Alex Muir; Mary Wynne Warner

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HOMOGENEOUS TOLERANCE SPACES

A. MUIR and M. W. WARNER, London

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INTRODUCTION

This paper discusses homogeneous tolerances spaces, classifying them as quotients of tolerance groups. In order to have a non-trivial theory it is necessary to modify a number of the usual definitions employed in the theory of tolerance spaces, particularly those of product space and function space.

Our interest in homogeneous spaces derives from the fact that many of the neural systems for which ZEEMAN ([4], [5]) initially proposed the theory do seem to exhibit a physical homogeneity in their structure. Probably some appropriate analogue of a topological manifold would be more suitable but unfortunately a usable definition is not immediately evident.

Poston, who has given the most detailed theory of tolerance spaces, employing the usual ideas of product and function space is only able to present a rather obscure homological definition, (POSTON, [3]). It seems to us that the idea of homogeneous space forms a usable concept which captures the essential uniformity of a patch of the retina or the set of touch receptors in a localized region of body surface.

Tolerance spaces were introduced by Zeeman in the context of perception theory. Discrimination of, say, points in the visual world is limited by the discreteness of retinal receptors. An indistinguishability of points therefore results and it is this that the tolerance concept formalizes.

It seems more consistent with Zeeman's intended application to accept that blurring of the image of a perceptual function is a property of the perceiving organism and not a result of a genuine indistinguishability in the objective world.

Thus two perceptions f and g are to be compared by reference to well-discriminated points in the world.

This initially motivated our change in the definition of function space tolerance since, as will be seen below, our definition does not employ a tolerance in the domain.

Furthermore, although it is commonplace to put the product topology on a Cartesian product of topological spaces arising in physical applications, this tendency derives more from mathematical niceties than from physical demands. Consider, for instance, a controlled dynamical system, the domain of interest being the direct product of the space of control functions and the space of states. There is no organic connection between these two spaces and hence no physical reason to intermingle their topologies.

Now ARBIB has pointed to the similarity between automata theory and control theory, linking the two through the concept of a tolerance automaton (Arbib, [1], [2]). This motivates us to keep separate the tolerances on the input space and the state space by making a definition of product tolerance which is finer than usual.

A tolerance space can be pictured as a graph with a loop at each point. For simplicity we omit the loops from the examples in the text.

The direction the theory takes, however, tends to be motivated by the geometric idea of nearness and consequently attempts to mirror the achievements of topology in this somewhat simpler setting.

1. TOLERANCE SPACES

1.1. A *tolerance space* (X, ϱ) is a set X with a symmetric, reflexive relation ϱ on X. We write $\varrho(x, x')$ as $x \varrho x'$; we say "x is within tolerance of x'" or, briefly, "x is near x'".

In the following, if the tolerance employed in a situation is clear from the context it will be denoted by ϱ, whatever the underlying set.

1.2. A *tolerance map* f from a tolerance space X to a tolerance space Y is a function $f : X \to Y$ such that $x \varrho x' \Rightarrow f(x) \varrho f(x')$.

Tolerance spaces and tolerance maps form a category, though this fact will not be used below.

1.3. A *homeomorphism* is a bijective tolerance map whose inverse is also a tolerance map.

1.4. The *monad* of a point x is the set of points within tolerance of x; it will be denoted $\mu(x)$.

1.5. Let X be a set, ϱ and ϱ_1 two tolerances on X. If $x \varrho x' \Rightarrow x \varrho_1 x'$, then we say ϱ is *finer* than ϱ_1 or that ϱ_1 is *coarser* than ϱ.

Finer means smaller monads; thus more discrimination between points.

1.6. Let X be a set, $\{Y_i\}$ a collection of tolerance spaces and $\{f_i : X \to Y_i\}$ a collection of functions.

The tolerance *induced* on X by these functions is the coarsest tolerance for which all the f_i are tolerance maps.

