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ON A CLASS OF FOURTH ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS

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I. Introduction. We are here concerned with a class of fourth order nonlinear differential equations of the form

$$\text{(E)} \qquad x^{(4)} + f(t)\, g(x, x'') = 0$$

where the functions f and g are subject to the following conditions:

(H) 1. f is positive and continuous in $[0, \infty)$;
2. $g(\lambda u, \lambda v) = \lambda g(u, v)$ for every λ, u, v;
3. $\operatorname{sgn} g(u, v) = \operatorname{sgn} u$; and
4. $g(u, v)$ is continuous for every u, v and satisfies conditions such that solutions of (E) are uniquely determined by initial conditions.

A solution of (E) is said *to be continuable* if it exists on $[0, \infty)$, and the term "solution" for the remainder of this work will always mean a continuable solution. With this understanding, (H) will be satisfied if, for example, one requires that g is Lipschitzian in any compact subset of the u, v-plane. In particular, $g(u, v)$ may be chosen as u or $u^3/(u^2 + v^2)$ or $uv^2/(u^2 + v^2)$, etc.

Equation (E) is viewed as "half-linear" since it has many properties in common with the linear equation

$$\text{(L)} \qquad x^{(4)} + f(t)\, x = 0\,.$$

In particular, we can easily verify by direct substitution that if $x(t)$ is a solution of (E), so is any of its constant multiples. Furthermore, we see that the term $f(t)\, g(x, x'')$ has the same sign of $f(t)\, x(t)$ because of the sign conditions (H-1) and (H-3). There are other properties of (E) which are similar to those of (E) but we shall defer them to later sections. For the time being, we point out that both (L) and second order analogue of (E)

$$\text{(E')} \qquad x'' + f(t)\, g(x, x') = 0$$

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have been studied quite extensively. For related studies which provide background material and motivation for the present paper, we refer the readers to the works of ŠVEC [11], WHYBURN [12], SANSONE [10], LEIGHTON and NEHARI [8], KEENER [5], KREITH [7], BIHARI [1, 2], GUSTAFSON [3] and KARSATOS [4].

Our study will be accomplished with the aid of a pair of equations of the second order. This pair is introduced in the next section where some basic results can also be found. Section 3 is devoted to the nonoscillatory solutions of (E) while section 4 is devoted to oscillatory solutions. The last section provides additional properties of the oscillatory solutions of (E).

2. Preliminary Considerations. It is helpful to view equation (E) as a pair of second order equations

$$x'' = y\,, \quad y'' = -f(t)\,g(x, y)\,. \tag{S}$$

Clearly, (E) is equivalent to (S) in the sense that $x(t)$ is a solution of (E) if and only if $\{x(t), y(t)\} \equiv \{x(t), x''(t)\}$ is a solution of (S). The system (S) is introduced because of its evident geometrical significance. Let t be regarded as a parameter; then a solution $\{x(t), y(t)\}$ of the system (S) describes a curve in the x, y-plane. This curve is called *the integral curve* of (S) and its shape is evidently dependent upon the initial values as well as the functions f and g. We propose to study this curve. To facilitate discussion, we introduce the usual polar coordinates (R, θ) of a point on this curve.

Let $z(t) = \{x(t), y(t)\}$ be a nontrivial solution of (S) and let

$$V(t) = V[x(t), y(t)] = x'(t)\,y(t) - x(t)\,y'(t)\,. \tag{2.1}$$

Then

$$V' = x''y - xy'' = y^2 + xf\,g(x, y) \tag{2.2}$$

where the last equality is obtained by substituting $x''(t)$ and $y''(t)$ from (S). Upon integrating (2.2) from a to t, we obtain

$$V(t) = V(a) + \int_a^t [y^2(s) + x(s)f(s)\,g(x(s), y(s))]\,\mathrm{d}s\,. \tag{2.3}$$

Since $xf\,g(x, y)$ has the same sign of fx^2, we see that $V(t)$ is an increasing function of t. Hence, $V(t)$ can vanish at most once. Note furthermore that the vectors $\{x(t), y(t)\}$, $\{x(t), x'(t)\}$ and $\{y(t), y'(t)\}$ cannot be zero whenever $V(t) \neq 0$; thus these vectors also can vanish at most once.

We now recall from elementary calculus that

$$\theta'(t) = [y'(t)\,x(t) - x'(t)\,y(t)]/R^2(t) = -V(t)/R^2(t) \tag{2.4}$$

whenever $R^2(t) \neq 0$. But since $V(T)$ vanishes together with $R(t)$, and since $R(t)$ can vanish at most once, we therefore conclude that $\theta'(t)$ can vanish or fail to exist for at most one value of t. Call this exceptional value d if it exists. On the interval $[0, d)$, $\theta'(t)$ is of one sign so that θ changes continuously in one sense and the corresponding integral curve must cross the x and y axes alternatively over $[0, d)$. Similarly, the integral curve crosses the x and y axes alternatively over the interval (d, ∞). In other words, we have proved the following

Lemma 2.1. *Let $\{x(t), y(t)\}$ be a nontrivial solution of* (S). *Then the zeros of $x(t)$ and $y(t)$ separate each other on $[0, \infty)$ with the possible exception of a neighborhood of one point at which $V[x(t), y(t)] = 0$.*

It is quite common that nonlinear equations possess solutions which have infinitely many zeros on compact intervals; nevertheless, such is not the case for our equation (E).

