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Karel Svoboda

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SEVERAL NEW CHARACTERIZATIONS OF THE 2-DIMENSIONAL SPHERE IN E^4

KAREL SVOBODA, Brno

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Solving the problem of the global characterization of the 2-dimensional sphere among surfaces in E^4, we have used, in [1], the property of the mean curvature vector field of the surface M being pseudoparallel. In the present paper, we give some new results concerning the global characterization of the sphere in E^4 and based again on the pseudoparallelness of the mean curvature vector field of M.

1. Let M be a surface in the 4-dimensional Euclidean space E^4 and ∂M its boundary. Let M be covered by open domains U_α in such a way that in each U_α there is a field of orthonormal frames $\{M; v_1, v_2, v_3, v_4\}$, $v_1, v_2 \in T(M)$, $v_3, v_4 \in N(M)$, $T(M), N(M)$ being the tangent and the normal bundles of M, respectively. Then

$$
\begin{aligned}
\mathrm{d}M &= \omega^1 v_1 + \omega^2 v_2 , \\
\mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 , \\
\mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 , \\
\mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 , \\
\mathrm{d}v_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 ;
\end{aligned} \tag{1}
$$

$$
\mathrm{d}\omega^i = \omega^k \wedge \omega_k^i , \quad \mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j , \quad \omega_i^j + \omega_j^i = 0 \quad (i, j, k = 1, 2, 3, 4) , \tag{2}
$$

$$
\omega^3 = \omega^4 = 0 .
$$

Differentiating the last equations of (2) and using Cartan's lemma, we get the existence of real-valued functions a_i, b_i, c_i; $\alpha_i, \beta_i, \ldots, \delta_i$; $A_i, B_i, \ldots, E_i$ $(i = 1, 2)$ on each U_α such that

$$
\begin{aligned}
\omega_1^3 &= a_1\omega^1 + b_1\omega^2 , \quad \omega_2^3 = b_1\omega^1 + c_1\omega^2 , \\
\omega_1^4 &= a_2\omega^1 + b_2\omega^2 , \quad \omega_2^4 = b_2\omega^1 + c_2\omega^2 ;
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
(4)\qquad & da_1 - 2b_1\omega_1^2 - a_2\omega_3^4 = \alpha_1\omega^1 + \beta_1\omega^2 , \\
& db_1 + (a_1 - c_1)\,\omega_1^2 - b_2\omega_3^4 = \beta_1\omega^1 + \gamma_1\omega^2 , \\
& dc_1 + 2b_1\omega_1^2 - c_2\omega_3^4 = \gamma_1\omega^1 + \delta_1\omega^2 , \\
& da_2 - 2b_2\omega_1^2 + a_1\omega_3^4 = \alpha_2\omega^1 + \beta_2\omega^2 , \\
& db_2 + (a_2 - c_2)\,\omega_1^2 + b_1\omega_3^4 = \beta_2\omega^1 + \gamma_2\omega^2 , \\
& dc_2 + 2b_2\omega_1^2 + c_1\omega_3^4 = \gamma_2\omega^1 + \delta_2\omega^2 ;
\end{aligned}
$$

$$
\begin{aligned}
(5)\qquad & d\alpha_1 - 3\beta_1\omega_1^2 - \alpha_2\omega_3^4 = A_1\omega^1 + (B_1 - b_1K - \tfrac{1}{2}a_2k)\,\omega^2 , \\
& d\beta_1 + (\alpha_1 - 2\gamma_1)\,\omega_1^2 - \beta_2\omega_3^4 = (B_1 + b_1K + \tfrac{1}{2}a_2k)\,\omega^1 + \\
& \quad + (C_1 + a_1K - \tfrac{1}{2}b_2k)\,\omega^2 , \\
& d\gamma_1 + (2\beta_1 - \delta_1)\,\omega_1^2 - \gamma_2\omega_3^4 = (C_1 + c_1K + \tfrac{1}{2}b_2k)\,\omega^1 + \\
& \quad + (D_1 + b_1K - \tfrac{1}{2}c_2k)\,\omega^2 , \\
& d\delta_1 + 3\gamma_1\omega_1^2 - \delta_2\omega_3^4 = (D_1 - b_1K + \tfrac{1}{2}c_2k)\,\omega^1 + E_1\omega^2 , \\
& d\alpha_2 - 3\beta_2\omega_1^2 + \alpha_1\omega_3^4 = A_2\omega^1 + (B_2 - b_2K + \tfrac{1}{2}a_1k)\,\omega^2 , \\
& d\beta_2 + (\alpha_2 - 2\gamma_2)\,\omega_1^2 + \beta_1\omega_3^4 = (B_2 + b_2K - \tfrac{1}{2}a_1k)\,\omega^1 + \\
& \quad + (C_2 + a_2K + \tfrac{1}{2}b_1k)\,\omega^2 , \\
& d\gamma_2 + (2\beta_2 - \delta_2)\,\omega_1^2 + \gamma_1\omega_3^4 = (C_2 + c_2K - \tfrac{1}{2}b_1k)\,\omega^1 + \\
& \quad + (D_2 + b_2K + \tfrac{1}{2}c_1k)\,\omega^2 , \\
& d\delta_2 + 3\gamma_2\omega_1^2 + \delta_1\omega_3^4 = (D_2 - b_2K - \tfrac{1}{2}c_1k)\,\omega^1 + E_2\omega^2
\end{aligned}
$$