In particular, the inclusion function of a subset of a tolerance space induces the *subspace* tolerance. Also, a tolerance space structure is induced on the Cartesian product of a collection of tolerance spaces by the projections onto each factor. This is, indeed, the categorial product in the tolerance category mentioned in § 1.2. However, for our purposes an alternative definition of tolerance product will be better; we will refer, where necessary, to the categorial product as the "usual product".

1.7. Let X, Y be two tolerance spaces. We then define

(i) the *usual product* tolerance on $X \times Y$ by $(x, y)\, \varrho\, (x', y')$ iff $x\, \varrho\, x'$ and $y\, \varrho\, y'$.

(ii) the *product* tolerance on $X \times Y$ by $(x, y)\, \varrho\, (x', y')$ iff $x\, \varrho\, x'$ and $y = y'$, or $x = x'$ and $y\, \varrho\, y'$.

Figure 1 illustrates the difference in the simplest non-trivial case. The dotted lines show "nearnesses" which the usual product requires and the product does not; it is evident from the definitions that the product space has a finer tolerance than the usual

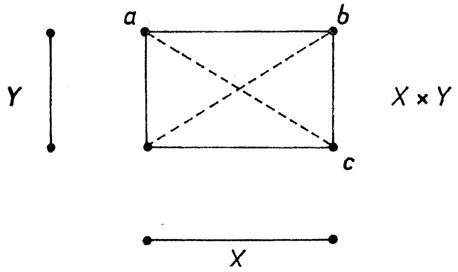

Fig. 1.

product space and, as we will see, leads to a less trivial theory of tolerance groups. However, the only joins omitted from the usual product are those of the kind ac, where there exists a point b such that $a\, \varrho\, b$, $b\, \varrho\, c$.

1.8. Let $\{X_i\}$ be a collection of tolerance spaces, $\{f_i : X_i \to Y\}$ a collection of functions to a set Y. The tolerance *co-induced* on Y is the finest tolerance for which all f_i are tolerance maps.

1.9. For $n > 1$ write $x\, \varrho^n\, y$ iff there is a finite sequence $\{z_0, \ldots, z_n\}$ such that $z_0 = x$, $z_n = y$ and $z_i\, \varrho\, z_{i+1}$.

The n-th *order monad* of a point x is defined by

$$\mu_n(x) = \{y : x\, \varrho^n\, y\}\,, \quad \mu_1(x) = \mu(x)\,.$$

The *component*, $C(x)$, of a point x is defined by

$$C(x) = \bigcup_{n>0} \mu_n(x)\,.$$

A tolerance space X is *connected* if $X = C(x)$ for some $x \in X$ (and hence for all $x \in X$).

A tolerance space X is *totally connected* if $x \varrho x'$ for all $x, x' \in X$.

1.10. If X, Y are tolerance spaces, define a tolerance on Y^X, the set of tolerance maps from X to Y by $f \varrho g$ iff $f(x) \varrho g(x)$ for all $x \in X$.

We will call this the *function space* tolerance; note that it is coarser than the definition employed by some other authors, e.g. Poston ([3]), which we shall call the *usual function space* tolerance, namely

$$f \varrho g \quad \text{iff } f(x) \varrho g(x') \quad \text{for all} \quad x \varrho x' .$$

The difference here is parallel to that in topology between point-wise convergence and compact-open topologies. The point-wise convergence topology does not employ the topology in the domain and, correspondingly, our function space tolerance does not use the tolerance on X. But again, the only joins omitted from the graph of the usual function space are those linking points a, b such that $a \varrho^2 b$.

1.11. Theorem. *If X, Y are tolerance spaces, let Y^X have the function space tolerance ($f \varrho g$ iff $f(x) \varrho g(x)$ for all $x \in X$), and let $Y^X \times X$ have the product tolerance ($(f, x) \varrho (f', x')$ iff $f = f'$, $x \varrho x'$ or $f \varrho f'$, $x = x'$). Then the evaluation map* ev : $Y^X \times X \to Y$, ev$(f, x) = f(x)$, *is a tolerance map and an identification; that is the tolerance on Y is that co-induced by* ev.

