \to^{\eta} \Pi_\Delta G(g)/M_\lambda .$$

Thus $G(g)\eta$ is archimedean and has each l-ideal closed and hence each l-ideal is a polar [1] and so $G(g)\eta$ is a cardinal sum of o-groups [1], but this means that $G(g)\eta$ is finite valued, a contradiction.

Remark. If we consider all l-groups instead of restricting our attention to $\mathbf{N}$, then $\mathbf{Fv} \subseteq \Delta(P_1)$ but whether or not we get equality is an open question. For an arbitrary l-group G the above proof shows that the following are equivalent

a) G is normal valued and each homomorphic image of each principal convex l-subgroup has all of its convex l-subgroups closed.

b) G is finite valued.

Let $\mathbf{D}$ be the torsion class of all l-groups whose regular subgroups satisfy the DCC or equivalently each prime subgroup is regular, and let $\mathbf{C}$ be the torsion class of all l-groups G such that each G^γ/G_γ is cyclic. A study of $\mathbf{D}$ and $\mathbf{C}$ is contained in [13] and [12] respectively.

Suppose that $G \in \Delta(P_i)$ and M is a prime, then $G/M \in P_i$ so the minimal prime subgroup must satisfy P_i. Then for $i = 2$ M is regular and for $i = 5$ M is regular and M^*/M is cyclic. Thus $\Delta(P_2) \subseteq \mathbf{D}$ and $\Delta(P_5) \subseteq \mathbf{D} \cap \mathbf{C}$ and since the other inclusion are clear we have the following Theorem.

Theorem 2.3 $\Delta(P_2) = \mathbf{D}$ *and* $\Delta(P_5) = \mathbf{D} \cap \mathbf{C}$ *the torsion class of all l-groups with each prime regular and discrete.*

Corollary. $\Delta(P_3) = \mathbf{Fv} \cap \mathbf{D}$ *the torsion class of all l-groups such that each prime subgroup is special.*

Corollary. *An abelian divisible l-group G belongs to* $\Delta(P_3)$ *iff* $G \cong \Sigma(\Gamma, G^\gamma/G_\gamma)$ *and* Γ *satisfies the DCC. In particular, a vector lattice belongs to* $\Delta(P_3)$ *iff* $G \cong \Sigma(\Gamma, R)$ *and* Γ *satisfies the DCC.*

This follows from the corresponding embedding theorem in [5]. For a proof see [10] page 4.21.

In [13] it is shown that $\mathbf{D} \cap \mathbf{F} = \mathbf{R}^* \cap \mathbf{Fv} = \mathbf{A}^* \cap \mathbf{Fv}$, where R^* is the completion of the torsion class $\mathbf{R}$ of cardinal sums of archimedian o-groups and A^* is the completion of the torsion class of hyper-archimedian l-groups.

A convex l-subgroup C of G is a *lex-subgroup* if it is a proper lexicographic extension of a convex l-subgroup D of G. If in addition, there does not exist a proper lex extension of C in G then C is a *maximal* lex-subgroup. The set $M(G)$ of all the maximal lex subgroup of G forms a root system with respect to inclusion [7]. Thus if a and b are special elements in G then $G(a)$ and $G(b)$ are lex-subgroups so either $a \wedge b = 0$ or $G(a)$ and $G(b)$ are comparable.

The main result in [7] Theorem 5.2 asserts that for an l-group G the following are equivalent.

a) G is a normal lex-sum of o-groups.
b) $G \in \mathbf{Fv}$ and $M(G)$ satisfies the DCC.
c) The lattice of all filets satisfies the DCC or equivalently the lattice of all principal polars satisfies the DCC.
d) Each filet chain is finite.

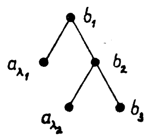
Remark. It has been pointed out by Steve McCleary that Proposition 4.1 in [7] is false and so the proof of Theorem 5.1 is not valid. This has been corrected in [10]. Moreover the definition of lex-sum has been simplified.

The next theorem shows that the class of all the normal lex-sums of o-groups is a torsion class.

