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FUNCTIONAL SEPARATION OF INDUCTIVE LIMITS AND REPRESENTATION OF PRESHEAVES BY SECTIONS PART TWO EMBEDDING OF PRESHEAVES INTO PRESHEAVES OF COMPACT SPACES

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1. EMBEDDINGS IN CUBES

Given a closured presheaf $\mathcal{S} = \{\mathcal{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$, suppose that $\mathcal{S}$ can be embedded into a larger one $\mathcal{S}^{**} = \{\mathcal{C}_\alpha | r_{\alpha\beta} | \langle A \leqq \rangle\}$. It means that for each $\alpha \in A$ there is a continuous 1-1 map $e_\alpha : \mathcal{X}_\alpha \to \mathcal{C}_\alpha$ such that the following diagram is commutative for any $\alpha, \beta \in A$, $\alpha \leqq \beta$:

(2.1.1)
$$\begin{array}{ccc} \mathcal{X}_\alpha & \xrightarrow{\varrho_{\alpha\beta}} & \mathcal{X}_\beta \\ \downarrow{\scriptstyle e_\alpha} & & \downarrow{\scriptstyle e_\beta} \\ \mathcal{C}_\alpha & \xrightarrow{r_{\alpha\beta}} & \mathcal{C}_\beta \end{array}$$

If $\mathcal{S}$ can be embedded into some $\mathcal{S}^{**}$ then by 1.1.1, there is a continuous 1-1 map e of $\mathcal{I} = \varinjlim \mathcal{S}$ into $\mathcal{J} = \varinjlim \mathcal{S}^{**}$. If $\mathcal{J}$ is f.s., then so is $\mathcal{I}$. This gives us a way how to prove the functional separatedness of $\mathcal{I}$. We shall study embeddings of $\mathcal{S}$ into some presheaves of compact spaces. These embeddings will be used in the last part of the paper.

2.1.2. Definition. A. A closured family $\mathcal{S} = \{\mathcal{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ (see 0.12) is called *topological* (T_1, *regular, completely regular, normal, compact*, ...) if $\mathcal{X}_\alpha$ is topological (T_1, *regular*, ...) for all $\alpha \in A$.

B. A *hull* (*weak hull*) of $\mathcal{S}$ is a pair $[\mathcal{C}, \mathcal{Z}]$, where $\mathcal{C} = \{\mathcal{C}_\alpha | r_{\alpha\beta} | \langle A \leqq \rangle\}$ is a topological inductive family and $\mathcal{Z} = \{e_\alpha : \mathcal{X}_\alpha \to \mathcal{C}_\alpha \mid \alpha \in A\}$, where each e_α is a continuous open 1-1 map (a continuous 1-1 map) into $\mathcal{C}_\alpha$ such that diagram 2.1.1 is commutative for any $\alpha, \beta \in A$, $\alpha \leqq \beta$.

Where possible, we omit the set $\mathcal{Z}$ saying that $\mathcal{C}$ is the hull of $\mathcal{S}$.

Notice that e_α is a homeomorphism of the topological modification $m\mathscr{X}_\alpha$ of $\mathscr{X}_\alpha$ into $\mathscr{C}_\alpha$ (see 0.9, 0.15) if $\mathscr{C}$ is a hull of $\mathscr{S}$. Thus if $\mathscr{S}$ is topological, then $e_\alpha : \mathscr{X}_\alpha \to \mathscr{C}_\alpha$ is a homeomorphism into $\mathscr{C}_\alpha$.

C. *A compact* (*weak compact*) hull of $\mathscr{S}$ is a compact inductive family $\mathscr{C}$ which is a hull (weak hull) of $\mathscr{S}$.

D. Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be from an i.c. category $\mathfrak{L}$. We say that $\mathscr{C}$ is a hull (weak hull) of $\mathscr{S}$ if $\mathscr{C}$ is *a hull* (*weak hull*) *of* cl $\mathscr{S}$ – see 0.9. (By 0.9, $\mathfrak{L}$ is a subcategory of CLOS or of SEM or of PROX (see 0.5, 0.10), so every $\mathscr{X}_\alpha$ is a closure or semiuniform or proximal space (X_α, τ_α). We denote by cl τ_α the closure generated in X_α by τ_α and put cl $\mathscr{S} = \{(X_\alpha, \text{cl}\,\tau_\alpha) | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$. Then cl $\mathscr{S}$ is from CLOS).

In the next lemma we collect together some well known facts and find some new ones, which will be used later.

2.1.3. Lemma. A. *Let $\mathscr{X} = (X, t)$ be a closure space, Q the compact unit interval, $F(X) \subset C(\mathscr{X} \to Q)$. We define a map $e_X : X \to Q^{F(X)}$ as follows: If $a \in X$ then $e_X(a) \in Q^{F(X)}$ is the map $e_X(a) : F(X) \to Q$, which to any $f \in F(X)$ assigns the number $e_X(a) f = f(a)$ (e_X is called the evaluation map.). We put $\mathscr{C}_X = (C_X, \vartheta_X) = Q^{F(X)}$, where ϑ_X is the product topology in $Q^{F(X)}$.*

(a) *If $F(X)$ separates points (points and points from closed sets) of $\mathscr{X}$, then $e_X : \mathscr{X} \to \mathscr{C}_X$ is* 1-1 *and continuous (a continuous open* 1-1 *map onto* ($e_X(X)$, ind ϑ_X), *hence a homeomorphism of $m\mathscr{X}$ into $\mathscr{C}_X$). Further, $\mathscr{X}$ is a T_1-space (i.e. the points of $\mathscr{X}$ are closed).*

(b) *If $\mathscr{Y} = (Y, t')$ is another closure space, $h : \mathscr{X} \to \mathscr{Y}$ a map such that the dual map h^* carries $F(Y)$ into $F(X)$, then the dual map h^{**} of h^* carries $\mathscr{C}_X$ continuously into $\mathscr{C}_Y = Q^{F(Y)}$ and this diagram is commutative:*

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\;h\;} & \mathscr{Y} \\ \downarrow e_X & & \downarrow e_Y \\ \mathscr{C}_X & \xrightarrow{\;h^{**}\;} & \mathscr{C}_Y \end{array}$$

(c) *If h^* carries $F(Y)$ onto $F(X)$, then h^{**} is* 1-1, *hence a homeomorphism of $\mathscr{C}_X$ into $\mathscr{C}_Y$.*

(d) *For $f \in F(X)$ let p_f be the f-th projection of $\mathscr{C}_X = Q^{F(X)}$ onto Q (p_f is defined for $\psi \in Q^{F(X)}$ by $p_f(\psi) = \psi(f)$). If $a \in X$, $\psi = e_X(a)$, then $p_f(e_X(a)) = e_X(a) f = f(a)$. Thus $p_f \circ e_X = f$ and p_f is an extension of $f \circ e^{-1}$ from $e_X(X)$ to the whole of $\mathscr{C}_X$. Setting $P\,F(X) = \{p_f \,|\, f \in F(X)\}$, we have $P\,F(X) \subset C(\mathscr{C}_X \to Q)$ and $P\,F(X)$ separates points of $\mathscr{C}_X$. If $g \in F(Y)$ and $h^*(g) \in F(X)$, then $h^{***} p_g = p_{h^* g}$. Thus if h^* carries $F(Y)$ into (onto) $F(X)$, then the dual map h^{***} of h^{**} carries $P\,F(Y)$ into (onto) $P\,F(X)$.*

B. *Let $\mathscr{X} = (X, t)$ be a closure space, $f \in C(X) = C(\mathscr{X} \to R)$, $F(X) \subset C(X)$. Put $Af = \frac{1}{2}(1 + (2/\pi) \operatorname{arctg} f)$, $A\,F(X) = \{Af \,|\, f \in F(X)\}$. Then $A\,F(X) \subset C(\mathscr{X} \to Q)$*

(Q *is the compact unit interval*), *and if* $F(X)$ *distinguishes points* (*points from closed sets*) *of* $\mathscr{X}$, *then so does* $A\,F(X)$. *If* $\mathscr{Y} = (Y, t')$ *is another closure space,* $h : \mathscr{X} \to \mathscr{Y}$ *continuous,* $f \in C(X)$, $g \in C(Y)$ *then* $h^*Ag = Af$ *if* $h^*g = f$. *Thus if* $F(X) \subset C(X)$, $F(Y) \subset C(Y)$ *are such that* h^* *carries* $F(Y)$ *into* (*onto*) $F(X)$, *then* h^* *carries* $A\,F(Y)$ *into* (*onto*) $A\,F(X)$.

More generally, if $\mathscr{X}$ *is an object of an i.c. category* $\mathfrak{L}$ *and* $f \in C(\mathscr{X} \to R \mid \mathfrak{L})$, *then* $Af \in C(\operatorname{cl} \mathscr{X} \to Q)$ (see 0.9). *Furthermore, let* C *be the field of complex numbers and* $X = |\mathscr{X}|$, $D(X) \subset C(\mathscr{X} \to C \mid \mathfrak{L})$ $(D^*(X) = C^*(\mathscr{X} \to C \mid \mathfrak{L}))$ *the set of all* (*bounded*) $\mathfrak{L}$ – *morphisms between* $\mathscr{X}$ *and* C (see 0.11). *If* $f \in D(X)$, $F(X) \subset D(X)$ *then* $f = f_1 + if_2$, *where* $f_1, f_2 \in C(\mathscr{X})$. *Putting* $Af = Af_1 + iAf_2$, $A\,F(X) = \{Af \mid f \in F(X)\}$, *we have* $A\,F(X) \subset D^*(X)$ *and similar statements as in the real case hold.*

C. *Let* X, Y, Z *be three sets and* $f : X \to Y$, $g : Y \to Z$, $h : X \to Z$ *maps,* $h = g \circ f$. *Let* $F(X), F(Y), F(Z)$ *be some sets of functions on* X, Y, Z *and* f^*, g^*, h^* *the dual maps. If* $f^*\,F(Y) \subset F(X)$, $g^*\,F(Z) \subset F(Y)$, *then* $h^*\,F(Z) \subset F(X)$ *and* $h^* = f^* \circ g^*$.