where

$$(6)\qquad K = a_1c_1 - b_1^2 + a_2c_2 - b_2^2 , \quad k = (a_1 - c_1)\,b_2 - (a_2 - c_2)\,b_1 ,$$

K being the Gauss curvature of M. Denote further by

$$(7)\qquad \xi = (a_1 + c_1)\,v_3 + (a_2 + c_2)\,v_4$$

the mean curvature vector field and by

$$(8)\qquad H = |\xi|^2 = (a_1 + c_1)^2 + (a_2 + c_2)^2$$

the mean curvature of M.

Let $\xi \neq 0$ on M. Denote by $P_m(M)$ the union of $T_m(M)$ and ξ_m for each $m \in M$ and by $P(M)$ the corresponding vector bundle over M. The vector field ξ is said to be pseudoparallel in $P(M)$, if $t\xi \in P(M)$ for each vector field $t \in T(M)$.

As mentioned in [1], ξ is pseudoparallel in $P(M)$ if and only if, according to (7),

$$
\begin{aligned}
(9)\qquad & (a_1 + c_1)(\alpha_2 + \gamma_2) - (a_2 + c_2)(\alpha_1 + \gamma_1) = 0 , \\
& (a_1 + c_1)(\beta_2 + \delta_2) - (a_2 + c_2)(\beta_1 + \delta_1) = 0 .
\end{aligned}
$$

Further, ξ being pseudoparallel in $P(M)$, we have

$$(\alpha_1 + \gamma_1)(\beta_2 + \delta_2) - (\beta_1 + \delta_1)(\alpha_2 + \gamma_2) = 0 \tag{10}$$

and, by differentiation of (9), when using (4), (5), (10), we obtain $k = 0$ on M and

$$\begin{aligned} &(a_1 + c_1)(A_2 + C_2 + c_2K) - (a_2 + c_2)(A_1 + C_1 + c_1K) = 0\,, \\ &(a_1 + c_1)(B_2 + D_2) - (a_2 + c_2)(B_1 + D_1) = 0\,, \\ &(a_1 + c_1)(C_2 + E_2 + a_2K) - (a_2 + c_2)(C_1 + E_1 + a_1K) = 0\,. \end{aligned} \tag{11}$$

2. Consider a real-valued function F on M. We define its covariant derivatives F_i, $F_{ij} = F_{ji}\,(i, j = 1, 2)$ with respect to the given field of orthonormal frames over U_α by means of the formulas

$$\mathrm{d}F = F_1\omega^1 + F_2\omega^2\,, \tag{12}$$

$$\mathrm{d}F_1 - F_2\omega_1^2 = F_{11}\omega^1 + F_{12}\omega^2\,, \quad \mathrm{d}F_2 + F_1\omega_1^2 = F_{12}\omega^1 + F_{22}\omega^2\,.$$

Thus, for the mean curvature H and the Gauss curvature K of M introduced by (6), (8), respectively, we have, according to (12) and using (4), (5),

$$\begin{aligned} \tfrac{1}{2}H_1 &= (a_1 + c_1)(\alpha_1 + \gamma_1) + (a_2 + c_2)(\alpha_2 + \gamma_2)\,, \\ \tfrac{1}{2}H_2 &= (a_1 + c_1)(\beta_1 + \delta_1) + (a_2 + c_2)(\beta_2 + \delta_2)\,; \end{aligned} \tag{13}$$