Theorem 2.4. *$\Delta(P_4)$ consists of those l-groups that satisfy property* (b). *Thus $\Delta(P_4)$ is the torsion class of normal lex-sums of o-groups.*

Proof. Consider $G \in B =$ the class of l-groups that satisfy (b). Then G has a basis $\{a_\lambda | \lambda \in \Lambda\}$; see the proof of Theorem 5.2 in [7].

1) $G \in P_4$. For suppose (by way of contradiction) M is a minimal prime that is not a polar. Then $M \supseteq \Sigma a''_\lambda$ and since $G \in \mathbf{Fv}$ we can pick $0 < b_1$ special and not in M. Now $b_1 \wedge a_{\lambda_1} > 0$ for some $\lambda_1 \in \Lambda$ and since both are special it follows that $G(b_1) \supset - G(a_{\lambda_1})$. Since $a_{\lambda_1} \in M$ a minimal prime we have $a_{\lambda_1} \wedge b_2 = 0$ for some $b_2 \notin M$ and, once again, we may assume b_2 is special. Now $G(b_1)$ and $G(b_2)$ must be comparable, otherwise $b_1 \wedge b_2 = 0$ and hence $b_1 \in M$ or $b_2 \in M$. It follows that $G(b_1) \supset \supset G(b_2)$. Continuing in this way we get an infinite descending chain in $M(G)$.



$$b''_1 \supset b''_2 \supset \dots$$

2) B is closed with respect to l-homomorphic images. Let C be an l-ideal of G. If $C + a$ is special in G/C then $C + a = C + b$, where b is special in G. For (without loss of generality) $0 < a = a_1 + \ldots + a_t$ where the a_i are special and disjoint. Now all but one of the a_i must belong to C. For if $a_1, a_2 \notin C$ then A_1/C and A_2/C are distinct values of $C + a$ where A_i is the value of a_i $(i = 1, 2)$.

If $a, b \in G \setminus C$ are special and $(C + a)^{**} \subset (C + b)^{**}$ then $a'' \subset b''$ where $*$ $(')$ is the polar operation in $G/C(G)$. (See [10] p. 3.22).

Thus since $B \subseteq P_4$ and B is closed with respect to l-homomorphic images we have $B \subseteq \Delta(P_4)$.

Now suppose that $G \in \Delta(P_4)$ then by Theorem 2.2 $G \in \mathbf{Fv}$ so to complete the proof we must only show that $M(G)$ satisfies the DCC. Suppose (by way of contradiction) this is false, then we have an infinite chain $a_1'' \supset a_2'' \supset \ldots$ in $M(C)$, where a_i is special with value γ_i for $i = 1, 2, \ldots$. Since $a_{i+1} \in G_{\gamma_i}$ we have $G_{\gamma_i} \supseteq G(a_{i+1})$ so $\bigcap G_{\gamma_i} \supseteq$ $\supseteq \bigcap G(a_i) = K$. Now $\bigcap G_{\gamma_i}$ is prime so $\bigcap G_{\gamma_i}/K$ is prime in G/K. Let M/K be a minimal prime contained in $\bigcap G_{\gamma_i}/K$. Then since $G/K \in P_4$

$$M/K = (K + b)^*$$

where $K + b$ is basic in G/K so without loss of generality b is special in G with value β. If $\beta \parallel \gamma_j$ the $b \wedge a_j = 0$ so $K + a_j \in M/K$ and hence $a_j \in M \subseteq \bigcap G_{\gamma_i}$ a contradiction. Therefore β is comparable with each γ_i. If $\beta \leqq \gamma_i$ for all i then $b \in K$ so $\beta > \gamma_i$ for some i. Now $a_i'' \supset a_{i+1}''$ so there exists a special element x such that $x \wedge a_{i+1} = 0$ and $x < a_i$. Thus $b > a_i > a_{i+1}$ and x so $b \geqq a_{i+1}$ and x, and a_{i+1} and x are disjoint modulo K. But $a_{i+1} \notin K$ and $x \notin G(a_{i+1})$ so $x \notin K$. Therefore $K + b$ exceeds the strictly positive disjoint elements $K + a_{i+1}$ and $K + x$, but this contradicts the fact that $K + b$ is basic.