Theorem 2.2. *If $\{x(t), y(t)\}$ is a nontrivial solution of* (S), *then neither $x(t)$ nor $y(t)$ can vanish infinitely often on a compact subinterval I of $[0, \infty)$.*

Proof. Assume to the contrary that $x(t)$ vanishes infinitely often on I and let $t = b$ be a limit point of its zeros. By Lemma 2.1 and Rolle's theorem, b is also a limit point of the zeros of $y(t)$, $x'(t)$ and $y'(t)$. Since $x(t)$ and $y(t)$ are of class $C^{(2)}$, we have $x(b) = y(b) = x'(b) = y'(b) = 0$ which in turn implies, by (H-4), that $x(t) = y(t) \equiv 0$. This contradicts our assumption and completes the proof. Q.E.D.

In later discussions, the integrated form of (S) will often be used and we write it down here for future convenience

$$x'(t) = x'(a) + \int_a^t y(s)\,ds\,, \tag{2.5}$$

$$y'(t) = y'(a) - \int_a^t f(s)\,g(x(s), y(s))\,ds\,. \tag{2.6}$$

3. Nonoscillatory Solutions. A scalar function $h(t)$ defined on $[0, \infty)$ is said *to be oscillatory* if it has arbitrary large zeros and nonoscillatory otherwise. Let K be a nonempty subset of the plane. A vector valued function $z(t) = \{x(t), y(t)\}$ is said to be *K-nonoscillatory* on an interval I if the set $\{z(t) \mid t \in I\}$ is contained in K. Denote the i-th open quadrant of the plane by K_i. Suppose $z(t) = \{x(t), y(t)\}$ is a solution of (S) such that either $x(t)$ or $y(t)$ is nonoscillatory. Then in view of Lemma 2.1, there exists some integer i, $1 \leqq i \leqq 4$, such that $z(t)$ is K_i-nonoscillatory for large t. The converse is also true. Thus we may in our subsequent discussions identify a nonoscillatory solution of (E) with a K_i-nonoscillatory solution of (S) as equivalent concepts, although in general the former may not imply the latter.

Theorem 3.1. *Suppose $\{x(t), y(t)\}$ is a solution of* (S) *such that either $x(t)$ or $y(t)$ is nonoscillatory. Then $xx'yy' \neq 0$ for large t. Furthermore, if $x(t) > 0$ for large t, then $\{x'(t), y'(t)\}$ is K_1-nonoscillatory for large t.*

Proof. Since both $x(t)$ and $y(t)$ are of one sign for large t, and since $x'' = y$ and $y'' = -f\,g(x, y)$, it then follows from Rolle's theorem that $x'(t)$ and $y'(t)$ are of one sign for large t. If $x(t) > 0$ for large t, then either $y(t) > 0$ or $y(t) < 0$ for large t. Assume first that $x(t) > 0$ and $y(t) > 0$ on some interval $[b, \infty)$, $b \geqq 0$. It is easily seen that $y'(t) > 0$ on $[b, \infty)$. Indeed, if $y'(c) \leqq 0$ for some $c \geqq b$, then letting $a = c$ in (2.6), we see that $y'(t) < 0$ for $t > c$. But then $y(t) > 0$, $y'(t) < 0$ and $y''(t) = -f(t)\,g(x, y) < 0$ on (b, ∞), which is impossible. We now assert that $x'(t) > 0$ for large t. Note that since $y'(t) > 0$ on $[b, \infty)$, if we let $a = b$ in (2.6), we see that $x'(t) \geqq x'(b) + y(b)\,(t - b)$. The desired conclusion follows immediately by letting t approach infinity.

Next we assume that $x(t) > 0$ and $y(t) < 0$ on some interval $[b, \infty)$, $b \geqq 0$. The fact that $x'(t) > 0$ on $[b, \infty)$ can be proved similarly to the proof given above of $y'(t) > 0$. Now assume to the contrary that $y'(c) \leqq 0$ for some $c \geqq b$. Then by letting $a = c$ in (2.5), we see that $y'(t) < 0$ for $t > c$. Furthermore, since $y'' = -f\,g(x, y) < 0$ for $t \geqq b$, $y(t) \leqq -ht$ for $t \geqq d \geqq c$, where $k = -y'(d) > 0$. Thus by letting $a = d$ in (2.5), we see that $x'(t) \leqq x'(d) - \int_d^t ks\,\mathrm{d}s$, which implies $x'(t) < 0$ for large t, contrary to our earlier assertion. Q.E.D.

