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GENERALIZATIONS OF THE RIEMANN-LEBESGUE AND CANTOR-LEBESGUE LEMMAS

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I. Introduction. The purpose of this paper is to consider the problem of evaluating limits of the type

$$\lim_{\lambda \to +\infty} \int_I f(t)\, \beta(\lambda t)\, \mathrm{d}t \tag{0.1}$$

under various assumptions regarding the functions f and β and the interval I. Perhaps the most familiar example of such a limit occurs in the Riemann-Lebesgue lemma which asserts that

$$\lim_{\lambda \to +\infty} \int_I f(t) \sin(\lambda t)\, \mathrm{d}t = 0 \tag{0.2}$$

provided that f is an integrable function over the interval I; and so our results may be viewed as a generalization of that well-known lemma. These results, for I infinite and finite, will be stated and proved in Sections 1 and 2, respectively. We will then apply them in Section 3 to establish a generalization of the Cantor-Lebesgue lemma. In the final section of the paper we will briefly consider the evaluation of (0.1) in the higher dimensional case.

1. Infinite Intervals. We begin with the following

Theorem 1. *Let $\beta \in L_\infty[0, \infty)$, then the necessary and sufficient condition for*

$$\lim_{\lambda \to +\infty} \int_0^\infty f(t)\, \beta(\lambda t)\, \mathrm{d}t \tag{1.1}$$

to exist for every function $f \in L_1[0, \infty)$ is that β have a mean value $M(\beta)$ in the sense that

$$M(\beta) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \beta(t)\, \mathrm{d}t \tag{1.2}$$

exist. This being the case, the limit (1.1) *is then given by the formula*

$$\lim_{\lambda\to+\infty}\int_0^\infty f(t)\,\beta(\lambda t)\,\mathrm{d}t = \left(\int_0^\infty f(t)\,\mathrm{d}t\right) M(\beta)\,. \tag{1.3}$$

Corollary 1. *Let* $\beta \in L_\infty(-\infty, +\infty)$, *then the necessary and sufficient condition for*

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{+\infty} f(t)\,\beta(\lambda t)\,\mathrm{d}t \tag{1.4}$$

to exist for every function f in $L_1(-\infty, +\infty)$ *is that the functions*

$$\beta_+(t) = \beta(t) \quad \text{for} \quad t \geqq 0\,, \quad \beta_-(t) = \beta(-t) \quad \text{for} \quad t \geqq 0\,,$$

which belong to $L_\infty[0, +\infty)$, *have mean values*

$$M(\beta_\pm) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \beta_\pm(t)\,\mathrm{d}t\,. \tag{1.5}$$

When these mean values exist, the limit (1.4) *is given by*

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{+\infty} f(t)\,\beta(\lambda t)\,\mathrm{d}t = \left(\int_0^{+\infty} f(t)\,\mathrm{d}t\right) M(\beta_+) + \left(\int_{-\infty}^0 f(t)\,\mathrm{d}t\right) M(\beta_-)\,. \tag{1.6}$$

Remarks. Clearly Theorem 1 and Corollary 1 contain the usual statement for the evaluation of the limit (0.2) in the Riemann-Lebesgue lemma. We mention some other well-known limits which are obviously subsumed under the Theorem or Corollary

$$\lim_{\lambda\to\infty}\int_a^b f(t)\,|\sin\lambda t|\,\mathrm{d}t = \frac{2}{\pi}\int_a^b f(t)\,\mathrm{d}t$$

due to Fejér [3]; more generally, along the same lines, assuming f and β to be periodic functions of period 2π with f integrable and β bounded

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(t)\,\beta(nt)\,\mathrm{d}t = \left(\frac{1}{2\pi}\int_0^{2\pi} f(t)\,\mathrm{d}t\right)\left(\frac{1}{2\pi}\int_0^{2\pi}\beta(t)\,\mathrm{d}t\right)$$

which appears in Zygmund [7, p. 49]. A further example is

$$\lim_{\lambda\to+\infty}\int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t = 0$$

which is familiar from Laplace transform theory; this generalizes as

$$\lim_{\lambda\to+\infty}\int_0^\infty \alpha(\lambda t)\,f(t)\,\mathrm{d}t = 0$$

provided that $\alpha(t) \to 0$ as $t \to +\infty$. Some additional examples can be found in Polya-Szegö [5].