An l-group G is *compactly generated* if for each subset A of G for which $a = \bigvee A$ exists there exists a finite subset of A with join a. An l-group G is *discrete* if each $0 < g \in G$ exceeds an atom or equivalently if G has a basis of atoms. In [8] it is shown that for an (arbitrary) l-group C the following are equivalent.

1) G is compactly generated.
2) G is discrete and each minimal prime is a polar.
3) If M is a minimal prime subgroup of G then there exists an atom $a \notin M$.
4) Each ultrafilter of G is principal.
5) Each minimal prime subgroup of G is closed and the lattice of all right cosets is discrete.

Note that 5) is P_6 so $\Delta(P_6) =$ *all l-groups G such that each l-homomorphic image is compactly generated.* The fact that this is a torsion class is rather surprising.

An l-group G is *ω-discrete* if each homomorphic image is discrete or equivalently if each homomorphic image has a basis of atoms. Let $\mathbf{Z}$ be the torsion class of cardinal sums of cyclic groups and let $\mathbf{Z}^*$ be the complete torsion class generated by $\mathbf{Z}$ (see [15]). Let $\mathbf{W}$ be the class of all ω-discrete l-groups.

Proposition 2.5. $\mathbf{W} = \mathbf{Z}^*$ *so* $\mathbf{W}$ *is a complete torsion class.*

Proof. If $G \in \mathbf{Z}^*$, $G = \mathbf{Z}^*(G) \subseteq \mathbf{Z}(G)''$ [15] so $\mathbf{Z}(G)$ contains a basis of atoms for G. Now $\mathbf{Z}^*$ is a torsion class so if K is an l-ideal of G then $G/K \in \mathbf{Z}^*$ and so has a basis of atoms. Therefore $\mathbf{Z}^* \subseteq \mathbf{W}$.

Conversely if $G \in \mathbf{W}$ then G has a basis of atoms $\{a_\lambda | \lambda \in \Lambda\}$ and $Z(G) = \Sigma[a_\lambda]$. Then $G/Z(G)$ has a basis of atoms so it follows that $G \in \mathbf{Z}^*$ and hence $\mathbf{W} \subseteq \mathbf{Z}^*$.

Theorem 2.6. $\Delta(P_6) = \mathbf{C} \cap \mathbf{D} \cap \mathbf{Fv} = \mathbf{W} \cap \mathbf{Fv}$.

Proof. $\Delta(P_6) = \Delta(P_1) \cap \Delta(P_5) = \mathbf{Fv} \cap \mathbf{D} \cap \mathbf{C}$. Now since each compactly generated l-group has a basis of atoms it follows that $\Delta(P_6) \subseteq \mathbf{W}$ so $\Delta(P_6) \subseteq \mathbf{W} \cap \mathbf{Fv}$. If $G \in \mathbf{W}$ then $G \in \mathbf{R}^* \subseteq \mathbf{D}$ (see [13]) and since each $G/G_\gamma \in \mathbf{W}$, G^γ/G_γ is cyclic so $G \in \mathbf{C}$. Thus $\mathbf{W} \subseteq \mathbf{C} \cap \mathbf{D}$ and hence $\mathbf{W} \cap \mathbf{Fv} \subseteq \mathbf{C} \cap \mathbf{D} \cap \mathbf{Fv} = \Delta(P_6)$.

Theorem 2.7. *For an abelian l-group G the following are equivalent.*

1) $G \in \Delta(P_6)$.
2) *Each l-homomorphic image is compactly generated.*
3) *G is ω-discrete and finite valued.*
4) *Each component of G is cyclic, G is finite valued and* $\Gamma(G)$ *satisfies the DCC.*
5) $G \cong \Sigma(\Gamma, Z)$ *where* Γ *satisfies the DCC.*

Proof. We know that (1) through (4) are equivalent and Proposition 3.2 in [12] asserts that (3) and (5) are equivalent.