Corollary 3.2. *Suppose $\{x(t), y(t)\}$ is a solution of* (S) *which is K_4-nonoscillatory for large t, then $y(t)/x(t) \to 0$ as $t \to \infty$.*

In view of Theorem 3.1 and the fact that a constant multiple of a solution of (S) is also a solution, we may classify nonoscillatory solutions if (E) into two classes. We say that a nonoscillatory solution $x(t)$ of (E) is Class *I* if some constant multiple of $\{x(t), x''(t)\}$ is K_1-nonoscillatory for large t, and Class II if some constant multiple of $\{x(t), x''(t)\}$ is K_4-nonoscillatory for large t.

If $x(t)$ is a Class I solution of (E) and if $x(t) > 0$ for large t, then by Theorem 3.1, $x'(t) > 0$ and hence $x(t) \geqq x(b) > 0$ for large t. On the other hand, if $x(t)$ is a Class II solution of (E) and $x(t) > 0$ for large t, then by Theorem 3.1, there is some positive number b such that $x(t) > 0$, $x'(t) > 0$, $x''(t) < 0$ and $x^{(3)}(t) > 0$ on $[b, \infty)$. If we integrate $x^{(4)} = -f\,g(x, y) < 0$ four times, we see that

$$x(t) \leqq x(b) + x'(b)\,(t - b) + |x''(b)|\,(t - b)^2/2 + x^{(3)}(b)\,(t - b)^3/6\,.$$

Thus $x(t) \leqq kt^3$ for large t, where k is a suitably chosen positive constant. The following is now clear.

Theorem 3.3. *Suppose $x(t)$ is a nonoscillatory solution of* (E). *Then there are positive constants c_1 and c_2 such that $c_1 \leqq |x(t)| \leqq c_2 t^3$ for large t.*

We say that a nonoscillatory solution $x(t)$ of (E) is asymptotically constant if there exists some constant $c_1 \neq 0$ such that $x(t) \to c_1$ as $t \to \infty$, and asymptotically cubic if there exists some constant $c_2 \neq 0$ such that $t^{-3}\,x(t) \to c_2$ as $t \to \infty$. According to Theorem 3.3, we may regard asymptotically cubic solutions as "maximal" and asymptotically constant solutions as "minimal". We now discuss some necessary conditions and sufficient conditions for their existence.

Theorem 3.4. *A necessary condition for* (E) *to have an asymptotically constant solution* $x(t)$ *is that*

$$\int^{\infty} s^3 f(s) < \infty . \tag{3.1}$$

Proof. Let $x(t)$ be an asymptotically constant solution of (E) and assume without loss of generality that $x(t) > 0$ for large t. $x(t)$ is Class II and thus there are positive numbers a_1, a_2 and b such that $x'(t) > 0$, $x''(t) < 0$, $x^{(3)}(t) > 0$ and $a_1 \leqq x(t) \leqq a_2$ for $t \geqq b$. Upon multiplying (E) by t^3 and integrating from b to t, we obtain

$$0 < \int_b^t s^3 f(s)\, g(x(s), x''(s))\, ds = - \int_b^t s^3 x^{(4)}(s)\, ds = \tag{3.2}$$

$$= -t^3 x^{(3)}(t) + 3t^2 x''(t) - 6t\, x'(t) + x(t) + c$$

where c is a constant. But since $-t^3 x^{(3)} + 3t^2 x'' - 6tx' < 0$ for $t \geqq b$, we have

$$\int^{\infty} s^3 f(s)\, g(x(s), x''(s))\, ds < \infty . \tag{3.3}$$

Recall from Corollary 3.2 that $x''(t)/x(t) \to 0$ as $t \to \infty$. Thus for any positive constant $\eta < g(1, 0)$, there exists a point d, $d \geqq b$, such that for $t \geqq d$,

$$g(1, 0) - \eta \leqq g(1, x''/x) \leqq g(1, 0) + \eta . \tag{3.4}$$

Consequently, we have

$$a_1[g(1, 0) - \eta] \int_d^{\infty} s^3 f(s)\, ds \leqq [g(1, 0) - \eta] \int_d^{\infty} s^3 f(s)\, x(s)\, ds \leqq$$

$$\leqq \int_d^{\infty} s^3 f(s)\, x(s)\, g(1, x''(s)/x(s))\, ds = \int_d^{\infty} s^3 f(s)\, g(x(s), x''(s))\, ds < \infty .$$

as required. Q.E.D.

It is not clear whether the converse of Theorem 3.4 holds or not. Nevertheless, if an extra condition is imposed in addition to (3.1), the existence of an asymptotically constant solution of (E) can be demonstrated.

