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A HILLE-WINTNER COMPARISON THEOREM FOR SECOND ORDER DIFFERENTIAL SYSTEMS

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1. Let $\mathscr{H}$ be a Hilbert space, let $\mathscr{B} = \mathscr{B}(\mathscr{H}, \mathscr{H})$ be the B^*-algebra of bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$ with the uniform operator topology, and let $\mathscr{S}$ be the subset of $\mathscr{B}$ consisting of the self-adjoint operators. This paper is concerned with the second order, selfadjoint differential equation

$$[P(x)\,Y']' + Q(x)\,Y = 0 \tag{1}$$

on $\mathsf{R}^+ = [0, \infty)$, where $P, Q : \mathsf{R}^+ \to \mathscr{S}$ are continuous and $P(x)$ is positive definite for all $x \in \mathsf{R}^+$. Appropriate discussions of the concepts of integration and differentiation of $\mathscr{B}$-valued functions, as well as the existence and uniqueness of solutions $Y : R^+ \to \mathscr{B}$ of (1), can be found in a variety of texts. See, for example E. HILLE [7, Chapters 6 and 9]. In particular, it is well known that when suitable initial conditions are specified for (1), the resulting initial value problem has a unique solution which exists on R^+.

We shall assume throughout this paper that $\mathscr{H}$ is a Hilbert space over the reals R, with the inner product on $\mathscr{H}$ denoted by $\langle\,,\rangle$ and norm $\|\ \| = \langle\,,\rangle^{1/2}$. It will be apparent, however, that the methods and results apply equally as well when $\mathscr{H}$ is a complex Hilbert space. We assume, also, that the B^*-algebra $\mathscr{B}$ is topologized by the operator norm

$$\|A\| = \sup_{\|\alpha\|=1} \|A\alpha\| \,.$$

The symbol I used for the identity element of $\mathscr{B}$. The symbol 0 is used indiscriminately for the zero element, with the proper interpretation of 0 being clear from the context. If $A \in \mathscr{S}$, the selfadjoint elements of $\mathscr{B}$, then the notation $A > 0$ $(A \geqq 0)$ is used to signify that A is positive (nonnegative) definite. If $A > 0$, then $A^{-1} \in \mathscr{S}$ and $A^{-1} > 0$.

Let $Y = Y(x)$ be a solution of equation (1). Then it is easy to verify by differentiation that

$$Y^*[PY'] - [PY']^*\,Y \equiv C \text{ (constant)}$$

on R^+. The solution Y is *conjoined* (*prepared*) if $C = 0$. The solution Y is *nonsingular at a*, $a \in \mathsf{R}^+$, if

(i) $Y(a) : \mathscr{H} \to \mathscr{H}$ is onto, and

(ii) $Y(a)$ has a bounded inverse.

If either of these conditions fails to hold, then Y is *singular at a*. In particular, Y is said to have an *algebraic singularity at a* if $Y(a)$ is not one-to-one. In the finite dimensional case, i.e., $\mathscr{H} = R_n$, Euclidean n-space, and $\mathscr{B} = \{A \mid A \text{ is an } n \times n \text{ matrix}\}$, it is clear that the solution Y can have only algebraic singularities, and, moreover, Y is singular at a if and only if $\det[Y(a)] = 0$. In the general B^*-algebra case conditions (i) and (ii) are equivalent to having $Y^{-1}(a) \in \mathscr{B}$. The solution Y is *nontrivial* if there is at least one point $a \in \mathsf{R}^+$ such that $Y(a)$ is nonsingular. In the finite dimensional case it is well known that a nontrivial conjoined solution of (1) can have at most a finite number of singular points on any compact subset of R^+. This property does not hold in the general B^*-algebra case. T. L. HAYDEN and H. C. HOWARD [5] have shown that while the set of singularities of a nontrivial solution Y of (1) is a closed set, it is possible for the set of singularities to have a finite limit point.