In this section we have shown:

$\Delta(P_1) = \mathbf{Fv}$ = each regular subgroup special

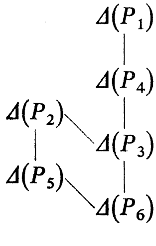
$\Delta(P_2) = \mathbf{D}$ = each prime regular

$\Delta(P_3) = \mathbf{Fv} \cap \mathbf{D}$ = each prime special

$\Delta(P_4)$ = normal lex-sums of o-groups

$\Delta(P_5) = \mathbf{D} \cap \mathbf{C}$ = each prime regular and discrete

$\Delta(P_6) = \mathbf{C} \cap \mathbf{D} \cap \mathbf{Fv} = \mathbf{W} \cap \mathbf{Fv}$
= each prime discrete and special
= each l-homomorphic image compactly generated.



Once again in this section we shall consider only l-groups that belong to the torsion class $\mathbf{N}$. We shall investigate the following subclasses of $\mathbf{N}$.

Q_1: $G = 0$ or G has a minimal prime subgroup that is closed.
Q_2: $G = 0$ or G has a minimal prime subgroup that is regular.
Q_3: $G = 0$ or G has a minimal prime subgroup that is special.
Q_4: $G = 0$ or G has a minimal prime subgroup that is a polar.
Q_5: $G = 0$ or G/M is discrete for some minimal prime M.
Q_6: $G = 0$ or G/M is discrete for some closed minimal prime M.

It is clear that none of the Q_i is a pre-torsion class, but we shall show that each $\Delta(Q_i)$ is a torsion class. Note that

$G \in Q_4$ iff $G = 0$ or G contains a basic element.
$G \in Q_3$ iff $G = 0$ or G contains a basic element g such that $G(g)$ is archimedian.
$G \in Q_5$ iff $G = 0$ or G contains an atom.

Lemma 3.1. 1) *If* $0 \neq C$ *is a convex l-subgroup of G that belongs to* Q_i *then* $G \in Q_i$ *for* $i = 1, 2, \ldots, 6$.

2) Q_i *and hence* $\Delta(Q_i)$ *is closed with respect to the join of convex l-subgroups for* $i = 1, 2, \ldots, 6$.

Proof. 1) Let N be a minimal prime subgroup of C that satisfies Q_i. Then $N = M \cap C$, where M is a minimal prime in G and by Theorem 1.1 M satisfies Q_i.

2) Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a set of convex l-subgroups of G each of which belongs to Q_i. Then by (1) $\bigvee C_\lambda \in Q_i$ and hence by I of Section 2 $\Delta(Q_i)$ is closed with respect to joins of convex l-subgroups.

Proposition 3.2. $\Delta(Q_i)$ *is a torsion class for* $i = 1, 2, \ldots, 6$.

Proof. We first show that if $G \in \Delta(Q_i)$ then each convex l-subgroup of G belongs to Q_i. For if $0 < g \in G$ then $G/g' \in Q_i$ so there is a minimal prime M/g' that satisfies Q_i. By Theorem 1.3 M satisfies Q_i and since $g \notin M$ we have by Theorem 1.1 that $M \cap G(g)$ satisfies Q_i. Therefore $G(g) \in Q_i$ and hence by Lemma 3.1 each convex l-subgroup of G belongs to Q_i.

Now suppose that C is a convex l-subgroup of $G \in \Delta(Q_i)$ and let K be an l-ideal of C. Then $G/K \in \Delta(Q_i)$ so $C/K \in Q_i$. Therefore $C \in \Delta(Q_i)$ and hence $\Delta(Q_i)$ is closed with respect to convex l-subgroups.