Proof of Theorem 1. According to the principle of uniform boundedness, for the limit (1.1) to exist for every $f \in L_1[0, \infty)$ it is first of all necessary that $\int_0^\infty f(t) \, . \, \beta(\lambda t) \, dt$, regarded as a collection of linear functionals on $L_1[0, \infty)$, be uniformly bounded for $\lambda > 0$. But this is certainly the case since the estimate

$$\left| \int_0^\infty f(t) \, \beta(\lambda t) \, dt \right| \leqq \|f\|_1 \, \|\beta\|_\infty \quad (\lambda > 0),$$

where $\| \;\|_1$ and $\| \;\|_\infty$ denote the $L_1[0, \infty)$ and $L_\infty[0, \infty)$ norms, respectively, clearly holds. Accordingly, the necessary and sufficient condition for the limit (1.1) to exist for all $f \in L_1[0, \infty)$, now is that it exists on a dense set of functions in $L_1[0, \infty)$. Since the span of the set of characteristic functions $\chi_{[0,b]}$ of intervals $[0, b]$, with arbitrary $b > 0$, is dense in $L_1[0, \infty)$, we need only veiify the existence of the limit (1.1) for $f = \chi_{[0,b]}$, $(b > 0)$:

$$\lim_{\lambda \to +\infty} \int_0^\infty \chi_{[0,b]}(t) \, \beta(\lambda t) \, dt = \lim_{\lambda \to +\infty} \int_0^b \beta(\lambda t) \, dt = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \int_0^{\lambda b} \beta(t) \, dt \, ;$$

and the existence of these limits is tantamount to the existence of the limit (1.2) defining $M(\beta)$. In fact

$$\lim_{\lambda \to +\infty} \int_0^\infty \chi_{[0,b]}(t) \, \beta(\lambda t) \, dt = b \, M(\beta) \, . \tag{1.7}$$

Thus, when the mean value $M(\beta)$ exists, the formula (1.3) for the limit (1.1) holds in the case where $f = \chi_{[0,b]}$. By linearity, it will then hold for f any element in the span of the functions $\chi_{[0,b]}$ $(0 < b)$; and since these are dense in $L_1[0, \infty)$, it follows, by an obvious approximation argument, that (1.3) holds, as well, for f an arbitrary function in $L_1[0, \infty)$.

2. Finite Intervals. We now want to consider the limit problem (0.1) under the assumption that I is a finite interval lying in $[0, \infty)$, i.e. we wish to consider

$$\lim_{\lambda \to +\infty} \int_a^b f(t) \, \beta(\lambda t) \, dt \quad (0 \leqq a < b < \infty) \, . \tag{2.1}$$

By so doing we will be able to deal with β's that are not necessarily bounded. Specifically, we will assume β to be a function on $[0, \infty)$ which is locally in L_q, i.e. whose restrictions to any finite subinterval of $[0, \infty)$ are in L_q. Using the notation $L_q^{\text{loc}}[0, \infty)$ to denote this class of locally q-integrable functions, we will establish the following result.

Theorem 2. *Let $\beta \in L_q^{\text{loc}}[0, \infty)$, $q > 1$. Then, in order that the limit (2.1) exist for every $f \in L_p[a, b]$, with p the Hölder conjugate of q, it is necessary and sufficient that*

(i) *the averages*

$$\frac{1}{T}\int_0^T |\beta(t)|^q \, \mathrm{d}t$$

be bounded as $T \to +\infty$; *and that*
(ii) β *have a mean value*

$$M(\beta) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \beta(t) \, \mathrm{d}t .$$

These conditions (i) *and* (ii) *holding, the limit* (2.1) *is then given by*

$$\text{(2.2)} \qquad \lim_{\lambda\to+\infty} \int_a^b f(t)\,\beta(\lambda t)\,\mathrm{d}t = \left(\int_a^b f(t)\,\mathrm{d}t\right) M(\beta) .$$

Remark. In case β is a periodic function in L_q, the conditions (i) and (ii) are automatically fulfilled.