Our primary interest in this paper is in the oscillation of nontrivial conjoined solutions of equation (1) and, hereafter, the term "solution" shall be interpreted to mean "nontrivial conjoined solution". The solution $Y = Y(x)$ of (1) is *oscillatory* if for each $a \in \mathsf{R}^+$ there is a point b, $b \geqq a$, such that $Y(b)$ is singular. The solution Y is *nonoscillatory* if it is not oscillatory. In the finite dimensional case, a solution Y is oscillatory if and only if $\det[Y]$ has an infinite number of zeros on R^+. Also, it is a consequence of Morse's generalization of the Sturm separation theorem that if (1) has an oscillatory solution, then all solutions are oscillatory. The following simple example shows that this property does not carry over to the general B^*-algebra case

Example. Let $\mathscr{H} = l_2$, and let $P(x) \equiv I$, $Q(x) \equiv 0$ in (1), i.e., consider the equation $Y'' = 0$ on R^+. Every solution Y of the equation has the form $Y(x) = Ax + B$, $A, B \in \mathscr{B}$, and Y is conjoined if $A^*B = B^*A$. The solution Y satisfying $Y(0) = I$, $Y'(0) = 0$ is $Y \equiv I$. Clearly Y is conjoined and nonsingular on R^+. On the other hand, the solution Z given by $Z(x) = Ax + I$, where $A = \operatorname{diag}[-1, -1/2, -1/3, \ldots]$, is conjoined and has an algebraic singularity at each positive integer n.

To maintain the relationship with the finite dimensional case, we say that equation (1) is *oscillatory* if and only if every solution is oscillatory.

Studies of the oscillatory behavior of solutions of second order equations of the form (1) have been made by several authors, including Hille [7, Chapter 9], Hayden and Howard [5], ETGEN and PAWLOWSKI [2, 3], Etgen and LEWIS [1], E. S. NOUSSAIR [9], and C. M. WILLIAMS [15]. Equation (1) in the finite dimensional case is the familiar self-adjoint matrix differential equation which has been investigated in great detail by a large number of authors. In this regard, we refer to the texts by P. HARTMAN [4], Hille [7], W. T. REID [10, 11] and C. A. SWANSON [13]. Each of these texts provides comprehensive bibliographies and references to the research literature.

An examination of the literature concerning sufficient conditions for equation (1) to be oscillatory (in both the finite and the infinite dimensional cases) reveals that most oscillation criteria involve assumptions which are generalizations of the Leighton-Winter oscillation criterion [8, 16] for the scalar version of (1). That is, most oscillation criteria for (1) involve assumptions of the form "$\int_0^\infty P^{-1}(x)\,dx = \int_0^\infty Q(x)\,dx = \infty$". The purpose of this paper is to present oscillation criteria for (1) in the case where $\int_0^\infty Q(x)\,dx$ "converges". These rough statements will be made more precise in Section 3.

2. The comparison theorem. The Hille-Wintner comparison theorem relates the oscillatory behavior of the solutions of the two scalar equations

$$y'' + f(x)\,y = 0\,, \tag{2}$$

$$y'' + g(x)\,y = 0\,. \tag{3}$$

It states:

Theorem 1. *Let f and g be continuous functions on R^+ such that the integrals $\int_x^\infty f(t)\,dt$, $\int_x^\infty g(t)\,dt$ converge (possibly just conditionally), and*

$$0 \leqq \int_x^\infty f(t)\,dt \leqq \int_x^\infty g(t)\,dt$$

on $[a, \infty)$ for some $a \in R^+$. If (2) is oscillatory, then (3) is oscillatory. Equivalently, if (3) is disconjugate on $[b, \infty)$ for some $b \geqq a$, then (2) is disconjugate on $[b, \infty)$.

Hille's version of this theorem [6, p. 245] contained the additional hypotheses $f(x) \geqq 0$, $g(x) \geqq 0$ on R^+. The theorem quoted above removes the nonnegativity conditions on the coefficients, and was published by Wintner [17] in 1957. Wintner, however, was apparently unaware of the following comparison theorem established by C. T. Taam [14] in 1952. Taam's result compares the two equations

$$[p(x)\,y']' + f(x)\,y = 0\,, \tag{4}$$

$$[r(x)\,y']' + g(x)\,y = 0\,, \tag{5}$$

and states (in a slightly modified form):

Theorem 2. *Let f, g be continuous functions on R^+ such that the integrals $\int_x^\infty f(t)\,dt$, $\int_x^\infty g(t)\,dt$ converge (possibly just conditionally), and*

$$\int_x^\infty g(t)\,dt \geqq \left|\int_x^\infty f(t)\,dt\right|$$

on $[a, \infty)$ *for some* $a \in \mathrm{R}^+$. *Let* p, r *be positive continuous functions on* R^+ *such that* $r(x) \leqq p(x)$, *and* $r(x) \leqq k$ *(constant) on* $[b, \infty)$ *for some* $b \in \mathrm{R}^+$. *If* (4) *is oscillatory, then* (5) *is oscillatory. Equivalently, if* (5) *is disconjugate on* $[c, \infty)$ *for some* $c \in \mathrm{R}^+$, *then* (4) *is disconjugate on* $[c, \infty)$.