Proposition 3.3. $\Delta(Q_i)$ *is the torsion class of all l-groups such that in each l-homomorphic image the intersection of all the minimal primes that satisfy* Q_i *is zero.*

Proof. If $G \in \Delta(Q_i)$ and $0 < g \in G$ the $G/g' \in Q_i$ and so contains a minimal prime M/g' that satisfies Q_i. Then by Theorem 1.3 M is a minimal prime in G that satisfies Q_i and $g \notin M$ so the intersection of all such minimal primes is zero. The other inclusion is clear.

$$\Delta(Q_2) = \mathbf{D} = \Delta(P_2).$$

For if $G \in \Delta(Q_2)$ and M is prime then the minimal prime in G/M is regular and hence M is regular.

$$\Delta(Q_5) = \mathbf{D} \cap \mathbf{C} = \Delta(P_5).$$

For if $G \in \Delta(P_5)$ and M is prime then the minimal prime in G/M is regular and discrete and hence M is regular and discrete.

$$\Delta(Q_1) \cap \Delta(Q_2) = \Delta(Q_3).$$

For if $G \in \Delta(Q_1) \cap \mathbf{D}$ then the closed minimal primes have zero intersection and each prime is regular. Then by Proposition 1.2 the special minimal primes have zero intersection

$$\Delta(Q_1) \cap \Delta(Q_5) = \Delta(Q_6).$$

For if $G \in \Delta(Q_1) \cap \mathbf{D} \cap \mathbf{C}$ then the closed minimal primes have zero intersection and since $G \in \mathbf{C} \cap \mathbf{D}$ each such prime is regular and discrete.

Now let $\mathbf{O}(\mathbf{R})$ be the torsion class of cardinal sums of o-groups (archimedean o-groups) and let $\mathbf{O}^*(\mathbf{R}^*)$ be the complete torsion class generated by $\mathbf{O}(\mathbf{R})$.

$\Delta(Q_4)$ = all l-groups G such that each l-homomorphic image is 0 or has basis = $= \mathbf{O}^*$. For the first equality follows from the fact that G has a basis iff the set of all minimal primes that are polars has zero intersection. Then clearly $\Delta(Q_4) \subseteq \mathbf{O}^*$. Finally if $G \in \mathbf{O}^*$ then $G = O^*(G) \subseteq O(G)''$ [15] so $\mathbf{O}(G)$ is a basis for G and since $\mathbf{O}^*$ is a torsion class each l-homomorphic image must have a basis.

$\Delta(Q_3)$ = all l-groups G such that each l-homomorphic image is 0 or has a basis of archimedean elements = $\mathbf{R}^*$. For G has a basis of archimedean elements iff the set of minimal primes that are special has zero intersection.

$\Delta(Q_6)$ = all l-group G such that each l-homomorphic image is 0 or has a basis of atoms = $\mathbf{Z}^* = \mathbf{W}$.

In [13] it is shown that for an arbitrary l-group G $\Gamma(G)$ admits a minimal plenary subset iff the convex l-subgroup radical $R(G) = 0$ and G is completely distributive iff the intersection $D(C)$ of all the closed prime subgroups is zero. Now if G is normal valued then $R(C) = D(C)$ so G is completely distributive iff $\Gamma(G)$ admits a minimal plenary subset.

Proposition 3.4. *For a normal valued l-group G the following are equivalent.*

1) *G is completely distributive.*
2) *The closed minimal primes of G have intersection zero.*

Proof. $(1 \to 2)$ Let Δ be the minimal plenary subset of essential elements in $\Gamma(G)$ and consider $\delta \in \Delta$. We show that either $\delta > \gamma \in \Delta$ or δ is minimal in Γ and so by Proposition 1.2 $G_\delta = a'$ where $G(a) \prec R$.

If δ is not special then there exists $0 < g \in G$ all of whose values are $<\delta$ but at least one value γ of g must belong to Δ.

Suppose that δ is special and let $0 < a \in G^\delta \setminus G_\delta$ be special. Let L be the lex-kernel of $G(a)$. If $L = 0$ then $G(a) \prec R$ so a is basic and $a' = G_\delta$ is minimal. If $0 < l \in L$ then $a > nl$ for all positive integers n. Let G_γ be a value of l in Δ then clearly $a \notin G_\gamma$ so $\gamma < \delta$.

Theorem 3.5. *$\Delta(Q_1)$ is the torsion class of all l-groups such that each l-homomorphic image is completely distributive.*

One should investigate the structure of the l-groups in $\Delta(Q_1)$. The following is a start.