For the proof of Theorem 2 will need the following.

Lemma 1. *Suppose* $g(t) \in L_1^{\mathrm{loc}}[0, \infty)$ *and that* $\theta \in (0, 1)$. *We consider the averages*

$$A(T) = \frac{1}{T}\int_0^T g(t)\,\mathrm{d}t , \quad A_\theta(T) = \frac{1}{T - \theta T}\int_{\theta T}^T g(t)\,\mathrm{d}t .$$

Then the boundedness of either set of averages as $T \to +\infty$ *implies the same for the other set of averages. Similarly, the convergence of either set of averages as* $T \to +\infty$ *implies the convergence of the other set of averages, and to the same limit*:

$$\lim_{T\to+\infty} A(T) = \lim_{T\to+\infty} A_\theta(T) .$$

Proof of the Lemma. In one direction the lemma is easy to establish. Namely, the identity

$$A_\theta(T) = \frac{1}{1-\theta} A(T) - \frac{\theta}{1-\theta} A(\theta T)$$

allows us to conclude that $A_\theta(T)$ is bounded as $T \to +\infty$, if $A(T)$ is bounded as $T \to +\infty$. Furthermore, the convergence of $A(T)$ as $T \to +\infty$ implies the convergence of $A_\theta(T)$ as $T \to +\infty$ and to the same limit:

$$\lim_{T\to+\infty} A_\theta(T) = \frac{1}{1-\theta} \lim_{T\to+\infty} A(T) - \frac{\theta}{1-\theta} \lim_{T\to+\infty} A(T) = \lim_{T\to+\infty} A(T) .$$

In the other direction our proof will be based on showing that the boundedness of $A_\theta(T)$ from above as $T \to +\infty$ implies the like property for $A(T)$. More precisely, we will show that from

$$A_\theta(T) \leqq L \quad \text{for} \quad T > T_0 , \tag{2.3}$$

it follows that

$$A(T) \leqq L + \frac{1}{T}\left(|L|\, T_0 + \int_0^{T_0} |g(t)|\, \mathrm{d}t\right) \quad \text{for} \quad T > T_0 . \tag{2.4}$$

A similar result can be established with respect to boundedness from below; hence, the boundedness of $A_\theta(T)$ as $T \to +\infty$ will imply the same for $A(T)$.

Next, since (2.3) holds with $L = \overline{\lim}_{T \to +\infty} A_\theta(T) + \varepsilon$, ε an arbitrary positive number, we may apply (2.4) with L equal to this value, to conclude that $\overline{\lim}_{T \to +\infty} A(T) \leqq \overline{\lim}_{T \to +\infty} A_\theta(T) + \varepsilon$; and hence that

$$\overline{\lim_{T \to +\infty}} A(T) \leqq \overline{\lim_{T \to +\infty}} A_\theta(T) .$$

Similarly, we can show that

$$\underline{\lim_{T \to +\infty}} A_\theta(T) \leqq \underline{\lim_{T \to +\infty}} A(T) .$$

Consequently, if $A_\theta(T)$ converges as $T \to +\infty$, so also will $A(T)$ converge as $T \to +\infty$, and to the same limit: $\lim_{T \to +\infty} A(T) = \lim_{T \to +\infty} A_\theta(T)$.

It remains only to show that (2.4) follows from (2.3). For this purpose set

$$I(T) = \int_0^T g(t)\, \mathrm{d}t .$$

Then, since $I(T) - I(\theta T) = (T - \theta T)\, A_\theta(T)$, (2.3) gives

$$I(T) - I(\theta T) \leqq L(T - \theta T) , \quad (T > T_0) .$$

Replacing T by $\theta^k T$ in this inequality we have

$$I(\theta^k T) - I(\theta^{k+1} T) \leqq L(\theta^k T - \theta^{k+1} T) \tag{2.5}$$