$$\begin{aligned} \tfrac{1}{2}H_{11} &= (a_1 + c_1)(A_1 + C_1 + c_1K + \tfrac{1}{2}b_2k) + \\ &+ (a_2 + c_2)(A_2 + C_2 + c_2K - \tfrac{1}{2}b_1k) + (\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2\,, \\ \tfrac{1}{2}H_{12} &= (a_1 + c_1)(B_1 + D_1) + (a_2 + c_2)(B_2 + D_2) + \\ &+ (\alpha_1 + \gamma_1)(\beta_1 + \delta_1) + (\alpha_2 + \gamma_2)(\beta_2 + \delta_2)\,, \\ \tfrac{1}{2}H_{22} &= (a_1 + c_1)(C_1 + E_1 + a_1K - \tfrac{1}{2}b_2k) + \\ &+ (a_2 + c_2)(C_2 + E_2 + a_2K + \tfrac{1}{2}b_1k) + (\beta_1 + \delta_1)^2 + (\beta_2 + \delta_2)^2\,; \end{aligned} \tag{14}$$

$$\begin{aligned} K_1 &= (c_1\alpha_1 - 2b_1\beta_1 + a_1\gamma_1) + (c_2\alpha_2 - 2b_2\beta_2 + a_2\gamma_2)\,, \\ K_2 &= (c_1\beta_1 - 2b_1\gamma_1 + a_1\delta_1) + (c_2\beta_2 - 2b_2\gamma_2 + a_2\delta_2)\,; \end{aligned} \tag{15}$$

$$\begin{aligned} K_{11} &= (c_1A_1 - 2b_1B_1 + a_1C_1) + (c_2A_2 - 2b_2B_2 + a_2C_2) + \\ &+ 2(\alpha_1\gamma_1 - \beta_1^2) + 2(\alpha_2\gamma_2 - \beta_2^2) + \tfrac{3}{2}(a_1b_2 - b_1a_2)\,k + \\ &+ [(a_1c_1 - 2b_1^2) + (a_2c_2 - 2b_2^2)]\,K\,, \\ K_{12} &= (c_1B_1 - 2b_1C_1 + a_1D_1) + (c_2B_2 - 2b_2C_2 + a_2D_2) + \\ &+ (\alpha_1\delta_1 - \beta_1\gamma_1) + (\alpha_2\delta_2 - \beta_2\gamma_2) - \\ &- [(a_1 + c_1)\,b_1 + (a_2 + c_2)\,b_2]\,K\,, \end{aligned} \tag{16}$$

$$K_{22} = (c_1C_1 - 2b_1D_1 + a_1E_1) + (c_2C_2 - 2b_2D_2 + a_2E_2) + \\ + 2(\beta_1\delta_1 - \gamma_1^2) + 2(\beta_2\delta_2 - \gamma_2^2) + \tfrac{3}{2}(b_1c_2 - c_1b_2)\,k + \\ + [(a_1c_1 - 2b_1^2) + (a_2c_2 - 2b_2^2)]\,K\,.$$

To abbreviate the following formulas, let us introduce the functions

$$(17)\qquad \mathscr{H}_{11} = HH_{11} - \tfrac{1}{2}H_1^2\,,\quad \mathscr{H}_{12} = HH_{12} - \tfrac{1}{2}H_1H_2\,,\\ \mathscr{H}_{22} = HH_{22} - \tfrac{1}{2}H_2^2$$

and

$$(18)\qquad \mathscr{A} = a_1(a_1 + c_1) + a_2(a_2 + c_2)\,,\quad \mathscr{B} = b_1(a_1 + c_1) + b_2(a_2 + c_2)\,,\\ \mathscr{C} = c_1(a_1 + c_1) + c_2(a_2 + c_2)\,.$$

It is clear that, under this notation,

$$(19)\qquad H = \mathscr{A} + \mathscr{C}\,.$$

In what follows, we are going to prove

Lemma 1. *The functions*

$$(20)\qquad I = (\mathscr{A} - \mathscr{C})(\mathscr{H}_{11} - \mathscr{H}_{22}) + 4\mathscr{B}\mathscr{H}_{12}\,,$$

$$(21)\qquad J = \mathscr{C}\mathscr{H}_{11} - 2\mathscr{B}\mathscr{H}_{12} + \mathscr{A}\mathscr{H}_{22} - 2H^2(K_{11} + K_{22})$$

are invariant on M.

Proof. Consider another field $\{M; \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ of tangent frames, and denote all expressions related to these frames by a bar. Let

$$(22)\qquad v_1 = \varepsilon_1 \cos\varrho\,.\,\bar{v}_1 - \sin\varrho\,.\,\bar{v}_2\,,\quad v_3 = \varepsilon_2\cos\sigma\,.\,\bar{v}_3 - \sin\sigma\,.\,\bar{v}_4\,,\\ v_2 = \varepsilon_1\sin\varrho\,.\,\bar{v}_1 + \cos\varrho\,.\,\bar{v}_2\,,\quad v_4 = \varepsilon_2\sin\sigma\,.\,\bar{v}_3 + \cos\sigma\,.\,\bar{v}_4\,,\\ \varepsilon_1^2 = \varepsilon_2^2 = 1\,.$$