Proof. A: (a): If $F(X)$ distinguishes points of X, then by 1.1.1, there is a Hausdorff topology in X coarser than t, so $\mathscr{X}$ is T_1. For the rest see [9, Ch. 4, Lemma 5, p. 116]. (b): By [9, Ch. 5, Lemma 23, p. 152], h^{**} is continuous and into $Q^{F(Y)}$. If $a \in X$, we have $h^{**} \circ e_X(a) f = e_X(a)\, h^*f = h^*f(a) = f \circ h(a) = e_Y\, h(a) f$; hence $h^{**} \circ e_X = e_Y \circ h$ as desired.

(c): If $\xi \in Q^{F(X)}$, then ξ is a map of $F(X)$ into Q and $h^{**}(\xi) = \xi \circ h^* \in Q^{F(Y)}$, thus $h^{**}(\xi)$ is a map of $F(Y)$ into Q such that if $f \in F(Y)$, then $(\xi h^*) f = \xi(h^*f)$. Suppose $\varphi, \psi \in Q^{F(X)}$, $h^{**}(\varphi) = h^{**}(\psi)$. Then $\varphi \circ h^* = \psi \circ h^* \in Q^{F(Y)}$. It means that for any $f \in F(Y)$ we have $\varphi(h^*f) = \psi(h^*f)$. But if $g \in F(X)$ then $g = h^*f$ for some $f \in F(Y)$, hence $\varphi(g) = \psi(g)$ for any $g \in F(X)$, thus $\varphi = \psi$ and h^{**} is 1-1. Therefore, h^{**} is a homeomorphism since $\mathscr{C}_X$ is compact.

(d): The topology ϑ_X in C_X is projectively defined by the functions from $P\,F(X)$, hence $P\,F(X) \subset C(\mathscr{C}_X \to Q)$. If $\varphi, \psi \in \mathscr{C}_X$, $\varphi \neq \psi$, then there is $f \in F(X)$ with $\varphi(f) \neq \psi(f)$. Then $p_f(\varphi) = \varphi(f) \neq \psi(f) = p_f(\psi)$, whence $P\,F(X)$ separates points of $\mathscr{C}_X$.

Further, if $\varphi \in Q^{F(X)}$, $g \in F(Y)$, then $(h^{***}p_g)\,\varphi = p_g h^{**}\varphi = h^{**}\,\varphi(g) = \varphi(h^*g) = p_{h^*g}\varphi$, hence $h^{***}p_g = p_{h^*g}$ which proves (d). We have proven A, while B and C are clear.

2.1.4. Proposition. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ *be an inductive family from an i.c. category* $\mathfrak{L}$. *For every* $\alpha \in A$ *let us have a set* $F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$ *such that the dual map* $\varrho^*_{\alpha\beta}$ *carries* F_β *into* F_α *for all* $\alpha, \beta \in A$, $\alpha \leqq \beta$. *For* $\alpha \in A$ *let* $\mathscr{C}_\alpha = (C_\alpha, \vartheta_\alpha) = Q^{F_\alpha}$ *with the product topology* ϑ_α, $PF_\alpha = \{p_f \mid f \in F_\alpha\}$, $\mathscr{H} = \{PF_\alpha \mid \alpha \in A\}$, *and let* $e_\alpha : |\mathscr{X}_\alpha| \to C_\alpha$ *be the evaluation map* (*see* 2.1.3A). *For*

$\alpha, \beta \in A$ let $\varrho_{\alpha\beta}^{**}$ be the dual map of $\varrho_{\alpha\beta}^{*}$. We put $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$, $\mathscr{Z} = \{e_\alpha \mid \alpha \in A\}$. Then

(a) *If F_α separates points of $\mathscr{X}_\alpha$ (points, and points from closed sets of $\operatorname{cl} \mathscr{X}_\alpha$) for all $\alpha \in A$, then*

$$(2.1.5) \qquad \mathscr{S}^{**} = \langle \{\mathscr{C}_\alpha | \varrho_{\alpha\beta}^{**}| \langle A \leqq \rangle\}; \mathscr{Z} \rangle$$

is a weak compact (compact) hull of $\mathscr{S}$.

(b) *For $\alpha \in A$ we put $\mathscr{E}\mathscr{X}_\alpha = e_\alpha(\mathscr{X}_\alpha)^-$ – the ϑ_α-closure of $e_\alpha(\mathscr{X}_\alpha)$. Then $\varrho_{\alpha\beta}^{**}$ carries $\mathscr{E}\mathscr{X}_\alpha$ into $\mathscr{E}\mathscr{X}_\beta$ for $\alpha, \beta \in A$, $\alpha \leqq \beta$. Thus $\mathscr{E}\mathscr{S} = \{\mathscr{E}\mathscr{X}_\alpha | \varrho_{\alpha\beta}^{**}| \langle A \leqq \rangle\}$ is a weak compact (compact) hull of $\mathscr{S}$.*

(c) *If $\mathscr{G} = \{g_\alpha \mid \alpha \in A\}$ is a thread through $\mathscr{H}$, then $\mathscr{F} = \{f_\alpha = g_\alpha \circ e_\alpha \mid \alpha \in A\}$ is a thread through $\mathscr{E}$.*

Proof. If $\alpha \in A$ then $\mathscr{C}_\alpha$ is compact and the evaluations $e_\alpha : \operatorname{cl} \mathscr{X}_\alpha \to \mathscr{C}_\alpha$ are continuous 1-1 (continuous open 1-1) maps [9, Ch. 4, Lemma 5, p. 116], [9, Ch. 5, Th. 24, p. 103]. It remains to prove that $\mathscr{S}^{**}$ is an inductive family, i.e. that $\varrho_{\alpha\gamma}^{**} = \varrho_{\beta\gamma}^{**} \circ \varrho_{\alpha\beta}^{**}$ for $\alpha, \beta, \gamma \in A$, $\alpha \leqq \beta \leqq \gamma$. But this follows from 2.1.3C.

(b): Put $K_\alpha = \mathscr{E}\mathscr{X}_\alpha$, and for $M \subset C_\alpha$ let M^- be the ϑ_α – closure of M. We have to prove $\varrho_{\alpha\beta}^{**} K_\alpha \subset K_\beta$ for $\alpha, \beta \in A$, $\alpha \leqq \beta$. But it follows from the continuity of $\varrho_{\alpha\beta}^{**}$ as we have $e_\beta \varrho_{\alpha\beta}(\mathscr{X}_\alpha) \subset e_\beta(\mathscr{X}_\beta)$.

(c): For every $\alpha \in A$ there is $f_\alpha \in F_\alpha$ with $g_\alpha = p_{f_\alpha}$. If $\sigma, \delta \in A$, $\sigma \leqq \delta$ then we get $\varrho_{\sigma\delta}^{***} g_\delta = p_{\varrho^*_{\sigma\delta} f_\delta} = g_\sigma = p_{f_\sigma}$, $p_{f_\sigma} \circ e_\sigma = p_{\varrho^*_{\sigma\delta} f_\delta} \circ e_\sigma$ by 2.1.3Ad, hence $f_\sigma = \varrho_{\sigma\delta}^* f_\delta$.

2.1.6. Definition. The inductive family $\mathscr{S}^{**}$ or $\mathscr{E}\mathscr{S}$ is called the *$\mathscr{E}$-hull* or the *$\mathscr{E}$-closure of $\mathscr{S}$*, respectively. Each of them is a weak compact (compact) hull of $\mathscr{S}$ if F_α distinguishes points of $\mathscr{X}_\alpha$ (points, and points from closed sets of $\operatorname{cl} \mathscr{X}_\alpha$) for all $\alpha \in A$. In that case $\mathscr{S}^{**}$ is called *the $\mathscr{E}$-weak compact ($\mathscr{E}$-compact)* hull of $\mathscr{S}$. If $\mathscr{S}$ is completely regular and T_1, $F_\alpha = C(\mathscr{X}_\alpha \to Q)$ for $\alpha \in A$, $\beta = \{F_\alpha \mid \alpha \in A\}$, then $\beta\mathscr{X}_\alpha$ is the Stone-Čech compactification of $\mathscr{X}_\alpha$ [9, Ch. 5, p. 152], and $\beta\mathscr{S}$ is called *the Stone-Čech compact hull of $\mathscr{S}$.*

2.1.7. Theorem. *Given an i.c. category $\mathfrak{L}$, a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ from $\mathfrak{L}$ and a set $B \subset A$ such that $\langle B \leqq \rangle$ is well ordered, assume that*

(1) *Either B is cofinal in $\langle A \leqq \rangle$, or $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}$.*

(2) *$\mathscr{S}_B$ is endowed with a smooth and connected separating family $\tilde{\mathscr{E}} = \{\tilde{F}_\alpha \mid \alpha \in B\}$ (see 1.1.5) such that $\varrho_{\alpha\beta}^*$ carries $\tilde{F}_\beta$ into $\tilde{F}_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$.*

Put $\mathscr{E} = \{F_\alpha = \frac{1}{2}(1 + (2/\pi) \operatorname{arctg} \tilde{F}_\alpha) \mid \alpha \in B\}$ and denote by $\mathscr{T}$ the $\mathscr{E}$-hull of $\mathscr{S}_B$ (see 2.1.2D, 2.1.4A, 2.1.3B, 2.1.6). If each $\tilde{F}_\alpha$ distinguishes points of $\mathscr{X}_\alpha$, then $\mathscr{K} = \varinjlim \mathscr{T}$ is functionally separated. Furthermore, $\mathscr{J} = \varinjlim \mathscr{S}_B$ and $\mathscr{I} = \varinjlim \mathscr{S}$ are f.s. by $C(\mathscr{I} \to R \mid \mathfrak{L})$. If each $\tilde{F}_\alpha$ distinquishes points and points from closed sets of $\operatorname{cl} \mathscr{X}_\alpha$ then $\mathscr{T}$ is a compact hull of $\mathscr{S}_B$. If there is a countable confinal set

$C \subset B$ and if $\varrho^*_{\alpha\beta}(\tilde{F}_\beta) \subset \tilde{F}_\alpha$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$ then the assumption of connectedness of $\mathcal{E}$ may be omitted.