provided that $\theta^k T > T_0$. For given $T > T_0$. let n now be the unique integer for which $\theta^n T > T_0 \geqq \theta^{n+1} T$. Adding the inequalities (2.5) for $k = 0, 1, \ldots, n$ we obtain

$$I(T) - I(\theta^{n+1} T) \leqq LT - L\theta^{n+1} T . \tag{2.6}$$

Since $I(\theta^{n+1}T) \leqq \int_0^{\theta^{n+1}T} |g(t)|\, \mathrm{d}t \leqq \int_0^{T_0} |g(t)|\, \mathrm{d}t$, and $-L\theta^{n+1}T \leqq |L|\, \theta^{n+1} T \leqq |L|\, T_0$, (2.6) yields

$$I(T) \leqq LT + |L|\, T_0 + \int_0^{T_0} |g(t)|\, \mathrm{d}t$$

for $T > T_0$. Dividing this through by T we obtain the desired estimate (2.4) for $A(T) = I(T)/T$.

Proof of Theorem 2. The proof proceeds exactly as in the case of Theorem 1. We need, first of all, to assure that the integrals $\int_a^b f(t)\,\beta(\lambda t)\,dt$ regarded as a collection of linear functionals on $L_p[a, b]$ are uniformly bounded as $\lambda \to +\infty$. Since the norms of these functionals is given by

$$\left(\int_a^b |\beta(\lambda t)|^q\,dt\right)^{1/q} = (b-a)^{1/q}\left(\frac{1}{\lambda b - \lambda a}\int_{\lambda a}^{\lambda b} |\beta(t)|^q\,dt\right)^{1/q} =$$

$$= (b-a)^{1/q}\left(\frac{1}{T-\theta T}\int_{\theta T}^{T} |\beta(t)|^q\,dt\right)^{1/q}$$

where $T = \lambda b$ and $\theta = a/b$, the desired boundedness is assured by condition (i) together with Lemma 1.

Next, we need to check that the limit (2.1) exists for a dense set of functions in $L_p[a, b]$; and as this dense set of functions we take the span of the characteristic functions $\chi_{[a,c]}$ of intervals $[a, c]$ with $a < c \leqq b$. It will, therefore, be enough to ascertain the convergence of (2.1) for $f = \chi_{[a,c]}$ $(a < c \leqq b)$. A formal calculation yields:

$$\lim_{\lambda\to+\infty}\int_a^b \chi_{[a,c]}(t)\,\beta(\lambda t)\,dt = \lim_{\lambda\to+\infty}\int_a^c \beta(\lambda t)\,dt = (c-a)\left(\lim_{\lambda\to+\infty}\frac{1}{\lambda c - \lambda a}\int_{a\lambda}^{c\lambda}\beta(t)\,dt\right) =$$

$$= (c-a)\left(\lim_{T\to+\infty}\frac{1}{T-\theta T}\int_{\theta T}^{T}\beta(t)\,dt\right)$$

where $T = c\lambda$ and $\theta = a/c$ here. Thus, by Lemma 1, the existence of the limit (2.1) for $f = \chi_{[a,c]}$ is seen to be equivalent to the condition (ii), the existence of the mean value $M(\beta) = \lim_{T\to+\infty} (1/T)\int_0^T \beta(t)\,dt$. Moreover, when the mean value exists we obtain

$$\lim_{\lambda\to+\infty}\int_a^b \chi_{[a,c]}(t)\,\beta(\lambda t)\,dt = (c-a)\,M(\beta) = \left(\int_a^b \chi_{[a,c]}(t)\,dt\right)M(\beta)\,, \tag{2.7}$$

which is formula (2.2) for $f = \chi_{[a,c]}$. By the same kind of approximation argument mentioned in the proof of Theorem 1, we will then be able to extract formula (2.2) in the general case, for any $f \in L_p[a, b]$, out of the particular case (2.7).