By easy calculations, see [1], we obtain

$$(23)\qquad \bar{\omega}^1 = \varepsilon_1(\cos\varrho\,.\,\omega^1 + \sin\varrho\,.\,\omega^2)\,,\quad \bar{\omega}^2 = -\sin\varrho\,.\,\omega^1 + \cos\varrho\,.\,\omega^2\,;$$

$$(24)\qquad \bar{\omega}_1^2 = \varepsilon_1(\mathrm{d}\varrho + \omega_1^2)$$

and further

$$(25)\qquad \bar{a}_1 = \varepsilon_2(R_1\cos\sigma + R_2\sin\sigma)\,,\\ \bar{b}_1 = -\varepsilon_1\varepsilon_2(S_1\cos\sigma + S_2\sin\sigma)\,,$$

$$\bar{c}_1 = \varepsilon_2(T_1 \cos\sigma + T_2 \sin\sigma),$$
$$\bar{a}_2 = -(R_1 \sin\sigma - R_2 \cos\sigma),$$
$$\bar{b}_2 = \varepsilon_1(S_1 \sin\sigma - S_2 \cos\sigma),$$
$$\bar{c}_2 = -(T_1 \sin\sigma - T_2 \cos\sigma)$$

where

(26) $$R_i = a_i \cos^2\varrho + 2b_i \sin\varrho\cos\varrho + c_i \sin^2\varrho,$$
$$S_i = a_i \sin\varrho\cos\varrho + b_i(\sin^2\varrho - \cos^2\varrho) - c_i \sin\varrho\cos\varrho,$$
$$T_i = a_i \sin^2\varrho - 2b_i \sin\varrho\cos\varrho + c_i \cos^2\varrho \quad (i = 1, 2).$$

Because of (12) and (23), we get from $\bar{H} = H$

(27) $$\bar{H}_1 = \varepsilon_1(H_1 \cos\varrho + H_2 \sin\varrho),$$
$$\bar{H}_2 = -H_1 \sin\varrho + H_2 \cos\varrho.$$

Differentiating these equations and using (24), (27) and the relations of the form (12) corresponding to H, $\bar{H}$, we obtain

(28) $$\bar{H}_{11} = H_{11} \cos^2\varrho + 2H_{12} \sin\varrho\cos\varrho + H_{22} \sin^2\varrho,$$
$$\bar{H}_{12} = -\varepsilon_1(H_{11} - H_{22}) \sin\varrho\cos\varrho + \varepsilon_1 H_{12}(\cos^2\varrho - \sin^2\varrho),$$
$$\bar{H}_{22} = H_{11} \sin^2\varrho - 2H_{12} \sin\varrho\cos\varrho + H_{22} \cos^2\varrho.$$

In the same way we get

(29) $$\bar{K}_{11} = K_{11} \cos^2\varrho + 2K_{12} \sin\varrho\cos\varrho + K_{22} \sin^2\varrho,$$
$$\bar{K}_{12} = -\varepsilon_1(K_{11} - K_{22}) \sin\varrho\cos\varrho + \varepsilon_1 K_{12}(\cos^2\varrho - \sin^2\varrho),$$
$$\bar{K}_{22} = K_{11} \sin^2\varrho - 2K_{12} \sin\varrho\cos\varrho + K_{22} \cos^2\varrho.$$

Further, from $\bar{H} = H$, (27) and (28) it follows that

(30) $$\bar{\mathscr{H}}_{11} = \mathscr{H}_{11} \cos^2\varrho + 2\mathscr{H}_{12} \sin\varrho\cos\varrho + \mathscr{H}_{22} \sin^2\varrho,$$
$$\bar{\mathscr{H}}_{12} = -\varepsilon_1(\mathscr{H}_{11} - \mathscr{H}_{22}) \sin\varrho\cos\varrho + \varepsilon_1 \mathscr{H}_{12}(\cos^2\varrho - \sin^2\varrho),$$
$$\bar{\mathscr{H}}_{22} = \mathscr{H}_{11} \sin^2\varrho - 2\mathscr{H}_{12} \sin\varrho\cos\varrho + \mathscr{H}_{22} \cos^2\varrho$$

and from (25), (26) we obtain

(31) $$\bar{\mathscr{A}} = \mathscr{A} \cos^2\varrho + 2\mathscr{B} \sin\varrho\cos\varrho + \mathscr{C} \sin^2\varrho,$$
$$\bar{\mathscr{B}} = -\varepsilon_1(\mathscr{A} - \mathscr{C}) \sin\varrho\cos\varrho + \varepsilon_1 \mathscr{B}(\cos^2\varrho - \sin^2\varrho),$$
$$\bar{\mathscr{C}} = \mathscr{A} \sin^2\varrho - 2\mathscr{B} \sin\varrho\cos\varrho + \mathscr{C} \cos^2\varrho.$$

According to (20), (21), the relations (29), (30) and (31) yield the assertion.