Proof. Recall that $F_\alpha \subset C(\mathrm{cl}\, \mathcal{X}_\alpha \to Q)$ – see 0.9, 2.1.3B. Let $\mathcal{T} = \{\mathcal{C}_\alpha = (C_\alpha, \vartheta_\alpha) \,|\varrho^{**}_{\alpha\beta}|\, \langle B \leqq \rangle\}$ be the $\mathcal{E}$ – hull of $\mathcal{S}_B$ constructed by 2.1.4 (see 2 1.6). Here $C_\alpha = Q^{F_\alpha}$ and ϑ_α is the product topology. Let $e_\alpha : \mathrm{cl}\, \mathcal{X}_\alpha \to \mathcal{C}_\alpha$ be the evaluation map (see 2.1.3A) and put $\mathcal{H} = \{PF_\alpha \mid \alpha \in B\}$. Here $PF_\alpha = \{p_g \mid g \in F_\alpha\}$, where p_g is the g-th projection of Q^{F_α} onto Q. We prove that $\mathcal{T}$ and $\mathcal{H}$ fulfil the conditions of Th. 1.1.7. By 2.1.3Ad, $\mathcal{H}$ is leftward smooth for so is $\mathcal{E}$ by 2.1.3B. Now we prove the connectedness of $\mathcal{H}$. Given $\alpha \in B$ so that the predecessor $\alpha - 1$ of α in $\langle B \leqq \rangle$ does not exist, $\beta \in B - \mathcal{L}(\mathcal{T})$, and a thread $\mathcal{G} = \{g_\gamma \mid \gamma \in \langle \beta\alpha) \cap B\}$ thourgh $\mathcal{H}$, then $\mathcal{F} = \{f_\gamma = g_\gamma \circ e_\gamma \mid \gamma \in \langle \beta\alpha) \cap B\}$ is a thread through $\mathcal{E}$ by 2.1.4c. It follows easily from 2.1.3B, 1.1.5B that $\mathcal{E}$ is fully connected for so is $\tilde{\mathcal{E}}$. (Use the inverse map for $\frac{1}{2}(1 + (2/\pi) \operatorname{arctg} x)$.) Thus there is $f \in F_\alpha$ with $\varrho^*_{\gamma\alpha} f = f_\gamma$ for all $\gamma \in \langle \beta\alpha) \cap B$ (here we need $\mathcal{E}$ to be fully connected as β need not be from $B - \mathcal{L}(\mathcal{S}_B)$). By 2.1.3A,d, we get $\varrho^{***}_{\gamma\alpha} p_f = p_{f_\gamma} = g_\gamma$ for these γ. As $p_f \in PF_\alpha$, the connectedness of $\mathcal{H}$ is proven. As $F_\alpha \subset C(\mathrm{cl}\, \mathcal{X}_\alpha \to Q)$ for all $\alpha \in B$ (see 0.11), we get from 2.1.3A, b that $\varrho^{**}_{\alpha\beta} \mathcal{C}_\alpha \to \mathcal{C}_\beta$ are continuous, therefore $\mathcal{T}$ is from CLOS. Thus $\mathcal{T}$ and $\mathcal{H}$ satisfy the conditions of 1.1.7, hence $\mathcal{K}$ is f.s.

If $\xi_\alpha : \mathcal{X}_\alpha \to \mathcal{J}$ and $\eta_\alpha : \mathcal{C}_\alpha \to \mathcal{K}$ are the canonical maps for $\alpha \in B$, then η_α are 1-1. Indeed, if $\alpha, \beta \in B$, $\alpha \leqq \beta$, then the assumption (2) together with 2.1.3B and 1.1.6 implies yields that $\varrho^*_{\alpha\beta}$ carries F_β onto F_α. By 2.1.3A, c, $\varrho^{**}_{\alpha\beta} : \mathcal{C}_\alpha \to \mathcal{C}_\beta$ is 1-1. By 0.10 (3b), η_α are 1-1. Let $p, q \in \mathcal{J}$, $p \neq q$. There is $\alpha \in B$ such that there are representatives $a \in \mathcal{X}_\alpha$ of p and $b \in \mathcal{X}_\alpha$ of q, $a \neq b$. Setting $r = e_\alpha(a)$, $s = e_\alpha(b)$ we have $r \neq s$, $r, s \in \mathcal{C}_\alpha$, and by Th. 1.1.7 there is $f \in C(\mathcal{K} \to R)$ with $f \circ \eta_\alpha(s) \neq f \circ \eta_\alpha(r)$ and with $f \circ \eta_\gamma \in PF_\gamma$ for all $\gamma \in B(\alpha) = \{\gamma \in B \mid \gamma \geqq \alpha\}$. Since $\{f \circ \eta_\gamma \mid \gamma \in B(\alpha)\}$ is a thread through $\mathcal{H}_{B(\alpha)}$, we get from 2.1.4c that $\{f_\gamma = f \circ \eta_\gamma \circ e_\gamma \mid \gamma \in B(\alpha)\}$ is a thread through $\mathcal{E}_{B(\alpha)}$. We have $\mathcal{J} = \varinjlim \mathcal{S}_{B(\alpha)}$ since $B(\alpha)$ is confinal in $\langle B \leqq \rangle$. Thus there is $g \in C(\mathcal{J} \to R \mid \mathfrak{L})$ with $g \circ \xi_\gamma = f_\gamma$ for all $\gamma \in B(\alpha)$. We have $g(p) = g \circ \xi_\alpha(a) = f_\alpha(a) = f \circ \eta_\alpha \circ e_\alpha(a) = f \circ \eta_\alpha(s) \neq f \circ \eta_\alpha(r) = f \circ \eta_\alpha \circ e_\alpha(b) = f_\alpha(b) = g \circ \xi_\alpha(b) = g(q)$ as desired. The rest follows from 1.4.2. Using 1.2.5 to $\mathcal{S}_B$, C, and to the property $P_{\alpha\beta} : \varrho^*_{\alpha\beta} F_\beta = F_\alpha$; if D is the set from 1.2.5 of the type ω_0, then by the just proven part of 2.1.7 we get that $\varinjlim \mathcal{S}_D$ is f.s. by $C(\mathcal{J} \to R \mid \mathfrak{L})$. Now we again use 1.4.2. The theorem is proven.

2.1.8. Remark. Th. 1.5.2 and Corollaries 1.5.4, 1.5.5 follow directly from Th. 2.1.7.

2. EMBEDDINGS IN SPACES OF MAXIMAL IDEALS

Instead of embedding a topological, completely regular, T_1 space into a cube $Q^{F(X)}$, where $F(X)$ is a set of continuous functions on (X, t), we can embed $\mathcal{X}$ into the space of maximal ideals of a Banach algebra $\mathcal{A}$ of complex functions on $\mathcal{X}$, or into a space of continuous linear multiplicative functionals on $\mathscr{A}$, which is the same. We shall use this to embed a presheaf into a hull.

2.2.1. Definition. Let $\mathscr{A}, \mathscr{B}$ be complex Banach algebras. A linear map $f : \mathscr{A} \to \mathscr{B}$ is called *multiplicative* if $f(x \,.\, y) = f(x) \,.\, f(y)$ for $x, y \in \mathscr{A}$. If $\mathscr{B}$ is the field of complex numbers then f is called a *multiplicative linear function*. The set of all continuous maps or functions of this kind is denoted respectively by $ML(\mathscr{A} \to \mathscr{B})$ or $\mathscr{F}(\mathscr{A})$. (An algebra $\mathscr{A}$ with unity is called a *Banach algebra* if it is a complex Banach space where the multiplication $(x, y) \to x \,.\, y$ is a continuous map of $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$).