Theorem 3.5. *If* (3.1) *holds and if* $g(1, v) \leqq M$ *for some positive constant* M, *then* (E) *has an asymptotically constant solution.*

Proof. The required solution of (E) will be obtained with the aid of the following integral equation

$$(3.5)\qquad x(t) = 1 + \int_a^t (s^3/6) f(s)\, g(x(s), x''(s))\, ds +$$

$$+ \int_t^\infty \{t(s-t)^2/2 + t^2(s-t)/2 + t^3/6\} f(s)\, g(x(s), x''(s))\, ds .$$

As can be verified by differentiation, a solution of (3.5) is a solution of (E). To solve (3.5), we define a sequence of approximate solutions as follows (cf. [9, 13, 14]):

$$(3.6)\qquad x_0(t) \equiv 1$$

$$x_{n+1}(t) = 1 + \int_a^t (s^3/6) f(s)\, g(x_n(s), x_n''(s))\, ds +$$

$$+ \int_t^\infty \{t(s-t)^2/2 + t^2(s-t)/2 + t^3/6\} f(s)\, g(x_n(s), x_n''(s))\, ds .$$

Choose the point a so large that

$$(3.7)\qquad \int_a^\infty s^3 f(s)\, ds < 1/2M .$$

Using (3.7) in (3.6). we can show inductively that for all n, $t \geqq a$,

$$(3.8)\qquad 1 \leqq x_n(t) \leqq 2 .$$

Furthermore, from (3.7), we find

$$x_n'(t) = \int_t^\infty [(s-t)^2/2] f(s)\, g(x_{n-1}(s), x_{n-1}''(s))\, ds .$$

Hence, for $t \geqq a$,

$$(3.9)\qquad |x_n'(t)| \leqq \int_t^\infty (s^2/2) f(s)\, x_{n-1}(s)\, g(1, x_{n-1}''(s)/x_{n-1}(s))\, ds \leqq$$

$$\leqq M \int_t^\infty s^3 f(s)\, ds < \tfrac{1}{2} .$$

It follows from (3.8) and (3.9) that the sequence $\{x_n(t)\}$ defines a uniformly bounded and equicontinuous family on $[a, \infty)$. Hence, by the Arzela-Ascoli theorem, there is a subsequence $\{x_{n(k)}(t)\}$ uniformly convergent on every compact subinterval of $[a, \infty)$. Now a standard argument (see for example Wong [13] or Nehari [9]) yields a function $x(t)$ which is a solution of (3.5), which in turn is asymptotically constant. Q.E.D.

Theorem 3.6. *Suppose* $0 < m \leqq g(1, v)$ *for some constant m. Then* (3.1) *is a necessary condition for* (E) *to have an asymptotically cubic solution.*

Proof. Let $x(t)$ be an asymptotically cubic solution of (E) and assume without loss of generality that $x(t) > 0$ for large t. Then $x(t)$ is Class I and by Theorem 3.1, there exist positive numbers a_1, a_2 and b such that $x'(t) > 0$, $x''(t) > 0$, $x^{(3)}(t) > 0$ and $a_1 t^3 \leqq x(t) \leqq a_2 t^3$ for $t \geqq b$. Upon integrating (E) from b to t, we obtain

$$x^3(b) = x^3(t) + \int_b^t f(s)\, g(x(s), x''(s))\, ds > \int_b^t f(s)\, g(x(s), x''(s))\, ds ,$$

which implies

$$\int_b^\infty f(s)\, g(x(s), x''(s))\, ds < \infty .$$

Thus

$$a_1 m \int_b^\infty s^3 f(s)\, ds \leqq \int_b^\infty f(s)\, x(s)\, g(1, x''(s)/x(s))\, ds < \infty$$

as required. Q.E.D.

We have shown in Theorem 3.5 that the condition (3.1) and the condition $g(1, v) \leqq \leqq M$ are sufficient for (E) to have an asymptotically constant solution. It turns out that these two conditions are also sufficient for the existence of an asymptotically cubic solution. The proof is a slight modification of that of Theorem 11.2 in [8] and hence omitted.

Lemma 3.7. *If* $x(t)$ *is a solution of* (E) *such that* $x(t) > 0$ *for large t, then* $\lim 6t^{-3}\, x(t) = \lim x^{(3)}(t)$ *as* $t \to \infty$. *Both limits are finite.*

Theorem 3.8. *If* (3.1) *holds and if* $g(1, v) \leqq M$, *then* (E) *has an asymptotically cubic solution. One such solutitn is determined by the initial conditions* $x(a) = = x'(a) = x''(a) = 0$ *and* $x^{(3)}(a) = 1$, *where a is chosen so that*

$$\int_a^\infty s^3 f(s)\, ds \leqq 5/M .$$

It should be noted that the boundedness of the function $g(1, v)$ in Theorems 3.5, 3.6 and 3.8 is imposed for the sake of convenience. One may replace, for instance, the condition $m \leqq g(1, v)$ in Theorem 3.6 by the condition that $g(1, v)$ is non-decreasing in v. It is then easy to show that the conclusion of Theorem 3.6 still holds and the proof is left to the readers.

4. Oscillatory Solutions. We now consider the behavior of oscillatory solutions of (E). According to the considerations in Section 2, it is clear that $x(t)$ is an oscillatory solution of (E) if and only if $\{x(t), y(t)\} \equiv \{x(t), x''(t)\}$ is a solution of (S) and $|\theta(t)| \to \infty$ as $t \to \infty$. We say that a solution $\{x(t), y(t)\}$ of (S) is rotary if $\lim_{t\to\infty} |\theta(t)| = \infty$. Since $\theta'(t) = -V(t)/R^2(t)$ is of one sign for large t, we expect that the behavior of a rotary solution depends on whether $V(a) \geqq 0$ at some point $a \geqq 0$ or $V(t) < 0$ for all $t \geqq 0$.