Remark. By direct calculations we get, ξ being pseudoparallel,

$$J = 4H^2[\gamma_1(\gamma_1 - \alpha_1) + \gamma_2(\gamma_2 - \alpha_2) + \beta_1(\beta_1 - \delta_1) + \beta_2(\beta_2 - \delta_2)] + + 2H^2[(a_1 - c_1)^2 + (a_2 - c_2)^2 + 4(b_1^2 + b_2^2)] K, \tag{32}$$

so that J does not depend on $A_i, \ldots, E_i$ $(i = 1, 2)$. In fact, it is possible to show that, up to a multiplicative factor, J is the unique function with this property which can be obtained by the elimination of $A_i, \ldots, E_i$ $(i = 1, 2)$ from the equations (11), (14), (16).

3. In this section we are going to give some characterizations of the sphere in E^4. They will be proved by means of the maximum principle used in this form:

Let M be a surface in E^4, ∂M its boundary. Let F be a real-valued function on M and F_i, F_{ij} $(i, j = 1, 2)$ its covariant derivatives defined by (12). *Let* (1) $F \geqq 0$ *on* M, (2) $F = 0$ *on* ∂M, (3) *on* M, *let F satisfy the equation*

$$a_{11}F_{11} + 2a_{12}F_{12} + a_{22}F_{22} + a_1F_1 + a_2F_2 + a_0F = a$$

where $a_0 \leqq 0$, $a \geqq 0$ *and the quadratic form* $a_{ij}x^ix^j$ *is positive definite. Then* $F = 0$ *on* M.

In what follows, we use the function

$$f = H - 4K = (a_1 - c_1)^2 + (a_2 - c_2)^2 + 4b_1^2 + 4b_2^2 \tag{33}$$

which satisfies obviously $f \geqq 0$ on M and $f = 0$ at the umbilical points $(a_1 - c_1 = 0$, $a_2 - c_2 = 0,\ b_1 = 0,\ b_2 = 0)$ of M.

Using (4), (5) and (12), we easily see that

$$f_1 = 2(a_1 - c_1)(\alpha_1 - \gamma_1) + 2(a_2 - c_2)(\alpha_2 - \gamma_2) + 8(b_1\beta_1 + b_2\beta_2),$$

$$f_2 = 2(a_1 - c_1)(\beta_1 - \delta_1) + 2(a_2 - c_2)(\beta_2 - \delta_2) + 8(b_1\gamma_1 + b_2\gamma_2)$$

and

$$\begin{aligned} f_{11} = {} & 2(a_1 - c_1)(A_1 - C_1) + 2(a_2 - c_2)(A_2 - C_2) + 8(b_1B_1 + b_2B_2) + \\ & + 2(\alpha_1 - \gamma_1)^2 + 2(\alpha_2 - \gamma_2)^2 + 8(\beta_1^2 + \beta_2^2) - [k + 4(a_1b_2 - b_1a_2)]\,k - \\ & - 2[(a_1 - c_1)\,c_1 + (a_2 - c_2)\,c_2 - 4(b_1^2 + b_2^2)]\,K, \\ f_{12} = {} & 2(a_1 - c_1)(B_1 - D_1) + 2(a_2 - c_2)(B_2 - D_2) + 8(b_1C_1 + b_2C_2) + \\ & + 2(\alpha_1 - \gamma_1)(\beta_1 - \delta_1) + 2(\alpha_2 - \gamma_2)(\beta_2 - \delta_2) + 8(\beta_1\gamma_1 + \beta_2\gamma_2) + \\ & + 4[(a_1 + c_1)\,b_1 + (a_2 + c_2)\,b_2]\,K, \\ f_{22} = {} & 2(a_1 - c_1)(C_1 - E_1) + 2(a_2 - c_2)(C_2 - E_2) + 8(b_1D_1 + b_2D_2) + \\ & + 2(\beta_1 - \delta_1)^2 + 2(\beta_2 - \delta_2)^2 + 8(\gamma_1^2 + \gamma_2^2) - [k + 4(b_1c_2 - c_1b_2)]\,k + \\ & + 2[(a_1 - c_1)\,a_1 + (a_2 - c_2)\,a_2 + 4(b_1^2 + b_2^2)]\,K. \end{aligned} \tag{34}$$

Now, we formulate

Theorem 1. *Let M be a surface in E^4 and ∂M its boundary. Let*

(i) $K > 0$ *on* M;
(ii) ξ *be pseudoparallel in* $P(M)$;
(iii) $(\mathscr{A} - \mathscr{C})(\mathscr{H}_{11} - \mathscr{H}_{22}) + 4\mathscr{B}\mathscr{H}_{12} \geqq 0$ *on* M;
(iv) ∂M *consist of umbilical points.*

Then M is part of a 2-dimensional sphere in E^4.