2.2.2. Lemma. A. *Any $a \in \mathscr{A}$ can be assigned a complex function $\hat{a}$ on $\mathscr{F}(\mathscr{A})$ as follows: If $f \in \mathscr{F}(\mathscr{A})$, we put $\hat{a}(f) = f(a)$. In this way, we get a set $\mathscr{D}(\mathscr{A}) = \{\hat{a} \mid a \in \mathscr{A}\}$ of functions on $\mathscr{F}(\mathscr{A})$. Endowing $\mathscr{F}(\mathscr{A})$ with the topology $t_{\mathscr{A}}$ projectively defined by the functions from $\mathscr{D}(\mathscr{A})$, we get a compact space $(\mathscr{F}(\mathscr{A}), t_{\mathscr{A}})$. Further, $\mathscr{D}(\mathscr{A})$ distinguishes points and closed sets of $(\mathscr{F}(\mathscr{A}), t_{\mathscr{A}})$. As $(\mathscr{F}(\mathscr{A}), t_{\mathscr{A}})$ is compact, the points of $\mathscr{F}(\mathscr{A})$ are closed. Thus $\mathscr{D}(\mathscr{A})$ distinguishes points of $\mathscr{F}(\mathscr{A})$.*

B. *Let (X, t) be a closure space. Consider a Banach algebra $\mathscr{A}_X \subset C^*((X, t) \to C)$ with the sup-norm $\|f\| = \sup\{|f(x)| \mid x \in X\}$ (C is the field of complex numbers). We say that $\mathscr{A}_X$ is symmetric if $g \in \mathscr{A}_X$ yields $\bar{g} \in \mathscr{A}_X$ ($\bar{g}$ is the complex conjugate of g). We can assign to any $x \in X$ an element $i_X(x) \in \mathscr{F}(\mathscr{A}_X)$ as follows: If $f \in \mathscr{A}_X$ then $i_X(x) f = f(x)$. This evaluation map $i_X : (X, t) \to (\mathscr{F}(\mathscr{A}_X), t_{\mathscr{A}_X})$ is continuous. It is* 1-1 *if $\mathscr{A}_X$ distinguishes points of X. Moreover, i_X is open if $\mathscr{A}_X$ distinguishes points and closed sets.*

C. *If $a \in \mathscr{A}_X$, $x \in X$, then $\hat{a} \circ i_X = a$, hence the function $\hat{a} \in \mathscr{D}(\mathscr{A})$ from the statement* A *is a continuous extension of $a \circ i_X^{-1} : i_X(X) \to C$ to the whole of $(\mathscr{F}(\mathscr{A}_X), t_{\mathscr{A}_X})$.*

D. *Let $\mathscr{A}, \mathscr{B}$ be two Banach algebras, $f : \mathscr{B} \to \mathscr{A}$ a map. Let f^* be the dual map for f, defined for $\psi \in \mathscr{F}(\mathscr{A})$, $b \in \mathscr{B}$ by $f^* \psi(b) = \psi \circ f(b)$. Then f^* carries $\mathscr{F}(\mathscr{A})$ into $\mathscr{F}(\mathscr{B})$. If f^{**} is the dual map for f^* and $b \in \mathscr{B}$, then $f^{**}(\hat{b}) = \widehat{f(b)}$. Therefore f^{**} carries $\mathscr{D}(\mathscr{B})$ into (onto) $\mathscr{D}(\mathscr{A})$ if f carries $\mathscr{B}$ into (onto) $\mathscr{A}$.*

Proof. A: $\mathscr{D}(\mathscr{A})$ distinguishes points and points from closed sets in $(\mathscr{F}(\mathscr{A}), t_{\mathscr{A}})$ for $t_{\mathscr{A}}$ is projectively defined by $\mathscr{D}(\mathscr{A})$. The rest of A follows easily from [7, Ch. 4, § 3, Th. 4.15.2].

C: If $x \in X$, $a \in \mathscr{A}_X$ then $\hat{a} \circ i_X(x) = i_X(x)(a) = a(x)$ as desired.

B: As $t_{\mathscr{A}_X}$ is projectively defined by $\mathscr{D}(\mathscr{A}_X)$, the map $i_X : (X, t) \to (\mathscr{F}(\mathscr{A}_X), t_{\mathscr{A}_X})$ is continuous iff so is $\hat{a} \circ i_X : (X, t) \to C$ for any $\hat{a} \in \mathscr{D}(\mathscr{A}_X)$. But $\hat{a} \circ i_X = a$ by C, and $a : (X, t) \to C$ is continuous. The other statements can be proven likewise as in [9, Ch. 4, Lemma 5, p. 1.1.6].

D: If $b \in \mathscr{B}$ then $\hat{b} \in \mathscr{D}(\mathscr{B})$ and $f^{**}(\hat{b}) \in \mathscr{D}(\mathscr{A})$. If $\varphi \in \mathscr{F}(\mathscr{A})$ then $f^{**}(\hat{b})\, \varphi = \hat{b} f^* \varphi = f^* \varphi(b) = \varphi \circ f(b) = \widehat{f(b)}\,(\varphi)$, which proves D.

2.2.3. Lemma. A: *Let $\mathscr{A}, \mathscr{B}$ be two complex Banach algebras, $h \in ML(\mathscr{A} \to \mathscr{B})$. Then the dual map $h^* : (\mathscr{F}(\mathscr{B}), t_{\mathscr{B}}) \to (\mathscr{F}(\mathscr{A}), t_{\mathscr{A}})$ is continuous.*

B: *Let $\mathscr{X}$ be an object from an i.c. category $\mathfrak{L}$, $X = |\mathscr{X}|$, $\mathscr{A} \subset C^*(\mathscr{X} \to C \mid \mathfrak{L})$ a symmetric Banach algebra* (see 2.2.2B) *with the sup-norm. Then the map $m : \mathscr{A} \to$ $\to C^{***} = C^*((\mathscr{F}(\mathscr{A}), t_{\mathscr{A}}) \to C)$ defined by $m(a) = \hat{a}$ is an isometric isomorphism onto C^{***} (with the sup-norm). Further, $i_X(X)$ is dense in $(\mathscr{F}(\mathscr{A}), t_{\mathscr{A}})$. If $f \in C^{***}$, then $\widehat{f \circ i_X} = f$, hence $C^{***} = \mathscr{D}(\mathscr{A})$.*

C: *Let $\mathscr{X} = (X, t)$, $\mathscr{Y} = (Y, t')$ be two closure spaces, $h : X \to Y$ a map, $\mathscr{A}_X$, $\mathscr{A}_Y$ some Banach algebras of bounded continuous complex functions on $\mathscr{X}, \mathscr{Y}$ with the supnorm such that h^* carries $\mathscr{A}_Y$ into $\mathscr{A}_X$. Then $h^* \in ML(\mathscr{A}_Y \to \mathscr{A}_X)$.*

Proof. A: By 2.2.2A, $t_{\mathscr{A}}$ is projectively defined by the functions from $\mathscr{D}(\mathscr{A})$. Thus h^* is continuous iff $\hat{a} \circ h^*$ is for any $\hat{a} \in \mathscr{D}(\mathscr{A})$. By 2.2.2D, $\hat{a} \circ h^* = h^{**}\hat{a} =$ $= \widehat{h(a)}$ and $\widehat{h(a)} : (\mathscr{F}(\mathscr{B}), t_{\mathscr{B}}) \to C$ is continuous by the definition of $t_{\mathscr{B}}$.

B: By [7, Ch. 1, § 8, Th. 3, p. 62], m is an isometric isomorphism onto C^{***}. If M is the $t_{\mathscr{A}}$ – closure of $i_X(X)$ and M were not the whole $\mathscr{F}(\mathscr{A})$, then there would be $y \in \mathscr{F}(\mathscr{A}) - M$. By 2.2.2A, thre is $a \in \mathscr{D}(A)$ with $\hat{a}(y) \neq 0$, $\hat{a} = 0$ on M. By 2.2.2C, $\hat{a} \circ i_X = a$ hence $a = 0$. But we have $0 = \|a\| = \|m(a)\| = \|\hat{a}\| \neq 0$ – a contradiction. If $a \in \mathscr{A}$ then $\hat{a} \circ i_X = a$ by 2.2.2C. If $f \in C^{***}$ then there is $a \in \mathscr{A}$ with $f = m(a) = \hat{a}$, so $f \circ i_X = \hat{a} \circ i_X = a \in \mathscr{A}$. Further, $\widehat{f \circ i_X} = \hat{a} = f$. The rest and C are clear.

2.2.4. Lemma. *Let $\mathscr{X}, \mathscr{Y}$ be two closure spaces, $\mathscr{A}, \mathscr{B}$ some Banach algebras of continuous complex bounded functions on $\mathscr{X}, \mathscr{Y}$ with the sup-norm. Suppose $h : \mathscr{X} \to \mathscr{Y}$ is a map such that h^* carries $\mathscr{B}$ into $\mathscr{A}$. Then*

(a) *The dual map $h^{**} : (\mathscr{F}(\mathscr{A}), t_{\mathscr{A}}) \to (\mathscr{F}(\mathscr{B}), t_{\mathscr{B}})$ is continuous.*

(b) *If $h^*(\mathscr{B})$ is dense in $\mathscr{A}$, then h^{**} is* 1-1, *hence it is a homeomorphism into $(\mathscr{F}(\mathscr{B}), t_{\mathscr{B}})$.*

(c) *This diagram is commutative (i_X, i_Y are the canonical evaluations of $\mathscr{X}$ and $\mathscr{Y}$ into $\mathscr{F}(\mathscr{A})$ and $\mathscr{F}(\mathscr{B})$ – see* 2.2.2B):

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\quad h \quad} & \mathscr{Y} \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ (\mathscr{F}(\mathscr{A}), t_{\mathscr{A}}) & \xrightarrow{\quad h^{**} \quad} & (\mathscr{F}(\mathscr{B}), t_{\mathscr{B}}) \end{array}$$

(d) *If $\mathscr{B}$ is symmetric* (see 2.2.2B) *and $h^*(\mathscr{B})$ norm-dense in $\mathscr{A}$ then $h^*(\mathscr{B}) = \mathscr{A}$.*

Proof. (a): By 2.2.3C, $h^* \in ML(\mathscr{B} \to \mathscr{A})$. Now we use 2.2.3A.