Note that if $p \equiv r \equiv 1$ on R^+, then the hypotheses involving p and r in Taam's theorem are satisfied, and (4), (5) reduce to (2), (3), respectively. Note also that Taam's theorem does not require $\int_x^\infty f(t)\,\mathrm{d}t \geqq 0$ on $[a, \infty)$ so that even in the case $p \equiv r \equiv 1$, Taam's theorem is stronger than Wintner's.

Our comparison theorem is a generalization of Taam's result. The theorem involves the set of positive functionals on the B^*-algebra $\mathscr{B}$. A linear functional g on $\mathscr{B}$ is *positive* if $g(A^*A) \geqq 0$ for all $A \in \mathscr{B}$. Equivalently, g is positive if $g(B) \geqq 0$ for all $B \in \mathscr{B}$ such that $B \geqq 0$. C. E. Rickart [12] has shown that each positive functional g on $\mathscr{B}$ is bounded (i.e., continuous), with $\|g\| = g(I)$. Also each positive functional g satisfies a generalized Cauchy - Schwartz inequality

$$(6) \qquad [g(A^*B)]^2 \leqq g(A^*A)\, g(B^*B)$$

for all $A, B \in \mathscr{B}$. It follows from (6) that g is the zero functional if and only if $g(I) = 0$. If $g \neq 0$, then $g(I) > 0$ and, in general, $g(A) > 0$ whenever $A \in \mathscr{S}$, $A > 0$. It also follows from (6) that if $g \neq 0$, then

$$(7) \qquad g(B^*B) \geqq \frac{1}{g(I)}\,[g(B)]^2$$

for all $B \in \mathscr{B}$. Finally, since a positive functional g is continuous,

$$g\left[\int_a^x A(t)\,\mathrm{d}t\right] = \int_a^x g[A(t)]\,\mathrm{d}t$$

whenever $A : \mathrm{R}^+ \to \mathscr{B}$ is integrable, and

$$g[B'(x)] = \{g[B(x)]\}'$$

whenever $B : \mathrm{R}^+ \to \mathscr{B}$ is differentiable.

Let $\mathscr{G}$ be the set of positive functionals on $\mathscr{B}$. The fact that $\mathscr{G}$ does contain elements in addition to the zero functional can be verified by associating with each nonzero vector $\alpha \in \mathscr{H}$ the functional g_α on $\mathscr{B}$ defined by

$$(8) \qquad g_\alpha(A) = \langle A\alpha, \alpha\rangle\,, \quad A \in \mathscr{B}\,.$$

It is easy to show that g_α is a positive functional with $\|g_\alpha\| = g_\alpha(I) = \|\alpha\|^2$. There are also positive functionals which are not simply the "associates" of vectors in $\mathscr{H}$. For example, in the finite dimensional case $\mathscr{H} = R_n$, the linear functional "trace" is a positive functional. It can be verified, in general, that $\mathscr{G}$ is a positive cone in the space of continuous linear functionals on $\mathscr{B}$.

Our comparison theorem will compare equation (1) with the scalar equation

$$(9) \qquad (r(x)\, y')' + h(x)\, y = 0\,,$$

where $r, h : \mathrm{R}^+ \to \mathrm{R}$ are continuous and $r(x) > 0$ for all $x \in \mathrm{R}^+$.