Suppose then $\{x(t), y(t)\}$ is a solution of (S). Suppose further that a is a zero of $x(t)$ such that $V(t) \geqq 0$ and $y(t) > 0$ at $t = a$. Since $V(t) > V(a) \geqq 0$ for all $t > a$, it follows from (2.1) that $R^2(t) = (x^2 + y^2)(t) \neq 0$ on (a, ∞). Since $\theta'(a) = -V(a)/R^2(a) \leqq 0$ and $\theta'(t) = -V(t)/R^2(t) < 0$ for $t > a$, if the function $x(t)\,y(t)$ has a first zero b in (a, ∞), then the corresponding integral curve over the interval (a, b) is contained in K_1. In other words, b must be a zero of $y(t)$ and $x(t) > 0$ on $(a, b]$ and $y(t) > 0$ on (a, b). We assert further that at $t = b$, $y'(t) < 0$ and $x'(t) > 0$, i.e. $(R^2)'(t) = (2xx' + 2yy')(t) > 0$. Indeed, the first inequality is clear from the fact that $x(b) > 0$ and

$$V(b) = x'(b)\,y(b) - x(b)\,y'(b) = -x(b)\,y'(b) > 0\,,$$

while the second is clear from the fact that $V(a) = x'(a)\,y(a) \geqq 0$, $y(a) > 0$ and (2.5).

An analogous argument shows that if a is a zero of $y(t)$ such that $V(a) \geqq 0$ and $x(a) > 0$ and if the first zero b of $x(t)\,y(t)$ in (a, ∞) exists, then $y(t) < 0$ on $(a, b]$, $x(t) > 0$ on (a, b), $x'(b) < 0$ and $y'(b) < 0$. The following lemma now follows easily.

Lemma 4.1. *Suppose $\{x(t), y(t)\}$ is a solution of (S). If a, b $(0 \leqq a \leqq b)$ are consecutive zeros of the function $x(t)\,y(t)$ such that $V[x(a), y(a)] \geqq 0$ and $R^2(a) = (x^2 + y^2)(a) \neq 0$, then $V(b) > 0$, $R^2(b) \neq 0$ and $(R^2)'(b) = (2xx' + 2yy')(b) > 0$.*

Lemma 4.2. *Suppose $\{x(t), y(t)\}$ is a solution of (S). If a, b $(0 \leqq a \leqq b)$ are consecutive zeros of the function $x(t)\,y(t)$ such that $R^2(a) = (x^2 + y^2)(a) \neq 0$ and $(R^2)'(a) \geqq 0$, then $R^2(b) \neq 0$, $V(b) > 0$ and $(R^2)'(b) > 0$.*

Proof. Assume without loss of generality that $x(a) = 0$ and $y(a) > 0$. Then $(R^2)'(a) \geqq 0$ implies $y'(a) \geqq 0$. If $x'(a) \geqq 0$, then $V(a) = x'(a)\,y(a) \geqq 0$, and by Lemma 4.1, the proof is complete. Otherwise, assume $x'(a) < 0$. We claim that b is the first zero of $x(t)$ in (a, ∞). Indeed, if this were not true, then b would be the first zero of $y(t)$ and by Rolle's theorem, $y'(t)$ would have a zero d in $[a, b)$. Letting $t = b$ and $d = a$ in (2.6), we see that $y'(b) > 0$ since $x(t) < 0$ on (a, b). But this contradicts the fact $y'(b) \leqq 0$. Since b is the first zero of $x(t)$ in (a, ∞), we see from (2.6) that $y'(t) > 0$ for $a < t \leqq b$. Consequently, $y(t) > 0$ for $a < t \leqq b$. Furthermore, since $x(a) = x(b) = 0$ and $x(t) < 0$ in (a, b), letting e be the last zero of $x'(t)$ in (a, b), and substituting $e = a$ and $b = t$ in (2.5), we see that $x'(b) > 0$. This implies $V(b) = x'(b)\,y(b) > 0$ and $(R^2)'(b) = 2\,y(b)\,y'(b) > 0$ as required. Q.E.D.

Theorem 4.3. *Suppose* $\{x(t), y(t)\}$ *is a nontrivial solution of* (S). *If* a, b $(0 \leqq a < b)$ *are zeros of the function* $x(t)\, y(t)$ *such that either* $V[x(a), y(a)] \geqq 0$ *or* $(R^2)'(a) = (2xx' + 2yy')(a) \geqq 0$, *then*

$$R^2(b) = (x^2 + y^2)(b) \neq 0, \quad V(b) > 0 \quad and \quad (R^2)'(b) > 0.$$

Proof. Note first, in view of Theorem 2.2, that on the interval $[a, b]$, either $x(t)$ or $y(t)$ has only a finite number of zeros. If $R^2(a) \neq 0$, then the proof follows from Lemmas 4.1 and 4.2 and finite induction. If $x(a) = y(a) = 0$, then either $x'(a) \neq 0$ or $y'(a) \neq 0$ since we are assuming that $\{x(t), y(t)\}$ is nontrivial. A slight modification of the proof of Lemma 4.2 and finite induction will then complete the proof.