(b): If $f, g \in \mathscr{F}(\mathscr{A})$, $h^{**}f = h^{**}g$, then $f \circ h^* = g \circ h^*$, hence f and g coincide on $h^*(\mathscr{B})$. The continuity of f, g and the density of $h^*(\mathscr{B})$ yield $f = g$.

(c): Given $x \in X$, $b \in \mathscr{B}$ then $h^{**} \circ i_X(x)\, b = i_X(x)\, h^*(b) = b \circ h(x) = i_Y\, h(x)\, b$.

(d): If $a \in \mathscr{A}$ then $\hat{a} \in \mathscr{D}(\mathscr{A})$. By (b), $h^{**} : (\mathscr{F}(\mathscr{A}), t_{\mathscr{A}}) \to (\mathscr{F}(\mathscr{B}), t_{\mathscr{B}})$ is a homeomorphism into (see 0.15), so there is $g \in C^{***} = C^*((\mathscr{F}(\mathscr{B}), t_{\mathscr{B}}) \to C)$ such that $\hat{a} = h^{***}g$. By 2.2.3B, $l = g \circ i_Y \in \mathscr{B}$ and $\hat{l} = g$. By 2.2.2C, D, $a = \hat{a} \circ i_X = h^{***}g \circ i_X = h^{***}\hat{l} \circ i_X = \widehat{h^* l} \circ i_X = h^*(l)$ as desired.

2.2.5. Lemma. A. *Let F be a set of complex bounded functions on a set S. Then there is a smallest Banach algebra $\mathscr{A}_S(F)$ (a symmetric Banach algebra $Z_S(F)$) of bounded complex functions on S, with the sup-norm, which contains F; $\mathscr{A}_S(F)$. $(Z_S(F))$ is called the (symmetric) algebraic hull of F. If all $f \in F$ are real and F is an algebra over R, complete in the sup-norm, then $\mathscr{A}_S(F) = \{f + ig \mid f, g \in F\}$, and $\mathscr{A}_S(F)$ is symmetric. If S is an object from an i.c. category $\mathfrak{L}$ and $F \subset C^* = C^*(S \to C \mid \mathfrak{L})$ then $\mathscr{A}_S(F) \subset C^*$ $(Z_S(F) \subset C^*)$ if C^* is a Banach algebra (symmetric Banach algebra) with the usual sup-norm. If T, G is another pair of the same kind as S, F and if $h : S \to T$ is a map such that h^* carries G onto a norm dense subset of F, then h^* carries $\mathscr{A}_T(G)$ $(Z_T(G))$ onto a norm dense subset of $\mathscr{A}_S(F)$ $(Z_S(F))$.*

B. *Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ br a presheaf from an i.c. category $\mathfrak{L}$. Suppose that for every $\alpha \in A$ we have a Banach algebra $\mathscr{A}_\alpha \subset C^*(\mathscr{X}_\alpha \to C \mid \mathfrak{L})$ with the sup-norm, which separates points of $\mathscr{X}_\alpha$ (points, and points from closed sets of* cl $\mathscr{X}_\alpha$ *– see* 0.9) *and such that $\varrho^*_{\alpha\beta}$ carries $\mathscr{A}_\beta$ into $\mathscr{A}_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. For each $\alpha \in A$ let $\mathscr{F}_\alpha$ be the set of all continuous complex multiplicative linear functionals on $\mathscr{A}_\alpha$ with the topology t_α projectively defined by the functions from $\mathscr{D}(\mathscr{A}_\alpha)$, and let $i_\alpha : |\mathscr{X}_\alpha| \to (\mathscr{F}_\alpha, t)$ be the canonical evaluations. We put $\mathscr{E}^\beta = \{\mathscr{A}_\alpha \mid \alpha \in A\}$, $\mathscr{L}^\beta = \{i_\alpha \mid \alpha \in A\}$, $\mathscr{E}^\beta\mathscr{S} = \{(\mathscr{F}_\alpha, t_\alpha) | \varrho^{**}_{\alpha\beta} | \langle A \leqq \rangle\}$ $\mathscr{D}(\mathscr{E}^\beta) = \{\mathscr{D}(\mathscr{A}_\alpha) \mid \alpha \in A\}$. Then $\langle \mathscr{E}^\beta\mathscr{S}, \mathscr{L}^\beta \rangle$ is a weak compact (compact) hull of $\mathscr{S}$ – see* 2.1.2D. *If $\mathscr{F}' = \{f_\alpha \in \mathscr{D}(\mathscr{A}_\alpha) \mid \alpha \in A\}$ is a thread through $\mathscr{D}(\mathscr{E}^\beta)$, (it means $\varrho^{***}_{\alpha\beta} f_\beta = f_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$), then $\mathscr{F} = \{a_\alpha = f_\alpha \circ i_\alpha \mid \alpha \in A\}$ is a thread through $\mathscr{E}^\beta$* – see 1.1.5.

C. *Given an i.c. category $\mathfrak{L}$, $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from $\mathfrak{L}$, $\langle A \leqq \rangle$ well ordered, suppose that for every $\alpha \in A$ we have a Banach algebra $\mathscr{A}_\alpha \subset C^*(\mathscr{X}_\alpha \to C \mid \mathfrak{L})$ with the sup-norm, such that $\varrho^*_{\alpha\beta}$ carries $\mathscr{A}_\beta$ into $\mathscr{A}_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. If $\alpha \in A$ is such that there is no predecessor $\alpha - 1$, we put $G'_\alpha = \{f \in C^*(\mathscr{X}_\alpha \to C \mid \mathfrak{L}) \mid \varrho^*_{\gamma\alpha} f \in \mathscr{A}_\gamma$ for all $\gamma \in A[\alpha]\}$. Then G'_α is a Banach algebra (in the sup-norm). Further, if $\mathscr{A}_\alpha$ is symmetric for all $\alpha \in A$, then so are all the G'_α.*

Proof. Clearly $\mathscr{A}_S(F)$ $(Z_S(F))$ is the smallest (symmetric) subalgebra of the Banach algebra $\mathscr{A}$ of all bounded complex functions on S with the sup-norm. Thus $\mathscr{A}_S(F)$. $(Z_S(F))$ is the closure of $\mathscr{S}(F)$ in $\mathscr{A}$, where $\mathscr{S}(F)$ is the smallest (symmetric) algebra which contains F. It consists of all finite sums of the form $\lambda_1 m_1 + \ldots + \lambda_k m_k$ (and of all their complex conjugates) where λ_i are complex numbers and the m_i's are finite products of some elements $p^i_1, \ldots, p^i_{s_i}$ from F, $i = 1, \ldots, k$. Using 2.2.3C to $\mathscr{S}(F)$,

$\mathscr{S}(G)$, h^*, we get that $h^* \in ML(\mathscr{S}(G) \to \mathscr{S}(F))$ (recall that the sets are regarded as topological spaces with the discrete topology – see 0.9), which together with the continuity of addition and multiplication in $\mathscr{A}$ yields our statement about h^*. The rest is clear.

B. By 2.2.2A, D, $(\mathscr{F}_\alpha, t_\alpha)$ are compact and $\varrho_{\alpha\beta}^{**}$ carries $\mathscr{F}_\alpha$ into $\mathscr{F}_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$. By 2.2.4c, the diagram 2.1.1 is commutative for $\mathscr{E}^\beta(\mathscr{S})$, and by 2.2.2B the evaluations i_α are 1-1 and continuous (1-1, open and continuous), hence $\mathscr{E}^\beta(\mathscr{S})$ is a weak compact (compact) hull of $\mathscr{S}$ (see 1.1.2B). If $\mathscr{F}' = \{f_\alpha \in \mathscr{D}(\mathscr{A}_\alpha) \mid \alpha \in A\}$ is a thread through $\mathscr{D}(\mathscr{E}^\beta)$, $\alpha, \beta \in A$, $\alpha \leqq \beta$, then we put $\mathscr{F} = \{a_\alpha = f_\alpha \circ i_\alpha \mid \alpha \in A\}$. From $\varrho_{\alpha\beta}^{***} f_\beta = f_\alpha$ we get $a_\alpha = f_\alpha \circ i_\alpha = \varrho_{\alpha\beta}^{***} f_\beta \circ i_\alpha = f_\beta \circ \varrho_{\alpha\beta}^{**} \circ i_\alpha = f_\beta \circ i_\beta \circ \varrho_{\alpha\beta} = a_\beta \circ \varrho_{\alpha\beta} = \varrho_{\alpha\beta}^{*}(a_\beta)$ (we have $\varrho_{\alpha\beta}^{**} \circ i_\alpha = i_\beta \circ \varrho_{\alpha\beta}$ since the diagram 2.1.1 is commutative for $\mathscr{E}^\beta\mathscr{S}$). Thus $\mathscr{F}$ is a thread through $\mathscr{E}^\beta$.