3. An Application. As an application of the preceding material we will establish a generalization of the Cantor-Lebesgue lemma. The classical version of this lemma asserts that if

$$\lim_{n\to\infty}[a_n\cos(nt) + b_n\sin(nt)] = 0$$

at each point t of a set of positive measure, then a_n and $b_n \to 0$ as $n \to \infty$. The generalization we have in mind is the following:

Theorem 3. *Let $\phi_1(t), \ldots, \phi_\mu(t)$ be linearly independent periodic functions of* *period τ in L_r with $r > 1$. Suppose that as n runs through the positive integers*

$$\lim_{n\to\infty} \sum_{j=1}^{\mu} a_n^{(j)} \phi_j(nt) = 0 \tag{3.1}$$

at all points t of a set of positive measure, then

$$\lim_{n\to\infty} a_n^{(j)} = 0 \quad (j = 1, \dots, \mu)\,. \tag{3.2}$$

For the proof we require the following.

Lemma 2. *Given μ linearly independent functions $\phi_1, \dots, \phi_\mu$ in $L_\varrho[a, b]$, $1 \leqq \varrho < \infty$, there exist μ functions $\psi_1, \dots, \psi_\mu$ in $L_\sigma[a, b]$, σ the Hölder conjugate of ϱ, so that*

$$\det\left(\left[\int_a^b \phi_j(t)\,\psi_k(t)\,\mathrm{d}t\right]_{j,k=1}^{\mu}\right) \neq 0\,.$$

Proof. Let $[c_{j,k}]_{j,k=1}^{\mu}$ be any $\mu \times \mu$ matrix with non-zero determinant. Define linear functionals F_k, $k = 1, \dots, \mu$, on $\mathrm{Sp}(\phi_1, \dots, \phi_\mu)$, the span of $\phi_1, \dots, \phi_\mu$, by setting

$$F_k(\phi_j) = c_{j,k} \quad \text{for} \quad j, k = 1, \dots, \mu\,, \tag{3.3}$$

and then using the linearity to define F_k on the rest of the span, i.e. by putting $F_k(\phi) = \sum_{j=1}^{\mu} \alpha_j F_k(\phi_j)$ if $\phi = \sum_{j=1}^{\mu} \alpha_j \phi_j$; in view of the linear independence of the ϕ_j's, F_k is well-defined by this procedure.

Now, since $\mathrm{Sp}(\phi_1, \dots, \phi_\mu)$ can be regarded as a subspace of $L_\varrho[a, b]$, we may apply the Hahn-Banach theorem to extend each F_k as a bounded linear functional to all of $L_\varrho[a, b]$. By the Riesz representation theorem, there then exist uniquely determined functions $\psi_k \in L_\sigma[a, b]$, $k = 1, \dots, \mu$ which generate these functionals F_k according to the formula

$$F_k(\phi) = \int_a^b \phi(t)\,\psi_k(t)\,\mathrm{d}t \quad (k = 1, \dots, \mu)$$

for all $\phi \in L_\varrho[a, b]$. Hence, on account of (3.3),

$$\int_a^b \phi_j(t)\,\psi_k(t)\,\mathrm{d}t = c_{j,k}\,;$$

and this proves the Lemma, since the $c_{j,k}$'s were chosen so that their determinant is non-vanishing.

Proof of Theorem 3. We begin by applying Lemma 2 to the μ linearly independent periodic functions $\phi_1, \dots, \phi_\mu$ in L_r. Regarding their restrictions to $[0, \tau]$ as elements in $L_1[0, \tau]$, the Lemma then assures us of the existence of μ functions

$\psi_1, \ldots, \psi_\mu$ in $L_\infty[0, \tau]$ satisfying the condition

$$\det\left(\left[\int_0^\tau \phi_j(t)\,\psi_k(t)\,\mathrm{d}t\right]_{j,k=1}^{\mu}\right) \neq 0\,; \tag{3.4}$$

and which we then immediately extend periodically to $[0, \infty)$ as functions of period τ.