Proof. The condition (ii) is expressed by (9) and implies $k = 0$ on M. Consider the equations (9) and (13). As $H \neq 0$, there exists a unique solution of the system

$$\alpha_1 + \gamma_1 = \tfrac{1}{2}(a_1 + c_1) H^{-1} H_1 , \quad \beta_1 + \delta_1 = \tfrac{1}{2}(a_1 + c_1) H^{-1} H_2 ,$$

$$\alpha_2 + \gamma_2 = \tfrac{1}{2}(a_2 + c_2) H^{-1} H_1 , \quad \beta_2 + \delta_2 = \tfrac{1}{2}(a_2 + c_2) H^{-1} H_2 .$$

Hence

(35)
$$\tfrac{1}{4}H^{-1}H_1^2 = (\alpha_1 + \gamma_1)^2 + (\alpha_2 + \gamma_2)^2 ,$$
$$\tfrac{1}{4}H^{-1}H_1H_2 = (\alpha_1 + \gamma_1)(\beta_1 + \delta_1) + (\alpha_2 + \gamma_2)(\beta_2 + \delta_2) ,$$
$$\tfrac{1}{4}H^{-1}H_2^2 = (\beta_1 + \delta_1)^2 + (\beta_2 + \delta_2)^2$$

and using these relations and $k = 0$, the equations (14) have the form, according to (17),

(36)
$$(a_1 + c_1)(A_1 + C_1 + c_1K) + (a_2 + c_2)(A_2 + C_2 + c_2K) = \tfrac{1}{2}H^{-1}\mathscr{H}_{11} ,$$
$$(a_1 + c_1)(B_1 + D_1) + (a_2 + c_2)(B_2 + D_2) = \tfrac{1}{2}H^{-1}\mathscr{H}_{12} ,$$
$$(a_1 + c_1)(C_1 + E_1 + a_1K) + (a_2 + c_2)(C_2 + E_2 + a_2K) = \tfrac{1}{2}H^{-1}\mathscr{H}_{22} .$$

The system of equations (11) and (36) has, because of $H \neq 0$ and (35), the only solution

(37)
$$A_1 + C_1 = \tfrac{1}{2}(a_1 + c_1) H^{-2}\mathscr{H}_{11} - c_1K ,$$
$$B_1 + D_1 = \tfrac{1}{2}(a_1 + c_1) H^{-2}\mathscr{H}_{12} ,$$
$$C_1 + E_1 = \tfrac{1}{2}(a_1 + c_1) H^{-2}\mathscr{H}_{22} - a_1K ,$$
$$A_2 + C_2 = \tfrac{1}{2}(a_2 + c_2) H^{-2}\mathscr{H}_{11} - c_2K ,$$
$$B_2 + D_2 = \tfrac{1}{2}(a_2 + c_2) H^{-2}\mathscr{H}_{12} ,$$
$$C_2 + E_2 = \tfrac{1}{2}(a_2 + c_2) H^{-2}\mathscr{H}_{22} - a_2K .$$

Thus we have

$$(38)\qquad \begin{aligned} A_1 - E_1 &= \tfrac{1}{2}(a_1 + c_1) H^{-2}(\mathscr{H}_{11} - \mathscr{H}_{22}) + (a_1 - c_1) K, \\ B_1 + D_1 &= \tfrac{1}{2}(a_1 + c_1) H^{-2}\mathscr{H}_{12}, \\ A_2 - E_2 &= \tfrac{1}{2}(a_2 + c_2) H^{-2}(\mathscr{H}_{11} - \mathscr{H}_{22}) + (a_2 - c_2) K, \\ B_2 + D_2 &= \tfrac{1}{2}(a_2 + c_2) H^{-2}\mathscr{H}_{12}. \end{aligned}$$

Now, consider the function f defined by (33). According to (34), we have

$$(39)\qquad f_{11} + f_{22} - 2[f + 4(b_1^2 + b_2^2)] K = 2V + 2\Phi$$

where

$$(40)\qquad \begin{aligned} V = (\alpha_1 - \gamma_1)^2 + (\beta_1 - \delta_1)^2 + (\alpha_2 - \gamma_2)^2 + (\beta_2 - \delta_2)^2 + \\ + 4(\beta_1^2 + \gamma_1^2 + \beta_2^2 + \gamma_2^2), \end{aligned}$$

$$(41)\qquad \begin{aligned} \Phi = (a_1 - c_1)(A_1 - E_1) + (a_2 - c_2)(A_2 - E_2) + \\ + 4b_1(B_1 + D_1) + 4b_2(B_2 + D_2). \end{aligned}$$