C. Given $f_n \in G'_\alpha$, $f \in C^* = C^*(\text{cl}\, \mathscr{X}_\alpha \to C)$ (see 2.2.2D), $f_n \to f$ in the sup-norm, then the continuity of $\varrho_{\gamma\alpha}^*$ on C^* yields $g_n^\gamma = \varrho_{\gamma\alpha}^* f_n \to \varrho_{\gamma\alpha}^* f = g_\gamma \in \mathscr{A}_\gamma$ for all $\gamma \in A[\alpha]$ (we have $f_n \in C^*$). By 0.20 we get $f \in G'_\alpha$ as desired. The rest is clear.

2.2.6. Definition. The family $\mathscr{E}^\beta\mathscr{S}$ from 2.2.5B will be called *an $\mathscr{E}^\beta$ hull of $\mathscr{S}$* (the index β is added to distinguish the hull of $\mathscr{S}$ in 2.1.6, which consists of cubes, from that one which consists of $\mathscr{F}_\alpha$. $\mathscr{E}^\beta\mathscr{S}$ is the Stone-Čech compactification of $\mathscr{S}$ if $\mathscr{S}$ is T_1, completely regular and $\mathscr{A}_\alpha = C^*(\mathscr{X}_\alpha \to R)$ – see [7, Ch 8, § 43]).

2.2.7. Theorem. *Given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ from an i.c. category $\mathfrak{L}$ and $B \subset A$ such that $\langle B \leqq \rangle$ is well ordered, suppose that*

(1) *Either B is cofinal in $\langle A \leqq \rangle$, or $\langle A \leqq \rangle$ is ordered, $\langle B - A \leqq \rangle$ well ordered and $B - A \subset \mathscr{L}$.*

(2) *For every $\alpha \in A$ we have a separating set $F_\alpha \subset C^*(\mathscr{X}_\alpha \to C \mid \mathfrak{L})$ (see 0.11) which is either a symmetric Banach algebra with the sup-norm or an algebra of real functions over the field of real numbers, complete in the sup-norm, such that*

(a) *$\varrho_{\alpha\beta}^*$ carries F_β into F_α for all $\alpha, \beta \in A$, $\alpha \leqq \beta$.*

(b) *$\varrho_{\alpha\alpha+1}^*$ carries $F_{\alpha+1}$ onto a norm dense subset of F_α.*

(c) *The family $\mathscr{E}' = \{F_\alpha \mid \alpha \in B\}$ is connected.*

If $\mathscr{A}_\alpha$ is the symmetric algebraic hull of F_α (see 2.2.5A) for $\alpha \in B$, $\mathscr{E} = \{\mathscr{A}_\alpha \mid \alpha \in B\}$ and if $\mathscr{T}$ is the $\mathscr{E}^\beta$-hull of $\mathscr{S}_B$ (see 2.2.6) then $\mathscr{K} = \varprojlim \mathscr{T}$ is f.s. Furthermore, $\mathscr{J} = \varprojlim \mathscr{S}_B$ and $\mathscr{I} = \varprojlim \mathscr{S}$ are f.s. by $C^*(\mathscr{I} \to R \mid \mathfrak{L})$. If moreover, every $\mathscr{A}_\alpha$ separates points and points from closed sets of $\text{cl}\, \mathscr{X}_\alpha$ (wchich holds if so does every F_α) then $\mathscr{T}$ is a compact hull of $\mathscr{S}_B$.

If there is a countable cofinal set C in B and if $\varrho_{\alpha\beta}^*(F_\beta)$ is norm dense in F_α for any $\alpha, \beta \in B$, $\alpha \leqq \beta$ then the condition (2c) may be left out.

Proof. Let $\mathscr{T} = \{(\mathscr{F}_\alpha, t_\alpha) |\varrho_{\alpha\beta}^{**}| \langle A \leqq \rangle\}$ be the $\mathscr{E}^\beta$-hull of $\mathscr{S}_B$ (see 2.1.2D, 2.2.6). Here $\mathscr{F}_\alpha$ is the set of all continuous complex multiplicative linear functionals on $\mathscr{A}_\alpha$

with the topology t_α projectively defined by $\mathscr{D}(\mathscr{A}_\alpha)$ – see 2.2.5B. Let $i_\alpha : |\mathscr{X}_\alpha| \to \mathscr{F}_\alpha$ be the evaluations. We put $\mathscr{H} = \mathscr{D}(\mathscr{E}) = \{\mathscr{D}(\mathscr{A}_\alpha) \mid \alpha \in A\}$. By 2.2.3B and 2.2.5A, $\mathscr{D}(\mathscr{A}_\alpha) = C^*((\mathscr{F}_\alpha, t_\alpha) \to C)$ – see 0.14. We shall show that $\mathscr{T}$ and $\mathscr{H}$ fulfil the conditions of Th. 1.1.7.

By 2.2.2A, $\mathscr{D}(\mathscr{A}_\alpha)$ separates points of $\mathscr{F}_\alpha$, hence $\mathscr{H}$ is separating. By 2.2.4d and 2.2.5A, $\mathscr{E}$ is leftward smooth, thus by 2.2.2D, $\mathscr{H}$ is smooth for the $\mathscr{A}_\alpha$'s are symmetric. We prove the full connectedness of $\mathscr{H}$ (see 1.1.5A, B). Given $\alpha \in B$ such that $\alpha - 1$ does not exist, $\beta \in B[\alpha]$, and a thread $\mathscr{G} = \{g_\gamma \in \mathscr{D}(\mathscr{A}_\gamma) \mid \gamma \in \langle\beta\alpha) \cap B\}$ through $\mathscr{H}_{\langle\beta\alpha)\cap B}$, then $\mathscr{F} = \{f_\gamma = g_\gamma \circ i_\gamma \mid \gamma \in \langle\beta\alpha) \cap B\}$ is a thread through $\mathscr{E}_{\langle\beta\alpha)\cap B}$ by 2.2.5B. We can easily get from 2.2.5A – as $\mathscr{A}_\alpha = \{f + ig \mid f, g \in F_\alpha\}$ – and from 5.1.5B, that $\mathscr{E}$ is fully connected, whence there is $f \in \mathscr{A}_\alpha$ with $\varrho^*_{\gamma\alpha} f = f_\gamma$ for all $\gamma \in \langle\beta\alpha) \cap B$. By 2.2.2D, $\widehat{\varrho^*_{\gamma\alpha} f} = \hat{f}_\gamma = \varrho^{***}_{\gamma\alpha} \hat{f}$, thus $\mathscr{H}$ is connected for $\hat{f} \in \mathscr{D}(\mathscr{A}_\alpha)$. By 2.2.3A, all $\varrho^{**}_{\alpha\beta} : (\mathscr{F}_\alpha, t_\alpha) \to (\mathscr{F}_\beta, t_\beta)$ are continuous whence $\mathscr{T} \subset$ CLOS. Thus $\mathscr{H}$ and $\mathscr{T}$ fulfil the conditions of Th. 1.1.7, hence $\mathscr{K} = \varinjlim \mathscr{T}$ is f.s.

If $\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{J}$ and $\eta_\alpha : \mathscr{F}_\alpha \to \mathscr{K}$ are the canonical maps for $\alpha \in B$ then, because all $\varrho^{**}_{\alpha\beta} \mathscr{F}_\alpha \to \mathscr{F}_\beta$ are 1-1 by 2.2.4B, 2.2.5A, we get by 0.10 (3b) that η_α is 1-1 for all $\alpha \in B$. Let $p, q \in \mathscr{J}$, $p \neq q$. There is $\alpha \in B$ such that there are representatives $a \in \mathscr{X}_\alpha$ of p and $b \in \mathscr{X}_\alpha$ of q, $a \neq b$. Setting $r = i_\alpha(a)$, $s = i_\alpha(b)$, we have $r \neq s$, $r, s \in \mathscr{F}_\alpha$, and by Th. 1.1.7 there is $f \in C(\mathscr{K} \to R)$ with $f \circ \eta_\alpha(s) \neq f \circ \eta_\alpha(r)$ and with $f \circ \eta_\gamma \in \mathscr{D}(\mathscr{A}_\gamma)$ for all $\gamma \in B(\alpha) = \{\gamma \in B \mid \gamma \geqq \alpha\}$. Since $\{f \circ \eta_\gamma \mid \gamma \in B(\alpha)\}$ is a thread through $\mathscr{H}_{B(\alpha)}$, we get from 2.2.5B that $\{f_\gamma = f \circ \eta_\gamma \circ i_\gamma \mid \gamma \in B(\alpha)\}$ is a thread through $\mathscr{E}_{B(\alpha)}$. We have $\mathscr{J} = \varinjlim \mathscr{S}_{B(\alpha)}$ since $B(\alpha)$ is cofinal in $\langle B \leqq \rangle$. Thus there is $g \in C(\mathscr{J} \to R \mid \mathfrak{L})$ with $g \circ \xi_\gamma = f_\gamma$ for all $\gamma \in B(\alpha)$. We have $g(p) = g \circ \xi_\alpha(a) = f_\alpha(a) = f \circ \eta_\alpha \circ i_\alpha(a) = f \circ \eta_\alpha(r) \neq f \circ \eta_\alpha(s) = f \circ \eta_\alpha \circ i_\alpha(b) = f_\alpha(b) = g \circ \xi_\alpha(b) = g(q)$ as desired. The rest follows from 1.4.2. The last assertion follows from 1.2.6 because 2a, 2b, 2c yield that $\mathscr{J}$ is f.s. by $C(\mathscr{J} \to R \mid \mathfrak{L})$. Taking the condition 2a as Q in 1.2.6, we get that $\varinjlim \mathscr{S}_D = \mathscr{K}$ is f.s. by $C(\mathscr{K} \to R \mid \mathfrak{L})$ for a countable cofinal set $D \subset B$. Now 1.4.2 completes the proof.