Next, we may clearly suppose that (3.1) holds at all points of a set E of positive measure lying in $[0, \tau]$. By Egoroff's theorem, we can, therefore, find a subset F of E with positive measure on which $\sum_{j=1}^{\mu} a_n^{(j)}\,\phi_j(nt) \to 0$ uniformly as $n \to \infty$. Setting

$$b_n^{(k)} = \int_F \left[\sum_{j=1}^{\mu} a_n^{(j)}\,\phi_j(nt)\right]\psi_k(nt)\,\mathrm{d}t \quad (k = 1, \ldots, \mu) \tag{3.5}$$

it then follows immediately, bearing in mind the boundedness of ψ_k, that

$$b_n^{(k)} \to 0 \quad \text{as} \quad n \to \infty\,. \tag{3.6}$$

By introducing the matrix $\Gamma_n = [\gamma_n^{(j,k)}]_{j,k=1}^{\mu}$ whose elements are

$$\gamma_n^{(j,k)} = \int_F \phi_j(nt)\,\psi_k(nt)\,\mathrm{d}t \quad (j, k = 1, \ldots, \mu) \tag{3.7}$$

together with the column vectors $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$ whose i^{th} components are $a_n^{(i)}$ and $b_n^{(i)}$, respectively, we may re-write (3.5) in vector-matrix notation as

$$\boldsymbol{b}_n = \Gamma_n \boldsymbol{a}_n\,. \tag{3.8}$$

We are now going to show, by means of Theorem 2, that the matrix Γ_n just introduced converges. This is accomplished by recognizing the integrals in (3.7) defining $\gamma_n^{(j,k)}$ to be of the form $\int_0^\tau f(t)\,\beta(nt)\,\mathrm{d}t$ considered in Theorem 2, provided that we take $f(t) = \chi_F(t)$ and $\beta(t) = \phi_j(t)\,\psi_k(t)$. We now note that as ϕ_j and ψ_k are periodic functions in L_r and L_∞, respectively, their product $\phi_j \,.\, \psi_k = \beta$ is a periodic function in L_r. Thus $\beta \in L_q^{\text{loc}}[0, \infty)$ with $q = r > 1$, while $f(t) = \chi_F(t) \in L_\infty[0, \tau] \subset L_p[0, \tau]$ with $p = q'$, the Hölder conjugate of q, as required by the hypotheses of Theorem 2. Finally, the periodicity of β assures us that it satisfies conditions (i) and (ii) of Theorem 2, with the mean value of β being given by $M(\beta) = (1/\tau)\int_0^\tau \phi_j(t)\,\psi_k(t)\,\mathrm{d}t$. Applying the conclusion (2.2) of the Theorem, we, therefore find that

$$\lim_{n\to\infty} \gamma_n^{(j,k)} = m(F)\left(\frac{1}{\tau}\int_0^\tau \phi_j(t)\,\psi_k(t)\,\mathrm{d}t\right) = \gamma^{(j,k)}$$

where $m(F)$ denotes the Lebesgue measure of F. This proves that the matrix Γ_n converges to the matrix $\Gamma = [\gamma^{(j,k)}]_{j\,k=1}^{\mu}$.

Next, it is clear from condition (3.4), that the determinant of the limiting matrix Γ is non-zero. Hence Γ is invertible; and since $\Gamma_n \to \Gamma$ as $n \to \infty$, so also is Γ_n invertible for n sufficiently large; moreover

$$\lim_{n\to\infty} \Gamma_n^{-1} = \Gamma^{-1}\,. \tag{3.9}$$

Inverting (3.8) we find that

$$\boldsymbol{a}_n = \Gamma_n^{-1}\boldsymbol{b}_n,$$

for n sufficiently large; from which it follows, by passing to the limit as $n \to \infty$, taking (3.6) and (3.9) into account, that

$$\lim_{n\to\infty} \boldsymbol{a}_n = \left(\lim_{n\to\infty} \Gamma_n^{-1}\right) \lim_{n\to\infty} \boldsymbol{b}_n = \Gamma^{-1}\boldsymbol{0} = \boldsymbol{0},$$

the desired conclusion.

4. Higher Dimensions. As far as generalizations to higher dimensional situations are concerned, there do not appear to be simple conditions which are both necessary and sufficient for the existence of

$$\lim_{\lambda\to+\infty} \int_{E^n} f(t)\,\beta(\lambda t)\,\mathrm{d}t \tag{4.1}$$

for every $f \in L_1(E^n)$ assuming $\beta \in L_\infty(E^n)$. We can, however, give a sufficient condition on β which will assure the existence of the limits (4.1).