Proposition 3.6. *If $G \in \Delta(Q_1)$ then each $0 < g \in G$ has a special value.*

Proof. Let Λ be the minimal plenary subset of $\Gamma(G)$. Then

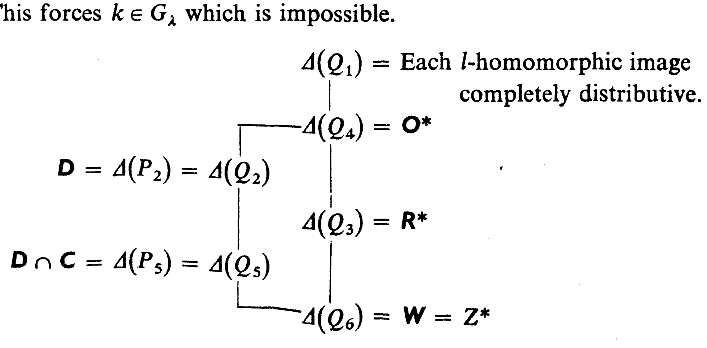
$$G_\lambda \to G_\lambda \cap G(g)$$

is a one to one map of the $G_\lambda \not\supseteq G(g)$ onto the minimal plenary subset Λ_g of $\Gamma(G(g))$. Let η be the natural l-homomorphism of $G(g)$ into $\Pi\, G(g)/(G_\lambda \cap G(g))$ all $\lambda \in \Lambda_g$ and let $N = \bigcap(G_\lambda \cap G(g))$. Then $G(g)\eta \cong G(g)/N$ is archimedean and completely distributive and hence has a basis.

Let $N + h$ be basic in $G(g)/N$ with $h > 0$. Now the maximal elements $G_\lambda \cap G(g)$ in Λ_g form a plenary subset of $\Gamma(G(g)/N)$ and $N + h$ has exactly one value in the set $-$ so $h \notin G_\alpha$ and $h \in G_\lambda$ and $\lambda \neq \alpha$ with $G_\lambda \cap G(g)$ maximal

$$G_\alpha + nh > G_\alpha + g \quad \text{for some} \quad n > 0\,, \quad \text{so}$$
$$k = (nh - g) \vee 0 = nh - (g \wedge nh) \in G(g) \setminus G_\alpha\,.$$

We show that k is special in G. It suffices to show that $G_\alpha \cap G(g)$ is the only value of k in Λ_g. Suppose (by way of contradiction) $\lambda \parallel \alpha$ is another value of k in Λ_g. Then $h \notin G_\lambda$ so G_λ is not maximal but then $\lambda < \beta \in \Lambda_g$ where β is maximal and hence $g \notin C_\beta$. This forces $k \in G_\lambda$ which is impossible.



From the diagram $\mathbf{R^*} = \mathbf{O^*} \cap \mathbf{D}$ and $\mathbf{W} = \mathbf{Z^*} = \mathbf{O^*} \cap \mathbf{D} \cap \mathbf{C} = \mathbf{R^*} \cap \mathbf{C}$. Note that

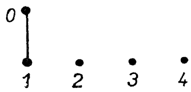
$$\begin{aligned} \mathbf{Fv} \cap \Delta(Q_1) &= \Delta(P_2), \\ \mathbf{Fv} \cap \Delta(Q_2) &= \mathbf{Fv} \cap \Delta(Q_3) = \Delta(P_3), \\ \mathbf{Fv} \cap \Delta(Q_5) &= \mathbf{Fv} \cap \Delta(Q_6) = \Delta(P_6), \quad \text{but} \\ \mathbf{Fv} \cap \Delta(Q_4) &= \mathbf{Fv} \cap \mathbf{O^*} \supset \Delta(P_4) \end{aligned}$$

so by intersection with $\mathbf{Fv} = \Delta(P_1)$ we get one new torsion class $-$ $\mathbf{Fv} \cap \mathbf{O^*}$.

EXAMPLES

1) An l-group G such that each minimal prime is a polar but G is not finite valued. Let G be compactly generated but not finite valued.