2.2.8. Theorem. *Given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from an i.c. category $\mathfrak{L}$ and $B \subset A$ such that $\langle B \leqq \rangle$ is well ordered and that there is a countable cofinal set $C \subset B$, suppose that the condition (1) of Th. 2.2.7 is fulfilled and (2) $\mathscr{S}_B$ is endowed with a rightward smooth separating family $\mathscr{G} = \{F_\alpha \subset C_\alpha = C^*(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in B\}$ (see 1.1.5) such that $\varrho^*_{\alpha\beta} F_\beta$ is norm dense in F_α for all $\alpha, \beta \in B$, $\alpha \leqq \beta$.*

If $\mathscr{A}_\alpha$ is the symmetric algebraic hull of F_α (see 2.2.5A) for $\alpha \in B$, $\mathscr{E} = \{\mathscr{A}_\alpha \mid \alpha \in B\}$ and $\mathscr{T} = \{(\mathscr{F}_\alpha, t_\alpha) | \varrho^{**}_{\alpha\beta} | \langle B \leqq \rangle\}$ is the $\mathscr{E}^\beta$ – hull of $\mathscr{S}_B$ (see 2.2.6) then $\mathscr{K} = \varinjlim \mathscr{T}$ is f.s. If moreover, for every $\alpha \in B$ the set C_α is a Banach algebra with the usual sup-norm, then $\mathscr{J} = \varinjlim \mathscr{S}_B$ and $\mathscr{J} = \varinjlim \mathscr{S}$ are f.s. by $C(\mathscr{J} \to R \mid \mathfrak{L})$. Further, if $\mathscr{E}$ is strongly separating then $\mathscr{T}$ is a compact hull of $\mathscr{S}_B$.

If every F_α *is a Banach algebra and* $\mathscr{W} = \{\mathscr{C}_\alpha | \varrho^{**}_{\alpha\beta}| \langle \beta \leqq \rangle\}$ *is the* $\mathscr{G}^\beta$ – *hull of* $\mathscr{S}_B$ *then* $\varprojlim \mathscr{W} = \mathscr{K}'$ *is f.s. If* $\mathscr{G}$ *is strongly separating then* $\mathscr{W}$ *is a compact hull of* $\mathscr{S}_B$.

Proof. Let the C_α's be Banach algebras. By 2.2.5A, $\mathscr{E}$ is rightward smooth separating, $\mathscr{A}_\alpha \subset C_\alpha$ and $\varrho^*_{\alpha\beta}\mathscr{A}_\beta$ is norm dense in $\mathscr{A}_\alpha$, hence $\mathscr{S}_B$ and $\mathscr{E}$ fulfil the conditions of Th. 2.2.7.

If the C_α's fail to be Banach algebras then 1.5.6B yields that $\mathscr{K}$ is f.s. as $\varrho^{**}_{\alpha\alpha+1}$ is 1-1 on $\mathscr{F}_\alpha$ for all $\alpha \in B$ by 2.2.4b.

If every F_α is a Banach algebra, then $\mathscr{W}$ is a weak compact hull of $\mathscr{S}$ by 2.2.5B (a compact one if $\mathscr{G}$ is trongly separating) and the functional separatedness of $\mathscr{K}'$ follows from the statement in 1.5.6B as $\varrho^{**}_{\alpha\alpha+1}$ is 1-1 on $\mathscr{C}_\alpha$ by 2.24b. The theorem is proven .

If the presheaf $\mathscr{S}$ is endowed with a leftward smooth, connected and separating family $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$, then the functional separatedness of $\varprojlim \mathscr{S}$ follows from Th. 1.1.7. However, that theorem does not work if $\mathscr{E}$ is not leftward smooth. Nevertheless, if every $\varrho^*_{\alpha\beta}$ sends F_β into F_α and every F_α is a symmetric Banach algebra such that $\varrho^*_{\alpha\alpha+1}F_{\alpha+1}$ is norm dense in F_α, then Th. 2.2.7 can be used. (By 2.2.4d, if the F_α's are symmetric then $\mathscr{E}$ is smooth so 2.1.7 and 1.1.7 work, too. Indeed, putting $\mathscr{R}\mathscr{E} = \{\mathscr{R}\mathscr{A}_\alpha \mid \alpha \in A\}$ where $\mathscr{R}\mathscr{A}_\alpha = \{f_1 \mid f_1 + if_2 \in \mathscr{A}_\alpha\}$, we see that $\mathscr{R}\mathscr{E}$ is smooth, connected and separating. From this it can be seen that in case of symmetric F_α's 2.2.7 follows from 2.1.7.) If A contains a countable cofinal subset then F_α's may be any separating sets of functions such that $\varrho^*_{\alpha\beta}F_\beta$ is norm-dense in F_α. Then 2.2.8 works and is, in this case, a generalization of 2.2.7, 2.1.7, 1.1.7. If the set B in 2.2.8 contains no countable cofinal set, then the connectedness of $\mathscr{E}$ makes difficulty even if we assume the connectedness of $\mathscr{G}$ (see 2.2.8). Thus we can see that if B is arbitrary, then 2.1.7 is more general than 2.2.7 for 2.2.7 follows from 2.1.7. But if there is a countable cofinal set in B, then 2.2.7 assumes the form of 2.2.8 and 2.2.8 implies 2.1.7.

2.2.9. Theorem. *Let* $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ *be a presheaf from an i.c. category* $\mathfrak{L}$, *whose canonical maps* $\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I} = \varprojlim \mathscr{S}$ *are* 1-1 (*this holds if all* $\varrho_{\alpha\beta}$ *are* 1-1 – *see* 0.10). *Suppose* $\mathscr{I}$ *is f.s. by* $C' = C(\mathscr{I} \to R \mid \mathfrak{L})$. *If we set* $C = \{\frac{1}{2}(1 + (2/\pi) \,.\, \operatorname{arctg} f) \mid f \in C'\}$, $\mathscr{E} = \{F_\alpha = \xi^*_\alpha C \mid \alpha \in A\}$ *then* $\mathscr{E}$ *is smooth and separating. If* $\mathscr{T}$ *is the* $\mathscr{E}$ – *weak compact hull of* $\mathscr{S}$ (*see* 2.1.1D, 2.1.4a), *then* $\mathscr{K} = \varprojlim \mathscr{T}$ *is f.s. If* $\mathscr{E}$ *is strongly separating* (*which holds if there is a strongly separating, smooth and connected family* $\mathscr{G} = \{G_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$ *for* $\mathscr{S}$), *then* $\mathscr{T}$ *is a compact hull of* $\mathscr{S}$.

Proof. As ξ_α are 1-1 so the family $\mathscr{E}$ is separating, smooth and $F_\alpha \subset C(\operatorname{cl} \mathscr{X}_\alpha \to Q)$ for all $\alpha \in A$ (see 2.1.2D, 2.1.3B; Q is the compact unit interval). Let $\mathscr{T} = \{\mathscr{C}_\alpha | \varrho^{**}_{\alpha\beta}| \,.\, \langle A \leqq \rangle\}$ be the $\mathscr{E}$ – hull of $\mathscr{S}$ by 2.1.6, $\mathscr{K} = \varprojlim \mathscr{T}$, $p, q \in K$, $p \neq q$. There is $\alpha \in A$ such that there are representatives $a, b \in \mathscr{C}_\alpha$ of p, q. We have $a \neq b$ and a, b are unique. Indeed, all $\varrho^*_{\alpha\beta}$ carry F_β onto F_α hence all $\varrho^{**}_{\alpha\beta}$ are 1-1 (see 2.1.3A, c), which together with 0.10 (3b) gives that all the canonical maps $\eta_\alpha : \mathscr{C}_\alpha \to \mathscr{K}$ are 1-1. For $\alpha \in A$ let $p_f : Q^{F_\alpha} = |\mathscr{C}_\alpha| \to Q$ be the f-th projection (see 2.1.3A, d) and $P\mathscr{E} = \{PF_\alpha \mid \alpha \in A\}$, where $PF_\alpha = \{p_f \mid f \in F_\alpha\}$. There is $f_\alpha \in F_\alpha$ with $p_{f_\alpha}(a) \neq p_{f_\alpha}(b)$ (see 2.1.3A, d), and $f \in C$ with $\xi^*_\alpha f = f_\alpha$. Then $F = \{f_\gamma = \xi^*_\gamma f \mid \gamma \in A(\alpha) = \{\beta \in A \mid \beta \geqq \alpha\}\}$ is a thread through $\mathscr{E}$. By 2.1.3A, d, $\mathscr{G}' = \{g_\gamma = p_{f_\gamma} \mid \gamma \in A(\alpha)\}$ is a thread through $P\mathscr{E}$ with $g_\gamma\, \varrho^{**}_{\alpha\gamma}(a) \neq g_\gamma\, \varrho^{**}_{\alpha\gamma}(b)$. Putting $g = \underrightarrow{\lim}\, \mathscr{G}'$, we have $g \in C(\mathscr{K} \to R)$ and $g(p) \neq g(q)$ as desired.

If $\mathscr{G}$ is strongly separating, smooth and connected, then $\mathscr{E}$ is strongly separating since $G_\alpha \subset \xi^*_\alpha C'$ for all $\alpha \in A$. Indeed, if $g \in G_\alpha$ then by induction we can make a thread $\mathscr{H} = \{g_\gamma \mid \gamma \in A(\alpha)\}$ through g with $g_\alpha = g$. Then $h = \underrightarrow{\lim}\, \mathscr{H} \in C'$ and $\xi^*_\alpha h = g$. The theorem is proved.