Proof. (i) Let $(f, x) \varrho (f', x')$. Then $f \varrho f'$, $x = x'$ or $f = f'$, $x \varrho x'$. In either case $f(x) \varrho f'(x')$ which implies ev is a tolerance map.

(ii) Let ϱ_1 be co-induced on Y by ev. Then from 1.5 and 1.8 $y \varrho_1 y' \Rightarrow y \varrho y'$.

Conversely, if $y \varrho y'$ let f, f' be constant maps from X to y, y' respectively. Then $f \varrho f'$, which implies $(f, x) \varrho (f', x)$ for all x. Thus ev$(f, x) \varrho_1$ ev(f', x) and so $y \varrho_1 y'$.

1.12. Theorem. (*Exponential Law.*) *If X, Y, T are tolerance spaces then $\phi : Y^{X \times T} \to (Y^X)^T$ defined by $\phi(f)(t)(x) = f(x, t)$ is a homeomorphism. Here the product and function space tolerances are those used in Theorem* 1.11.

Proof. Given that $f : X \times T \to Y$ is a tolerance map, we prove first that

(i) $\phi(f)(t) : X \to Y$ is a tolerance map,

(ii) $\phi(f) : T \to Y^X$ is a tolerance map.

(i) $x \varrho x' \Rightarrow f(x, t) \varrho f(x', t) \Rightarrow \phi(f)(t)(x) \varrho \phi(f)(t)(x')$.

(ii) $t \varrho t' \Rightarrow (x, t) \varrho (x, t')$ for all $x \Rightarrow f(x, t) \varrho f(x, t')$ for all $x \Rightarrow \phi(f)(t)(x) \varrho$ $\varrho\, \phi(f)(t')(x)$ for all $x \Rightarrow \phi(f)(t) \varrho \phi(f)(t')$.

(iii) ϕ is set theoretically $1-1$.

(iv) ϕ is onto. If $g \in (Y^X)^T$, let $f(x, t) = g(t)(x)$. Then $(x, t)\, \varrho\, (x', t') \Rightarrow x = x'$ and $t\, \varrho\, t'$ or $x\, \varrho\, x'$ and $t = t' \Rightarrow g(t)(x)\, \varrho\, g(t')(x')$ in either case. Hence $f \in Y^{X \times T}$ and $\phi(f) = g$.

(v) ϕ is tolerance. Let $f, f' \in Y^{X \times T}$, $f\, \varrho\, f'$. Then for all (x, t), $f(x, t)\, \varrho\, f'(x, t)$ which implies $\phi(f)\, \varrho\, \phi(f')$.

(vi) ϕ^{-1} is tolerance. Let $g, g' \in (Y^X)^T$, $g\, \varrho\, g'$. Then for all t, $g(t)\, \varrho\, g'(t)$ and for all x $g(t)(x)\, \varrho\, g'(t)(x)$. Hence $\phi^{-1}(g)(x, t)\, \varrho\, \phi^{-1}(g')(x, t)$ for all (x, t), which implies that $\phi^{-1}(g)\, \varrho\, \phi^{-1}(g')$.

2. HOMOGENEOUS SPACES

2.1. (i) A tolerance space X is homogeneous iff $\forall x, x' \in X$, there exists a homeomorphism $h: X \to X$ such that $h(x) = x'$.

(ii) A homogeneous space X is *very homogeneous* iff $\forall x, x' \in X$ such that $x\, \varrho\, x'$, there exists a homeomorphism h such that $h(x) = x'$ and $h\, \varrho\, 1$ in the function space X^X.

(iii) Note that this definition employs our coarse tolerance on the function space X^X; for comparison we also make an exactly parallel definition which uses the usual function space. A homogeneous space X is *most homogeneous*, iff $\forall x, x' \in X$ such that $x\, \varrho\, x'$, there exists a homeomorphism h such that $h(x) = x'$ and $h\, \varrho\, 1$ in the usual function space X^X.