Corollary 4.4. *Suppose* $\{x(t), y(t)\}$ *is a nontrivial solution of* (S) *such that* $V[x(a), y(a)] \geqq 0$ *at* $a \geqq 0$. *Then* $R^2(b) = (x^2 + y^2)(b) \neq 0$, $V(b) > 0$ *and* $(R^2)'(b) = (2xx' + 2yy')(b) > 0$ *at any zero* b *(except possibly the first one) of the function* $x(t)\, y(t)$ *in* (a, ∞).

As a consequence of Theorem 4.3 and Corollary 4.4, we see that if $x(t)$ is a nontrivial solution of (E) such that $V[x(a), x''(a)] \geqq 0$ at some point $a \geqq 0$, then $[x^2 + (x'')^2](b) \neq 0$, $V[x(b), x''(b)] > 0$ and $[2xx' + 2x''x^{(3)}](b) > 0$ at any zero b of $x(t)$ in (a, ∞).

It is easy to convert Theorem 4.3 into a result concerning the behavior of $\{x(t), y(t)\}$ at points $c < a$. If c is a nonnegative number, the substitution $t = a + c - s$ will transform (S) into a system of the same type in the independent variable s. An application of Theorem 4.3 then leads to the following result.

Corollary 4.5. *Suppose* $\{x(t), y(t)\}$ *is a solution of* (S). *If* c, a $(0 \leqq c < a)$ *are zeros of* $x(t)\, y(t)$ *such that* $R^2(a) = (x^2 + y^2)(a) \neq 0$, $(R^2)'(a) < 0$ *and* $V[x(a), y(a)] < 0$, *then* $R^2(c) \neq 0$, $(R^2)'(c) < 0$ *and* $V(c) < 0$.

We now turn our attention to those solutions of (S) such that $V(t) < 0$ for $t \geqq 0$.

Theorem 4.6. *Suppose* $\{x(t), y(t)\}$ *is a solution of* (S) *such that* $V[x(t), y(t)] < 0$ *for* $t \geqq 0$. *If* a, b $(0 \leqq a < b)$ *are zeros of the function* $x(t)\, y(t)$ *such that* $R^2(a) = (x^2 + y^2)(a) \neq 0$ *and* $(R^2)'(a) = (2xx' + 2yy')(a) < 0$, *then* $R^2(b) \neq 0$ *and* $(R^2)'(b) < 0$, *except possibly when* b *is the last zero of* $x(t)\, y(t)$ *in* (a, ∞).

Proof. Assume without loss of generality that $x(a) = 0$, $y(a) > 0$, $x'(a) < 0$ and $y'(a) < 0$. To prove the theorem it suffices to assume that b is the first zero of $x(t)\, y(t)$ in (a, ∞), because in view of Theorem 2.2, the proof can then be completed by finite induction. Clearly our assumptions imply $\theta'(t) = -V(t)/R^2(t) > 0$ for $t \geqq a$. Hence, the integral curve $\{x(t), y(t)\}$ is contained in K_2 for $a < t < b$. In other words, b is the first zero of $y(t)$ in (a, ∞) and $x(t) < 0$, $y(t) > 0$ on (a, b). This, together with the fact that $V(b) = -x(b)\, y'(b) < 0$ implies $y'(b) < 0$ and $x(b) < 0$. To complete the proof, we will show that if the function $x(t)\, y(t)$ has a second zero in (a, ∞), then $x'(b) > 0$ so that $(R^2)'(b) = (2xx')(b) < 0$. Assume to the contrary that

$x'(b) \leqq 0$; then either $x(t)$ has a first zero c in (b, ∞) or $y(t)$ has a first zero in (b, ∞). If the former holds, then $x'(c) \geqq 0$ and $y(t) < 0$ on (b, c). But letting $a = b$ and $t = c$ in (2.5), we see that $x'(c) < 0$, which is a contradiction. The latter cannot hold either since otherwise there is a point e in (b, d) such that the tangent of the integral curve $\{x(t), y(t)\}$ at $\{x(e), y(e)\}$ passes through the origin, that is, $\theta'(e) = -V(e) : : R^2(e) = 0$, which contradicts the fact that $\theta'(t) > 0$ for $t \geqq a$. Q.E.D.