Theorem 3. *Assume that there exists a number $a \in \mathrm{R}^+$ such that on $[a, \infty)$ the following conditions are satisfied:*

(i) $r(x)\, I - P(x) \geqq 0$,

(ii) *there exists a positive constant k such that $kI - P(x) \geqq 0$,*

(iii) *there exists a $g \in \mathscr{G}$, $g \neq 0$, such that the integral $\int_x^\infty g[Q(t)]\, \mathrm{d}t$ converges, possibly just conditionally,*

(iv) *the integral $\int_x^\infty h(t)\, \mathrm{d}t$ converges, possibly just conditionally, and*

$$(\mathrm{v}) \qquad \int_x^\infty g[Q(t)]\, \mathrm{d}t \geqq g(I) \left| \int_x^\infty h(t)\, \mathrm{d}t \right| .$$

If equation (1) is nonoscillatory, then equation (9) is nonoscillatory, in fact disconjugate on some interval $[c, \infty)$, $c \geqq a$. Equivalently, if equation (9) is oscillatory, than equation (1) is oscillatory.

Proof. Suppose (1) is nonoscillatory. Then there exists a solution Y which is nonsingular on the interval $[b, \infty)$, for some $b \in \mathrm{R}^+$. Let $c = \max\{a, b\}$, and let $S(x) = P(x)\, Y'(x)\, Y^{-1}(x)$ on $[c, \infty)$. The conjoined property of Y implies that for each $x \in [c, \infty)$, $S(x)$ is selfadjoint. An easy computation shows that

$$(10) \qquad S'(x) = -S(x)\, P^{-1}(x)\, S(x) - Q(x)\,,$$

on $[c, \infty)$. Fix any $x \in [c, \infty)$, and integrate S' from x to t to obtain the equation

$$S(t) + \int_x^t S(u)\, P^{-1}(u)\, S(u)\, \mathrm{d}u = S(x) - \int_x^t Q(u)\, \mathrm{d}u\,.$$

Let g be the positive functional specified in hypothesis (iii), and "apply" g to this equation. From the linearity and continuity of g, we get

$$(11) \qquad g[S(t)] + \int_x^t g[SP^{-1}S]\, \mathrm{d}u = g[S(x)] - \int_x^t g[Q]\, \mathrm{d}u\,.$$

It follows from hypothesis (iii) that the right-hand side of (11) has a finite limit L as $t \to \infty$. Thus

$$\lim_{t \to \infty} \left\{ g[S(t)] + \int_x^t g[SP^{-1}S]\, \mathrm{d}u \right\} = L.$$

Since $SP^{-1}S \geqq 0$ on $[c, \infty)$, we have $g[SP^{-1}S] \geqq 0$ on $[c, \infty)$, and so the function $\int_x^t g[SP^{-1}S]\, \mathrm{d}u$ is increasing on $[x, \infty)$. Suppose $\int_x^\infty g[SP^{-1}S]\, \mathrm{d}u = \infty$. Then

$g[S(t)] \to -\infty$ as $t \to \infty$. However, from hypothesis (ii), we have that $P^{-1}(x) - (1/k)I \geqq 0$ on $[c, \infty)$, and therefore

$$g[S(t)] + \int_x^t g[SP^{-1}S]\,du \geqq g[S(t)] + \frac{1}{k}\int_x^t g[S^2]\,du\,.$$

Also, from (7), $g[S^2] \geqq (1/g(I))\,(g[S])^2$, and so it follows that

$$g[S(t)] + \int_x^t g[SP^{-1}S]\,du \geqq g[S)t)] + \frac{1}{kg(I)}\int_x^t (g[S])^2\,du\,. \tag{12}$$

Now, since $g[S(t)] \to -\infty$ as $t \to \infty$, we have

$$\frac{1}{kg(I)}\int_x^t (g[S])^2\,du \to \infty \text{ as } t \to \infty\,,$$

and a straight forward argument shows that

$$\limsup_{t\to\infty} g[S(t)] + \frac{1}{kg(I)}\int_x^t (g[S])^2\,du = \infty\,.$$