2) Let Γ be the partially ordered set



Let $V(\Gamma, Z)$ be the l-group of all integer valued functions on Γ, and let G be the l-subgroup of V that consists of $\sum_{i=1}^{\infty} Z_i$ plus the constant functions. Then G is c-generated since the minimal prime subgroups are of the form

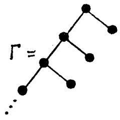
$$G_n = \{g \in G \mid g_n = 0\} \quad n = 1, 2, 3, \ldots$$

G belongs to $\mathbf{W} = \Delta(Q_6) = Z^*$, but not to $\Delta(P_1) = \mathbf{Fv}$.

Finally G belongs to $\Delta(Q_1)$ and all but one element in the minimal plenary subset of $\Gamma(G)$ is special.

3) If P is the class of all l-groups that are subdirect sums of reals, then $\Delta(P)$ is the torsion class of hyperarchimedian l-groups. Note that P is not a pretorsion class; for if G is archimedian the $G = \bigvee_{g \in G} G(|g|)$ and each $G(|g|) \in P$.

4) If $G \in P_4$ then G has a basis so $\Delta(P_4) \subseteq \mathbf{O^*} \cap \mathbf{Fv}$, but we need not have equality. For let $G = \Sigma(\Gamma, Z)$ where



Then $G \in \mathbf{O^*} \cap \mathbf{Fv}$ but $G \notin P_4$.

References

[1] *A. Bigard:* Contribution a la theorie des groupes reticules, These, University Paris, (1969).

[2] *R. Byrd:* Complete distributivity in lattice-ordered groups, Pacific J. Math. *20* (1967), 423—432.

[3] *R. Byrd* and *T. Lloyd:* Closed subgroups and complete distributivity in lattice-ordered groups, Math Zeitschr. *101* (1967) 123—130.

[4] *P. Conrad:* Some structure theorem for lattice-ordered groups, Trans. Amer. Math. Soc. *99* (1961), 212—240.

[5] *P. Conrad, J. Harvey* and *C. Holland:* The Hahn embedding theorem for abelian lattice ordered groups, Trans. Amer. Math. Soc. *108* (1963), 143—169.

[6] *P. Conrad:* The lattice of all convex l-subgroups of a lattice-ordered group, Czech. Math. J. *15* (1965), 101—123.

[7] *P. Conrad:* Lex-subgroups of lattice-ordered groups, Czech. Math. J. *18* (1968), 86—103.

[8] *P. Conrad, A. Bigard* and *S. Wolfenstein:* Compactly generated lattice-ordered groups, Math. Zeit. *107* (1968), 201—211.

[9] *P. Conrad* and *D. McAlister:* The completion of a lattice-ordered group, J. Australian Math. Soc. *9* (1969), 182—208.

[10] *P. Conrad:* Lattice-ordered groups, Lecture notes, Tulane Math. Library (1970).

[11] *P. Conrad* and *R. Bleier:* The lattice of closed ideals and a^*-extensions of an abelian l-group, Pacific J. Math. *47* (1973), 329—340.

[12] *P. Conrad* and *P. Montgomery:* Lattice-ordered groups with rank over components, Czech. Math. J. *25* (1975), 445—453.

[13] *P. Conrad:* Torsion radicals of lattice-ordered groups, Symposia Mathematica, Vol. *XXI* (1977), 479—513.

[14] *J. Martinez:* Varieties of lattice-ordered groups, Math. Zeitschr. *137* (1974), 265—284.

[15] *J. Martinez:* Torsion theory for lattice-ordered groups, Czech. Math. J. *25* (1975), 284—299.

[16] *J. Martinez:* Torsion theory for lattice-ordered groups, Part II: Homogeneous l-groups, To appear.

[17] *F. Petersen:* Contributions to the theory of regular and prime subgroups of lattice-ordered groups, Dissertation Tulane University (1967).

[18] *F. Šik:* Sous-groupes simples et ideaux simples des groupes réticulés, C. R. Acad. Sci. Paris *261* (1965), 2791—2793.

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