In order to simplify the statements of our latter results, we make the following definitions. A nontrivial rotary solution $\{x(t), y(t)\}$ of (S) is said to be Type I if $V[x(t), y(t)] < 0$ for $t \geqq 0$ and Type II if $V[x(a), y(a)] \geqq 0$ for some point $a \geqq 0$. A nontrivial rotary solution of (S) is either Type I or Type II. Furthermore, in view of our previous results in this Section, it is Type I if and only if its corresponding integral curve is a "positive spiral" and $R^2(t) \neq 0$, $(R^2)'(t) < 0$, $V(t) < 0$ at any time it crosses the x or y axis (see Figure 1). It is Type II (Fig. 2) if and only if its

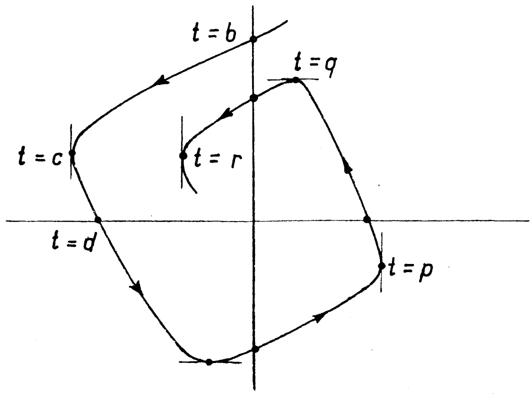

Figure 1.

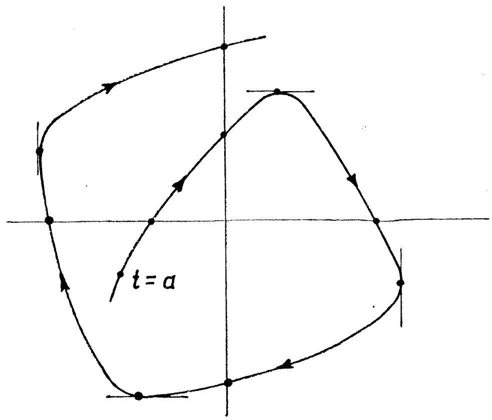

Figure 2.

corresponding integral curve over the interval $[a, \infty)$ is a "negative spiral" and $R^2(t) \neq 0$, $(R^2)'(t) > 0$, $V(t) > 0$ at any time $t > a$ it crosses the x or y axis (except possibly the first time).

In Figures 1 and 2, we have indicated the turning points of the integral curves, that is, the zeros of the functions $x'(t)$ and $y'(t)$. Furthermore, we have also indicated that between $t = b$ and $t = d$, $x'(t)$ has exactly one zero $t = c$, while $y'(t)$ has no zeros there. To see that this is indeed the case, we first note that $x'(b) < 0$ and $x'(d) > 0$ by Theorem 4.6. Hence, $x'(t)$ has at least one zero in (b, d). If $x'(t)$ has more than one zero, then by Rolle's theorem, $x''(t) = y(t)$ has a zero in (b, d), contrary to the fact that $y(t) \neq 0$ in (b, d). On the other hand, by Theorem 4.6, $y'(b) < 0$ and $y'(d) < 0$. If $y'(t)$ has a zero in (b, d), then it has at least two distinct zeros there since the existence of a double zero c' would imply $V(c') = 0$. But then $y''(t) = -f(t)\,g(x, y)$ would have a zero, which contradicts the fact that $x(t) < 0$ on (b, d) and $\operatorname{sgn} f\,g(x, y) = \operatorname{sgn} x$.

In the next theorem we give some sufficient conditions for all solutions of (S) to be rotary.

Theorem 4.7. *All nontrivial solutions of* (S) *are rotary provided* $0 < m \leqq g(1, v)$ *and*

$$\int^{\infty} t^2 f(t)\,\mathrm{d}t = \infty\,.$$

Proof. Assume to the contrary that (S) has a solution $z(t) = \{x(t), y(t)\}$ which is nonrotary. Then it is Class I or Class II. Suppose first that $z(t)$ is Class I. We may assume without loss of generality that $x(t) > 0$ for large t. By Theorem 3.1, there then exists some positive number T such that $x(t) > 0$, $x'(t) > 0$, $y(t) > 0$ and $y'(t) > 0$ for $t \geqq T$. Since $y(t) = x''(t) > 0$, it follows that

$$x(t) \geqq y(T)(t - T)^2 + x'(T)(t - T) + x(T) > y(T)(t - T)^2\,.$$

On the other hand, we have

$$y'(T) = y'(t) + \int_T^t f(s)\,g(x(s), y(s))\,\mathrm{d}s \geqq m \int_T^t f(s)\,x(s)\,\mathrm{d}s \geqq$$

$$\geqq m\,y(T) \int_T^t (s - T)^2 f(s)\,\mathrm{d}s\,.$$

Since the lefthand side is independent of t, we conclude that $s^2 f(s)$ is summable in (T, ∞), contrary to hypothesis.