Thus, from inequality (12), $g[S(t)] + \int_x^t g(SP^{-1}S)\,du$ is not bounded above on $[x, \infty)$, contradicting the fact that this expression has the finite limit L. (Note that

$$g[S(t)] + \frac{1}{kg(I)}\int_x^t (g[S])^2\,du$$

might not have limit ∞ as $t \to \infty$ as suggested in Taam's proof. See also Wintner's argument [17, p. 258.) Hence the integral $\int_x^\infty g(SP^{-1}S)\,du$ converges and $\lim_{t\to\infty} g[S(t)]$ exists. Now, by using hypothesis (ii), we have

$$\int_x^t g[S^2]\,du = \int_x^t g[SP^{-1}PS]\,du \leqq k\int_x^t g[SP^{-1}S]\,du \leqq k\int_x^\infty g[SP^{-1}S]\,du < \infty$$

for all $t \in [x, \infty)$. We can conclude, therefore, that $\lim_{t\to\infty} g[S(t)] = 0$, and from (11) we have

$$g[S(x)] = \int_x^\infty g[SP^{-1}S]\,du + \int_x^\infty g[Q]\,du \tag{13}$$

for all $x \in [c, \infty)$.

Define the function m on $[c, \infty)$ by

$$m(x) = g[S(x)] - \int_x^\infty \{g[Q] - g(I)\,h\}\,du$$

Equation (12) and hypothesis (v) imply that $g[S(x)] > 0$ and $|m(x)| \leqq g[S(x)]$ on $[c, \infty)$. Now, by differentiating m, and using the linearity and continuity of g, we get

$$m'(x) = -g[S(x) P^{-1}(x) S(x)] - g(I) h(x)$$

or

$$m'(x) + g[S(x) P^{-1}(x) S(x)] = -g(I) h(x) .$$

From hypothesis (i),

$$P^{-1}(x) - \frac{1}{r(x)} I \geqq 0 ,$$

and so we have

$$m'(x) + \frac{1}{r(x)} g[S^2(x)] \leqq -g(I) h(x) .$$

Finally, since

$$g[S^2(x)] \geqq \frac{1}{g(I)} (g[S(x)])^2 ,$$

and since $(g[S(x)])^2 \geqq m^2(x)$, we obtain the inequality

$$m'(x) + \frac{1}{g(I) r(x)} m^2(x) \leqq -g(I) h(x)$$

which, by Taam's result [14, Theorem 1] (also see Wintner [16]), implies that the second order equation

$$[g(I) r(x) y']' + g(I) h(x) y = 0 ,$$

is disconjugate on $[c, \infty)$. Obviously this equation is equivalent to (9), and so the theorem is established.

In equation (1) let $P(x) \equiv I$ on R^+, and in equation (9) let $r(x) \equiv 1$ on R^+. Then equations (1) and (9) become

$$Y'' + Q(x) Y = 0 , \tag{14}$$

$$y'' + h(x) y = 0 , \tag{15}$$

respectively. It is easy to see that with $P \equiv I$ and $r \equiv 1$, hypotheses (i) and (ii) of Theorem 3 are satisfied. Thus the following generalization of the Hille-Wintner comparison theorem, Theorem 1, is an immediate consequence of Theorem 3.

Theorem 4. *Assume that there exists a number $a \in \mathsf{R}^+$ such that on $[a, \infty)$ the following conditions are satisfied:*

(i) *there exists a* $g \in \mathscr{G}$, $g \neq 0$, *such that the integral* $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ *converges, possibly just conditionally,*

(ii) *the integral* $\int_x^\infty h(t)\,\mathrm{d}t$ *converges, possibly just conditionally, and*

(iii) $$\int_x^\infty g[Q(t)]\,\mathrm{d}t \geqq g(I)\left|\int_x^\infty h(t)\,\mathrm{d}t\right|.$$

If equation (14) *is nonoscillatory, then equation* (15) *is nonoscillatory, in fact disconjugate on* $[c, \infty)$ *for some* $c \geqq a$. *Equivalently, if equation* (15) *is oscillatory, then equation* (14) *is oscillatory.*

3. Hille-Wintner type oscillation criteria. As remarked in the introductory section most oscillation criteria for equation (1) involve assumptions of the form "$\int_0^\infty P^{-1}(x)\,\mathrm{d}x = \int_0^\infty Q(x)\,\mathrm{d}x = \infty$". To be more specific Etgen and Pawlowski [2,3] used the set of positive functionals on $\mathscr{B}$ to obtain oscillation criteria for (1), and showed that most of the known criteria in both the finite and infinite dimensional cases could be obtained by making suitable choices of positive functionals. For example, in [3] it is shown that if there exists a $g \in \mathscr{G}$ such that

$$\int_0^\infty \frac{\mathrm{d}x}{g[P(x)]} = \int_0^\infty g[Q(x)]\,\mathrm{d}x = \infty$$

then equation (1) is oscillatory. This is a generalization of the Leighton-Wintner oscillation criteria, and it is demonstrated that this result includes the oscillation criteria established by such authors as Allegretto and Erbe, Hayden and Howard, Kreith, and Noussair and Swanson. For specific references see the papers cited above.