(iv) It is evident that if a space is homogeneous then any two points have homeomorphic n-th order monads, for any n. In testing whether a given space is homogeneous, it is useful to look at monads of its points; if these are not identical, as graphs, then the given space is not homogeneous. If the n-th order monads of any two points are homeomorphic we say that the space is *n-homogeneous.*

2.2. Most homogeneous $\Rightarrow$ Very homogeneous $\Rightarrow$ homogeneous $\Rightarrow$ n-homogeneous.

We provide counter-examples to show that the implications cannot be reversed.

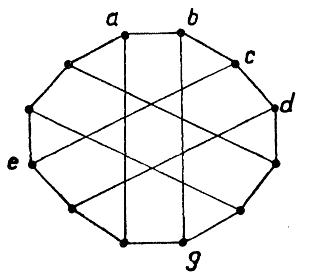

Fig. 2.

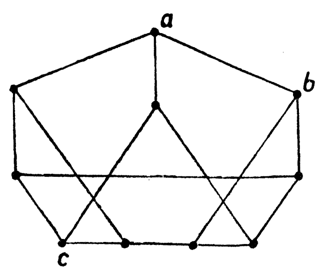

Fig. 3.

(i) The space of Figure 2 is homogeneous but not very homogeneous. A homeomorphism taking a to b is given by reflection in the vertical axis of symmetry. A homeomorphism taking a to c is a rotation through 60°. All other required homeomorphisms are composites of these two. Thus the space is homogeneous.

But if f is a homeomorphism taking b to c, then $f(a) = b$, d or e. Now a non ϱ d, a non ϱ e and if $f(a) = b$, then $f(g) = d$ or e. Since g non ϱ d, g non ϱ e, in either case there is an x such that $f(x)$ non ϱ x so f non ϱ 1 and the space is not very homogeneous.

(ii) The product space of Figure 1 is very homogeneous but not most homogeneous, since there is no homeomorphism h taking a to b such that for all $z \varrho z'$, $h(z) \varrho z'$.

(iii) The space of Figure 3 is 1-homogeneous but not 2-homogeneous and hence not homogeneous.

The monad of each point is the same. $\mu_2(a) =$ whole space, but $b \notin \mu_2(c)$.

It is evident that a totally connected space is most homogeneous.

2.3. Theorem. *Every connected most homogeneous space is totally connected.*

Proof. In the usual tolerance on X^X, $f \varrho 1$, $g \varrho 1 \Rightarrow f \,.\, g \varrho 1$.

Now for any $x, y \in X$ there is a sequence $\{x_0, x_1, \ldots, x_n\}$ such that $x_0 = x$, $x_n = y$ with $x_i \varrho x_{i+1}$. Hence, since X is most homogeneous, there exists a sequence $\{h_1, h_2, \ldots, h_n\}$ with $x_i = h_i(x_{i-1})$ and $h_i \varrho 1$ in the usual function space tolerance. The composition $h = h_n \,.\, h_{n-1} \ldots h_1 \varrho 1$ and $h(x) = y$, so $y \varrho x$.

This result makes the theory of most homogeneous spaces, and hence of groups as we shall see trivial in terms of the usual tolerance on function spaces. In the next section we first exhibit this fact in detail which motivates our pursuing of the nontrivial alternative definitions.

3. TOLERANCE GROUPS

3.1. Let $(G, \cdot, \varrho)$ be a group $(G, \cdot)$ with a tolerance ϱ. We call G a *tolerance group* if the product and inverse,

(i) $G \times G \to G$ given by $(g, g') \to g \,.\, g'$,

(ii) $G \to G$ given by $g \to g^{-1}$,

are tolerance maps.

Clearly this definition depends on the tolerance on $G \times G$. If we have the usual tolerance on $G \times G$, then we refer to a *usual tolerance group*. The alternative tolerance we have proposed only requires left and right translates in G to be tolerance maps.

3.2. Zelinka has proved that in a usual tolerance group the monad of the identity $\mu(1)$ is a totally connected normal subgroup and that cosets of $\mu(1)$ are homeomorphic components of G. (Zelinka, [6]). This can be strengthened to the following.

Lemma. *A usual tolerance group is most homogeneous (and hence totally connected, if connected, by* § 2.3)

Proof. If $x \varrho y$ in G, define the tolerance homeomorphism $h : G \to G$ by $h(g) = y . x^{-1} . g$.