Inserting (38) into (41) we get

$$\Phi = \tfrac{1}{2}H^{-2}(\mathscr{A} - \mathscr{C})(\mathscr{H}_{11} - \mathscr{H}_{22}) + 2H^{-2}\mathscr{B}\mathscr{H}_{12} + [(a_1 - c_1)^2 + (a_2 - c_2)^2] K$$

and thus, according to (20),

$$\Phi = \tfrac{1}{2}H^{-2}I + [(a_1 - c_1)^2 + (a_2 - c_2)^2] K,$$

so that the equation (39) is of the form

$$(42)\qquad f_{11} + f_{22} - 4Kf = 2V + H^{-2}I.$$

It is $a_0 = -4K < 0$ because of (i), $a = 2V + H^{-2}I \geqq 0$ according to (40) and (ii), and the quadratic form corresponding to $f_{11} + f_{22}$ is positive definite. Thus, the assumptions of the maximum principle are satisfied, and we have $f = 0$ on M, i.e. each point of M is umbilical.

Remark. Let $V_1, V_2 \in T(M)$ be orthonormal vector fields. Choose orthonormal frames on each U_α in such a way that $v_1 = V_1$, $v_2 = V_2$. Define normal vector fields V_{11}, V_{12}, V_{22} by the relations

$$V_{11} = (V_1V_1)^N, \quad V_{12} = (V_1V_2)^N, \quad V_{22} = (V_2V_2)^N,$$

$(X)^N$ denoting the normal component of the vector field X. Then it is easy to see that

$$(43)\qquad V_{11} = a_1v_3 + a_2v_4, \quad V_{12} = b_1v_3 + b_2v_4, \quad V_{22} = c_1v_3 + c_2v_4.$$

Thus, the condition (iii) of Theorem 1 can be written, using (18) and (43), in the form

(iii′)
$$(\mathscr{H}_{11} - \mathscr{H}_{22}) \langle V_{11} - V_{22}, V_{11} + V_{22} \rangle + \\ + 4\mathscr{H}_{12} \langle V_{12}, V_{11} + V_{22} \rangle \geqq 0 \quad \text{on} \quad M .$$

The following theorem is a generalization of the preceding result. To establish it, we use the already mentioned property of the invariant J that this function does not contain $A_i, \ldots, E_i$ $(i = 1, 2)$.

Theorem 2. *Let M be a surface in E^4, ∂M its boundary. Let*

(i) $K > 0$ *on* M;

(ii) ξ *be pseudoparallel in* $P(M)$;

(iii) $(2 - \lambda) [(\mathscr{A} - \mathscr{C}) (\mathscr{H}_{11} - \mathscr{H}_{22}) + 4\mathscr{B}\mathscr{H}_{12}] + \lambda H[(\mathscr{H}_{11} + \mathscr{H}_{22}) - 4H(K_{11} + K_{22})] \geqq 0$ *on* M, $\lambda : M \to \boldsymbol{R}$ *being a function with* $|\lambda| \leqq 2$;

(iv) ∂M *consist of umbilical points.*

Then M is part of a 2-dimensional sphere in E^4.

Proof. Following the proof of Theorem 1 we have the equation (42). From (32) we obtain, using (33),

$$2Kf = H^{-2}J - 4[\gamma_1(\gamma_1 - \alpha_1) + \beta_1(\beta_1 - \delta_1) + \gamma_2(\gamma_2 - \alpha_2) + \beta_2(\beta_2 - \delta_2)] .$$

Multiplying this equation by a function λ and adding it to (42), we get

(44)
$$f_{11} + f_{22} - 2(2 - \lambda) Kf = H^{-2}(I + \lambda J) + 2\, W(\lambda)$$

where

$$W(\lambda) = V - 2\lambda[\gamma_1(\gamma_1 - \alpha_1) + \beta_1(\beta_1 - \delta_1) + \gamma_2(\gamma_2 - \alpha_2) + \beta_2(\beta_2 - \delta_2)]$$

and further, according to (40),

$$W(\lambda) = [\alpha_1^2 - 2(1 - \lambda) \alpha_1\gamma_1] + [\delta_1^2 - 2(1 - \lambda) \beta_1\delta_1] + (5 - 2\lambda) (\beta_1^2 + \gamma_1^2) + \\ + [\alpha_2^2 - 2(1 - \lambda) \alpha_2\gamma_2] + [\delta_2^2 - 2(1 - \lambda) \beta_2\delta_2] + (5 - 2\lambda) (\beta_2^2 + \gamma_2^2) .$$