We now illustrate how the comparison theorems of the previous section can be used to obtain oscillation and nonoscillation criteria of the Hille-Wintner type. In particular, we shall use Theorem 4 to develop sufficient conditions for the oscillation of equation (14) and necessary conditions for the nonoscillation of (14).

Theorem 5. *If there exists a* $g \in \mathscr{G}$, $g \neq 0$, *such that* $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ *converges, possibly just conditionally, and if the scalar equation*

(16) $$y'' + \frac{1}{g(I)}\,g[Q(x)]\,y = 0$$

is oscillatory, then equation (14) *is oscillatory.*

Proof. It is easy to see that if

$$h(x) = \frac{1}{g(I)}\,g[Q(x)],$$

then the hypotheses of Theorem 4 are satisfied. Thus the oscillation of (16) implies the oscillation of (14).

Some specific oscillation criteria for (14) can now be obtained by using specific oscillation criteria for (16) together with specific choices of the positive functional g. For example, according to Wintner [17], if $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ converges, possibly just conditionally, and if

$$\frac{1}{g(I)}\int_x^\infty g[Q(t)]\,\mathrm{d}t > \gamma/x$$

for some $\gamma > \frac{1}{4}$, then (16) is oscillatory. Let $\mathscr{H} = R_n$, and let $S_{k,n}$ denote the collection of strictly increasing sequences of k integers chosen from the set $\{1, 2, \ldots, n\}$. For any $n \times n$ matrix A, and any $\sigma(k) = \{i_1, i_2, \ldots, i_k\} \in S_{k,n}$, let $\sum_\sigma A$ denote the sum of the entries of the $k \times k$ submatrix of A obtained by deleting all rows and columns of A except for the $i_1, i_2, \ldots, i_k$ rows and columns. If there exists $\sigma(k) \in S_{k,n}$ such that $\int_x^\infty [\sum_\sigma Q(t)]\,\mathrm{d}t$ converges, possibly just conditionally, and if

$$\frac{1}{k}\int_x^\infty [\sum_\sigma Q(t)]\,\mathrm{d}t > \gamma/x\,, \quad \gamma > \tfrac{1}{4}\,,$$

then equation (14) is oscillatory. In the special case $k = 1$ this criterion becomes: equation (14) is oscillatory if $\int_x^\infty q_{ii}(t)\,\mathrm{d}t$ converges, possibly just conditionally, and

$$\int_x^\infty q_{ii}(t) > \gamma/x\,,$$

for some diagonal element q_{ii} of Q and some $\gamma > \frac{1}{4}$. These criteria are obtained from Theorem 5 by noting that if $\sigma(k) \in S_{k,n}$, then

$$\sum_\sigma A = \langle A\alpha, \alpha\rangle = g_\alpha(A)\,, \quad (\text{see (8)})\,,$$

where α is the vector in R_n having ones in the $i_1, i_2, \ldots, i_k$ positions and zeros elsewhere. In the same manner additional oscillation criteria can be obtained for equation (14) simply by combining oscillation criteria for equation (16) with specific positive functionals.

Our final theorem gives a necessary condition for the nonoscillation of equation (14). Like Theorem 5, it is an immediate consequence of Theorem 4.

Theorem 6. *If equation* (14) *is nonoscillatory, and if* $g \in \mathscr{G}$, $g \neq 0$, *has the property* $\int_x^\infty g[Q(t)]\,\mathrm{d}t$ *converges, possibly just conditionally, then the scalar equation* (16) *is nonoscillatory.*

We conclude this paper with a simple application of Theorem 6. Let $\mathscr{H} = R_n$, and suppose that equation (14) is nonoscillatory. Then for each diagonal element q_{ii} of Q such that $\int_x^\infty q_{ii}(t)\,dt$ converges, possibly just conditionally, the scalar equation

$$y'' + q_{ii}(x)\,y = 0$$

is nonoscillatory.

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