Then $h(x) = y$ and $\forall g \; g' \in G$ with $g \varrho g'$, $h(g) \varrho g'$, so $h \varrho 1$.

This lemma shows that usual tolerance groups are uninteresting. Even the set of reals under addition with tolerance $a \varrho b \Leftrightarrow |a - b| < \varepsilon$ does not form a usual tolerance group. We now turn to the study of tolerance groups,

3.3. Lemma. *In a tolerance group G*

(i) *the monad of the identity is a self-conjugate subset which is closed under inverse.*

(ii) *the subgroup M' generated by $\mu(1)$ is a connected normal subgroup, whose cosets are homeomorphic components of G.*

Proof. (i) $x \varrho 1 \Rightarrow x^{-1} \varrho 1$. Also $x \varrho 1 \Rightarrow y . x . y^{-1} \varrho y . y^{-1} = 1$ for all $y \in G$.

(ii) By (i), every element of M' can be written in the form $z = x^\alpha . y^\beta \dots t^\gamma$ where $\{x, y, \dots, t\} \subseteq \mu(1)$ and $\alpha, \beta, \dots, \gamma$ are positive integers.

If $x \varrho^n 1$ and $y \varrho 1$, then $x . y \varrho^{n+1} 1$, so by induction z is in the $(\alpha + \beta + \dots + \gamma)$ – order monad of 1 and M' is connected.

M' is generated by a self-conjugate subset, so it is normal. Further, $z \varrho z' \Leftrightarrow g . z \varrho g . z'$ so $g . M' \cong M'$, and left cosets are homeomorphic.

Finally, let $z \in M'$, $g \varrho z$; then $z^{-1} . g \in \mu(1)$ so $g \in z . \mu(1) \subseteq zM' = M'$, so M' is a component of G.

3.4. Lemma. *A tolerance group is very homogeneous.*

Proof. Let $x \varrho y$ in G and define $h: G \to G$ by $h(g) = y . x .^{-1} g$. Then $h(x) = y$ and for all $g \in G$ $h(g) \varrho x . x .^{-1} g = g$ so $h \varrho 1$.

3.5. Lemma. *The function space of homeomorphisms of a tolerance space is a tolerance group.*

Proof. Let g, g' be homeomorphisms of a tolerance space X, with $g \varrho g'$. Then $\forall x \in X$, $g(x) \varrho g'(x)$, which implies that $x \varrho g^{-1} g'(x)$ for all x. Let $x' \in X$ and put $x' = g'(x)$. Then $g'^{-1}(x') = x \varrho g^{-1} g'(x) = g^{-1}(x')$, so $g'^{-1} \varrho g^{-1}$.

Also if for all x, $g(x) \varrho g'(x)$ then for any other homeomorphism f, $f . g(x) \varrho f . g'(x)$ so $f . g \varrho f . g'$.

Finally, if $x' \in X$, write $x = f(x')$ so that $g(x) = g . f(x') \varrho g' . f(x')$ and $g . f \varrho g' . f$.

4. CLASSIFICATION OF HOMOGENEOUS SPACES

4.1. Let G be a tolerance group, H a subgroup with subspace tolerance. Then the quotient set G/H may be given the identification tolerance coinduced by the projection $G \to G/H$.

Lemma. *G/H is very homogeneous*

Proof is straightforward.

4.2. Let X be a tolerance space, G its homeomorphism group and $H = \{g \in G: g(x_0) = x_0\}$ for some $x_0 \in X$. Define $\phi: G \to X$ by $\phi(g) = g(x_0)$.

Theorem. *ϕ induces a tolerance bijection $\tilde{\phi}$: $G/H \to X$* if *X is homogeneous. Iff, further, X is very homogeneous then ϕ is a homeomorphism.*

Proof. Denote elements of G/H, that is cosets $g \,.\, H$, by $\mathbf{g}$. Then define $\tilde{\phi}$ by $\tilde{\phi}(\mathbf{g}) = g(x_0)$.