Theorem 4. *Suppose that $\beta \in L_\infty(E^n)$ has radial mean values in almost every direction, i.e. for almost all ξ with $|\xi| = 1$ assume that the limit*

$$M(\beta)(\xi) = \lim_{R\to+\infty} \frac{1}{R}\int_0^R \beta(r\xi)\,\mathrm{d}r \tag{4.2}$$

exists. Then the limit (4.1) *exist for all $f \in L_1(E^n)$ and is given by*

$$\lim_{\lambda\to+\infty} \int_{E^n} f(t)\,\beta(\lambda t)\,\mathrm{d}t = \int_{E^n} f(t)\,M(\beta)\left(\frac{t}{|t|}\right)\mathrm{d}t. \tag{4.3}$$

Proof. We will avail ourselves of the change to "polar coordinates" formula for the evaluation of integrals over E^n (cf. [2, p. 1049]):

$$\int_{E^n} F(t)\,\mathrm{d}t = \int_{|\xi|=1} \int_0^\infty F(r\xi)\,r^{n-1}\,\mathrm{d}r\,\mathrm{d}\sigma(\xi), \tag{4.4}$$

here $t = r\xi$ with $r = |t|$ and $\xi = t/|t|$, while $\mathrm{d}\sigma(\xi)$ denotes the element of area on the unit sphere $|\xi| = 1$.

Once again, it suffices to prove (4.1) for a dense set of functions $f(t) \in L_1(E^n)$. For this dense set we take the span of the set of functions of the form

$$f(t) = g(r)\,\phi(\xi) \quad \text{with} \tag{4.5}$$

$$\int_0^\infty |g(r)|\,r^{n-1}\,\mathrm{d}r < \infty \quad \text{and} \quad \int_{|\xi|=1} |\phi(\xi)|\,\mathrm{d}\sigma(\xi) < \infty.$$

Making use of (4.4) for such functions we have

$$\int_{E^n} f(t)\,\beta(\lambda t)\,\mathrm{d}t = \int_{|\xi|=1} \left(\int_0^\infty g(r)\, r^{n-1}\, \beta(\lambda r\xi)\,\mathrm{d}r\right) \phi(\xi)\,\mathrm{d}\sigma(\xi)\,.$$

Applying Theorem 1 to the one-dimensional inner integral on the right, we find, in view of our hypothesis (4.2), that

$$\lim_{\lambda\to+\infty} \int_0^\infty g(r)\, r^{n-1}\, \beta(\lambda r\xi)\,\mathrm{d}r = \left(\int_0^\infty g(r)\, r^{n-1}\,\mathrm{d}r\right) M(\beta)\,(\xi)$$

holds for almost all ξ with $|\xi| = 1$. Hence, taking account of the estimate

$$\left|\left(\int_0^\infty g(r)\, r^{n-1}\, \beta(\lambda r\xi)\,\mathrm{d}r\right) \phi(\xi)\right| \leqq \left(\int_0^\infty |g(r)|\, r^{n-1}\,\mathrm{d}r\right) \|\beta\|_\infty\, |\phi(\xi)|\,,$$

the Lebesgue dominated convergence theorem allows us to conclude that

$$\lim_{\lambda\to+\infty} \int_{|\xi|=1} \left(\int_0^\infty g(r)\, r^{n-1}\, \beta(\lambda r\xi)\,\mathrm{d}r\right) \phi(\xi)\,\mathrm{d}\sigma(\xi) =$$

$$= \int_{|\xi|=1} \left(\int_0^\infty g(r)\, r^{n-1}\,\mathrm{d}r\right) M(\beta)\,(\xi)\,\phi(\xi)\,\mathrm{d}\sigma(\xi) = \int_{E^n} f(t)\, M(\beta)\left(\frac{t}{|t|}\right)\mathrm{d}t$$

(using (4.4) once more). This establishes (4.3) for functions of the form (4.5), and thereby completes the proof of the Theorem.

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