Hence

(45)
$$W(\lambda) = [\alpha_1 - (1 - \lambda) \gamma_1]^2 + [\delta_1 - (1 - \lambda) \beta_1]^2 + (4 - \lambda^2) (\beta_1^2 + \gamma_1^2) + \\ + [\alpha_2 - (1 - \lambda) \gamma_2]^2 + [\delta_2 - (1 - \lambda) \beta_2]^2 + (4 - \lambda^2) (\beta_2^2 + \gamma_2^2) ,$$

so that, λ being a function such that $|\lambda| \leqq 2$, we conclude $W(\lambda) \geqq 0$. As

$$(46) \qquad (2 - \lambda)\left[(\mathcal{A} - \mathcal{C})(\mathcal{H}_{11} - \mathcal{H}_{22}) + 4\mathcal{B}\mathcal{H}_{12}\right] + \\ + \lambda H[(\mathcal{H}_{11} + \mathcal{H}_{22}) - 4H(K_{11} + K_{22})] = 2(I + \lambda J),$$

the equation (44) satisfies all the assumptions of the condition (3) of the maximum principle. Thus, for $|\lambda| \leqq 2$, $f = 0$ on M and the proof is complete.

Remark. It is easy to see that Theorem 2 contains as a special case, namely for $\lambda = 0$, the assertion of Theorem 1.

Corollary 1. *Let M be a surface in E^4 and ∂M its boundary. Let the conditions* (i), (ii) *and* (iv) *of Theorem* 2 *be satisfied on M. Let*

(iii) $\mathcal{A}\mathcal{H}_{11} + 2\mathcal{B}\mathcal{H}_{12} + \mathcal{C}\mathcal{H}_{22} - 2H^2(K_{11} + K_{22}) \geqq 0$ *on M.*
Then M is part of a 2-dimensional sphere in E^4.

Proof. The assertion follows from Theorem 2 when putting $\lambda = 1$ and using (19).

Remark. When using the notation (43), we can write the condition (iii) of Corollary 1 in the form

$$(\mathrm{iii}') \qquad \mathcal{H}_{11}\langle V_{11}, V_{11} + V_{22}\rangle + 2\mathcal{H}_{12}\langle V_{12}, V_{11} + V_{22}\rangle + \\ + \mathcal{H}_{22}\langle V_{22}, V_{11} + V_{22}\rangle - 2H^2(K_{11} + K_{22}) \geqq 0 \quad \text{on} \quad M.$$

Corollary 2. *Let M be a surface in E^4, ∂M its boundary, M having the properties* (ii) *and* (iv) *of Theorem* 2. *Let*

$$(\mathrm{iii}_1) \qquad \mathcal{H}_{11} + \mathcal{H}_{22} - 4H(K_{11} + K_{22}) \geqq 0 \quad on \quad M$$

or

$$(\mathrm{iii}_2) \qquad H_{11} + H_{22} - 4(K_{11} + K_{22}) \geqq 0 \quad on \quad M.$$

Then M is part of a 2-dimensional sphere in E^4.

Proof. The corollary with the supposition (iii_1) follows immediately from Theorem 2 for $\lambda = 2$. Thus, we are going to prove it when considering that (iii_2) is true.

Putting $\lambda = 2$ into the assertion of Theorem 2, we get from (44)

$$f_{11} + f_{22} = H^{-2}(I + 2J) + 2\,W(2)$$

where, according to (45),

$$W(2) = (\alpha_1 + \gamma_1)^2 + (\beta_1 + \delta_1)^2 + (\alpha_2 + \gamma_2)^2 + (\beta_2 + \delta_2)^2$$

and, because of (46),

$$I + 2J = H[(\mathscr{H}_{11} + \mathscr{H}_{22}) - 4H(K_{11} + K_{22})].$$

Further, using (35) implied by the condition (ii), we obtain

$$W(2) = \tfrac{1}{4}H^{-1}(H_1^2 + H_2^2)$$

and hence, according to (17),

$$H^{-2}(I + 2J) + 2\,W(2) = H_{11} + H_{22} - 4(K_{11} + K_{22}).$$

This completes our proof.

Remark. In fact, a little more general theorem involving the condition (iii$_2$) of Corollary 2 is valid. As proved in [2], we can omit the assumption of the pseudo-parallelness of the mean curvature vector field ξ to get the same inequality on M.

References

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Author's address: 602 00 Brno, Gorkého 13, ČSSR (Katedra matematiky FS VUT).