(i) $\tilde{\phi}$ is well-defined, since $\mathbf{g} = \mathbf{g}' \Rightarrow g^{-1} \,.\, g'(x_0) = x_0 \Rightarrow g(x_0) = g'(x_0)$.

(ii) $\tilde{\phi}$ is a tolerance map, since $\mathbf{g} \,\varrho\, \mathbf{g}' \Rightarrow g(x) \,\varrho\, g'(x)$, $\forall x \in X \Rightarrow \tilde{\phi}(\mathbf{g}) \,\varrho\, \tilde{\phi}(\mathbf{g}')$.

(iii) $\tilde{\phi}$ is $1-1$, since $\tilde{\phi}(\mathbf{g}) = \tilde{\phi}(\mathbf{g}') \Rightarrow g(x_0) = g'(x_0) \Rightarrow \mathbf{g} = \mathbf{g}'$.

(iv) $\tilde{\phi}$ is onto, since for all $x \in X$, there exists $g \in G$ such that $x = g(x_0) = \tilde{\phi}(\mathbf{g})$ – by homogeneity of X.

(v) By Lemma 4.1 G/H is very homogeneous, so if $\tilde{\phi}$ is a homeomorphism X is very homogeneous.

Conversely, let X be very homogeneous and let $g(x_0) \,\varrho\, g'(x_0)$. Then there exists $h \in G$ such that $h \,.\, g(x_0) = g'(x_0)$ with $h \,\varrho\, 1$.

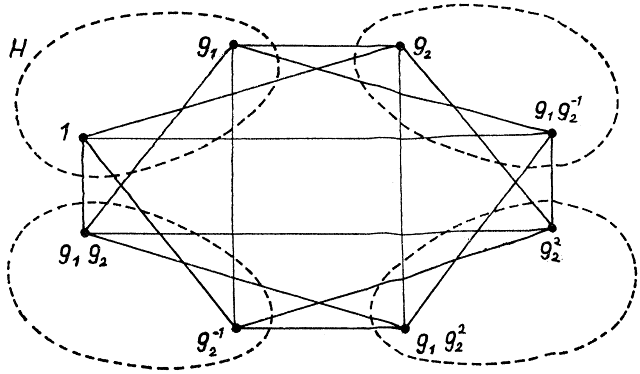

Fig. 4.

So $g \varrho h . g$ and hence $g \varrho h . g = g'$.

4.3. Example. The space of Figure 1 has $G =$ dihedral group, D_8 generated by g_1: $(a, b, c, d) \mapsto (a, d, c, b)$ g_2: $(a, b, c, d) \mapsto (b, c, d, a)$.

The tolerance group structure is pictured in Figure 4. If we choose $x_0 = a$, then $H = \{1, g_1\}$ and $G/H = \{1, g_2, g_2^2, g_2^{-1}\} \cong X$.

4.4. Tolerance groups can be re-constructed from their group properties and the monad of 1. All tolerance groups may therefore be obtained by choosing a group and selecting a subset closed under inverse and conjugation to be $\mu(1)$.

By Theorem 4.2 all very homogeneous spaces can be then obtained by quotients of tolerance groups and, further, all homogeneous spaces are coarsenings of such quotients.

By exactly similar reasoning we can prove that a most homogeneous space is homeomorphic to a quotient of usual tolerance groups, but as pointed out in 2.3 and 3.2 this result is trivial.

The corresponding theorem in topological spaces fails because the group operations are not in general continuous in the function space topology. The same problem arises in metric spaces unless we confine ourselves to "tolerance" maps f for which for all $\varepsilon > 0$, $d(x, y) < \varepsilon \Rightarrow d(f(x), f(y)) < \varepsilon$. Then the need for inverses also to possess this property confines us to isometries.

If a space X has bounded metric and is homogeneous in the sense that for all x, y in X, there exists an isometry of X mapping x to y, then with G the isometry group of X it is straightforward to show that $\tilde{\phi}$: $G/H \to X$ is a continuous bijection.

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Authors' address: The City University, St. John Street, London E.C.I., U.K.