Next we suppose $z(t)$ is Class II. We may assume without loss of generality that $x(t) > 0$ for large t. By Theorem 3.1 and Corollary 3.2, there exists some positive number T such that $x(t) > 0$, $x'(t) > 0$, $y(t) < 0$, $y'(t) > 0$ and $0 < g(1, 0) - \eta <$

$< g(1, y/x)$ for $t \geqq T$, where $\eta < g(1, 0)$ is a suitably chosen constant. After multiplying (E) by $(t - T)^2$ and integrating by parts, we obtain

$$2\,x'(T) = (t - T)^2\, x^{(3)}(t) - 2(t - T)\, x''(t) + 2\,x'(t) +$$

$$+ \int_T^t (s - T)^2 f(s)\, g(x(s), y(s))\, ds > \int_T^t (s - T)^2 f(s)\, x(s)\, g(1, y(s)/x(s))\, ds \geqq$$

$$\geqq x(T)\,[g(1, 0) - \eta] \int_T^t (s - T)^2 f(s)\, ds\,.$$

Again, we conclude from the above inequalities that $s^2 f(s)$ is summable in (T, ∞), contrary to hypothesis. Q.E.D.

It is known [8, 11] that if the linear equation (L) has an oscillatory solution, then all solutions are oscillatory; and if all solutions of (L) are oscillatory, then (L) has both Type I and Type II solutions. It is easy to see that if all solutions of (E) are oscillatory, then a Type II solution exists. Indeed, the solution $\{x(t), y(t)\}$ of (S) determined by the initial conditions $x(0) = x'(0) = y(0) = 0$ and $y'(0) = 1$ is Type II. The remaining two properties of (L), however, are related to linearity (as seen in [8, 11]). It therefore remains open the question whether analogous properties hold for (E).

5. Additional Properties of Rotary Solutions. First we observe that if we let

$$U(t) = U[x(t), y(t)] = x(t)\, y(t) - [x'(t)]^2 \tag{5.1}$$

where $\{x(t), y(t)\}$ is a solution of (S), then

$$U'(t) = -V(t)\,. \tag{5.2}$$

Theorem 5.1. *If* $\{x(t), y(t)\}$ *is a Type* II *solution of* (S), *then*

(i) $x'(t)$ *is unbounded*;

and

(ii) $$\int^\infty [x'(t)]^2\, dt = \infty\,.$$

Proof. Since $\{x(t), y(t)\}$ is Type II, $V[x(t), y(t)]$ is positive and bounded away from zero for large t. Clearly then $U(t) \downarrow -\infty$ as $t \to \infty$. Since $x(t)$ is oscillatory, (i) follows immediately by evaluating $U(t)$ along the zero of $x(t)$. To prove (ii), integrate $U(t)$ from a to t where a is a zero of $x(t)$. In doing so, we obtain

$$x(t)\, x'(t) - \int_a^t U(s)\, ds = 2 \int_a^t [x'(s)]^2\, ds\,.$$

Since $x(t)$ is oscillatory and since $\int_a^t U(s)\,ds \to -\infty$ as $t \to \infty$, (ii) follows. Q.E.D

Theorem 5.2. *If* $\{x(t), y(t)\}$ *is a Type* I *solution of* (S), *then* $x(t)\,y(t) < [x'(t)]^2$ *for* $t \geqq 0$.

Proof. Since $V[x(t), y(t)] < 0$ for $t \geqq 0$, $U'(t) > 0$ there. Furthermore, since $x(t)$ is oscillatory and since $U(b) < 0$ at any zero b of $x(t)$,

$$U(t) = x(t)\,y(t) - [x'(t)]^2 < 0$$

for all $t \geqq 0$ as required. Q.E.D.

Next we observe that the derivative of the function

$$W(t) = W[x(t), y(t)] = x'(t)\,y'(t)/f(t) + x^2(t)/2 \tag{5.2}$$

is

$$W' = y'(y - f'x'/f)/f + x'[x - g(x, y)]\,. \tag{5.4}$$

Theorem 5.3. *Suppose* $\{x(t), y(t)\}$ *is a Type* I *solution of* (S). *If* $g(u, 1) \leqq u$ *for all* u *and if* $f'(t)$ *is continuous and nonnegative, then* $|x(t)|$ *is decreasing at its successive maxima and minima.*

Proof. Let p, r be consecutive zeros of the function $x'(t)$ and assume without loss of generality that $x(t) > 0$ at $t = p$. According to the properties of a Type I solution (see Figure 1), we know that there exists a number q such that $y'(q) = 0$, $x(t) > 0$ on (p, q), $x'(t) < 0$ on (p, r), and $y(t) > 0$, $y'(t) < 0$ on (q, r). Clearly, $x(p) > x(q)$ since $x'(t) < 0$ on (p, q). Furthermore, since

$$W' = y'(y - f'x'/f)/f + x'y[x/y - g(x/y, 1)] < 0$$

on (q, r), we have

$$W(q) = x^2(q)/2 > x^2(r)/2 = W(r)\,.$$

Thus $|x(p)| > |x(q)| > |x(r)|$ as required. Q.E.D.

The proof of the following is similar.

Theorem 5.4. *Suppose* $\{x(t), y(t)\}$ *is a Type* II *solution of* (S). *If* $g(u, 1) \leqq u$ *for all* u *and if* $f'(t)$ *is continuous and nonpositive, then for large* t, $|x(t)|$ *is increasing at its successive maxima and minima.*

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