The family $P\mathscr{E}$ from the proof need not be connected but still we could prove that $\mathscr{K}$ is f.s.

We have proved in Ths. 2.1.7, 2.2.7, 2.2.8 that certain weak compact hulls of $\mathscr{S}$ have f.s. inductive limits. These hulls are the $\mathscr{E}$-hulls ($\mathscr{E}^\beta$-hulls) of $\mathscr{S}$ by certain fully connected separating families $\mathscr{E}$ of sets (algebras) of functions which depend on $\mathscr{S}$. In Th. 2.2.9 we have established the existence of a hull whose inductive limit is f.s. That hull was not made with the help of a connected family. Moreover, it depends on $\mathscr{I}$.

2.2.10. Proposition. *Given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from an i.c. category $\mathfrak{L}$, $\langle A \leqq \rangle$ well ordered, let us consider the statements*

(1) *There is a leftward smooth, connected and separating family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L}) \mid \alpha \in A\}$ for $\mathscr{S}$.*

(2) *$\mathscr{I} = \underrightarrow{\lim}\, \mathscr{S}$ is f.s.*

(3) *There is a weak compact hull $\mathscr{T}$ of $\mathscr{S}$ such that $\underrightarrow{\lim}\, \mathscr{T}$ is f.s.*

If each $\varrho_{\alpha\beta}$ is 1-1 then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) is proven in Th. 1.1.7 and in Th. 2.2.9, respectively.

3. EMBEDDINGS INTO ONE-POINT COMPACTIFICATIONS

2.3.1. Definition. Let $\mathscr{X} = (X, t)$ be a locally compact (shortly l.c.) topological space $f \in C(\mathscr{X} \to C)$ (C is the field of complex numbers). We say that f *has limit zero at infinity* if for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ on $X - K$. The set of all such functions is denoted by $\mathscr{Z}^\infty_X$. Let c be a complex number. We say that f has limit c at infinity if $f - c \in \mathscr{Z}^\infty_X$. We write $c = \lim f$. The set of all functions which have a limit at infinity is denoted by $\mathscr{L}^\infty_X$.

2.3.2. Lemma. *Let $\mathscr{X} = (X, t)$ be topological and locally compact. Then $\mathscr{L}^\infty_X$ is a symmetric Banach subalgebra of $\mathscr{A} = C^*(\mathscr{X} \to C)$ with the sup-norm, which distinguishes points and closed sets.*

Proof. Clearly $\mathscr{L}_X^\infty$ is a symmetric subalgebra of $\mathscr{A}$. If $f \in \mathscr{A}$, $f_n \in \mathscr{L}_X^\infty$, $f_n \to f$ in the sup-norm, then each f_n has a limit a_n at infinity. By a well known theorem concerning uniform convergence of functional sequences, there are finite $\lim a_n = a$, $\lim f = l$ and $a = l$. Thus $\mathscr{L}_X^\infty$ is closed in $\mathscr{A}$, hence it is a Banach algebra.

Given a closed $M \subset X$, $x \in X - M$, there is a one-point compactification of $\mathscr{X}$, i.e. a compact topological space $\mathscr{Y} = (Y, t')$ and a homeomorphism $e : \mathscr{X} \to \mathscr{Y}$ into $\mathscr{Y}$ (see 0.15) such that $Y - e(X)$ is one point p [9, Ch. 5, p. 150]. Then $e(x) \notin e(M)^-$, where $e(M)^-$ is the t'-closure of $e(M)$. There is $g \in C^*(\mathscr{Y} \to C)$ such that $g = 1$ on $e(M)^-$, $g \circ e(x) = 0$. Then $f = g \circ e$ distinguishes M and x. As the family $\{e(X - K) \mid K \subset X \text{ is compact}\}$ is a filter base of t'-neighborhoods of p [9, Ch 5, p. 156], we have $\lim f = g(p)$, hence $f \in \mathscr{L}_X^\infty$ as desired.

In this section, the words "locally compact" will be often shortened to "l.c.". and the word "neighborhood" to "nbd".

2.3.3. Lemma. *Let $\mathscr{X} = (X, t)$, $\mathscr{Y} = (Y, t')$ be l.c., $h : \mathscr{X} \to \mathscr{Y}$ a homeomorphism into $\mathscr{Y}$, $\mathscr{B} = \{h(X - K) \mid K \subset X$ compact$\}$. Then $\mathscr{B}$ is a filter base in Y.*

A. *A point $y \in Y$ is a cluster point of $\mathscr{B}$ iff for no compact t'-nbd K_y of y the set $h(X) \cap K_y$ is compact (y is a cluster* point *of $\mathscr{B}$ iff $y \in \bigcap\{M^- \mid M \in \mathscr{B}\}$, where M^- is the t'-closure of M).*

B. *Suppose $\mathscr{B}$ has no cluster point in Y. If L is a compact subset of Y, then $L \cap h(X)$ is compact.*

Proof. A: Let $h(X) \cap K_y$ be compact for a compact t'-nbd K_y of y. Take a compact t'-nbd L_y of y such that $L_y \subset \operatorname{int} K_y$. Then L_y does not intersect the set $M = h(X - h^{-1}(K_y))$, so $y \notin M^-$. But $h^{-1}(K_y)$ is compact in $\mathscr{X}$, hence y is not a cluster point of $\mathscr{B}$. Conversely, if y is not a cluster point of $\mathscr{B}$ then there is a compact $K \subset X$ with $y \notin (h(X - K))^-$, hence there is a compact t'-nbd K_y of y such that $K_y \cap h(X - K) = \emptyset$, thus $K_y \cap h(X) - K_y \cap h(K) = \emptyset$ and $N = K_y \cap h(X) \subset h(K)$. But N is closed in $h(X)$, $h(K)$ is compact, thus so is N which proves A.

B: By A, every point $x \in L$ has a compact t'-nbd K_x such that $K_x \cap h(X)$ is compact. Choose a finite cover $\{K_x \mid x \in F\}$ of L. Then $M = U\{K_x \mid x \in F\}$ and $h(X) \cap M$ is compact, $h(X) \cap L$ is closed in $h(X)$, hence it is compact, being a subset of $h(X) \cap M$.

2.3.4. Lemma. *Let $(X, t), (Y, t')$ be l.c., $h : (X, t) \to (Y, t')$ a homeomorphism into, (R, u), (S, v) the one point compactifications of (X, t), (Y, t'). We set $R - X = \{p\}$, $S - Y = \{q\}$. Then there is a continuous extension $\hat{h} : (R, u) \to (S, v)$ of h iff either $\mathscr{B} = \{h(X - K) \mid K \subset X$ compact$\}$ has no cluster point in (Y, t'), or $\mathscr{B}$ has a limit point in (Y, t') (we write $b = \lim \mathscr{B}$). Further, $\hat{h}(p) = q$ iff $\mathscr{B}$ has no cluster point in Y, and $\hat{h}(p) \in Y$ iff there is $b = \lim \mathscr{B} \in Y$ (in this case $b = \hat{h}(p)$, $b \notin h(X)$). If there exists $\hat{h}$ then it is* 1-1.

Proof. Necessity: We may suppose $X \subset R$, $Y \subset S$. The set $\{X - K \mid K \subset X$ compact$\}$ is a filter base of u-nbds of p [9, Ch. 5, p. 150], hence h has a continuous extension $\hat{h} : (R, u) \to (S, v)$ iff $\hat{h}(p) = \lim \mathscr{B}$. If $\hat{h}(p) \in Y$ then $\lim \mathscr{B} = \hat{h}(p) \in Y$. If $\hat{h}(p) \notin Y$ then $\hat{h}(p) = q$ and clearly $\mathscr{B}$ has no cluster point in (Y, t') as $\hat{h}(p)$ is the only cluster point of $\mathscr{B}$ in S. This proves the necessity. Conversely, if there is $b = \lim \mathscr{B} \in Y$, we may put $\hat{h}(p) = b$, $\hat{h} = h$ on X and the map $\hat{h} : (R, u) \to (S, v)$ is continuous. If $\mathscr{B}$ has no cluster point in (Y, t), then q is the limit of $\mathscr{B}$ in (S, v). Indeed, by 2.3.3B, if $L \subset Y$ is compact, then $K = L \cap h(X)$ as well as $h^{-1}(K)$ are compact and $h(X - h^{-1}(K)) \subset Y - L$ as desired. We put $\hat{h}(p) = q$, $\hat{h} = h$ on X. Then $\hat{h} : (R, u) \to (S, v)$ is continuous which proves the sufficiency. Suppose that $\hat{h}$ exists. Then either $\mathscr{B}$ has no cluster point – and then $\hat{h}(p) = q$ and $\hat{h}$ is 1-1 – or there is $\lim \mathscr{B} = b$. If it were $b \in h(X)$, then we should have $\hat{h}(R) = h(X)$. Thus $h(X) \cap U$ is compact for any compact t'-nbd U of b. By 2.3.3A, $U \cap h(X)$ is not compact for any such U – a contradiction which completes the proof.

2.3.5. Lemma. *Let (X, t), (Y, t') be l.c., $h : (X, t) \to (Y, t')$ a homeomorphism into (Y, t').*

A. *$h^* \mathscr{L}_Y^\infty \subset \mathscr{L}_X^\infty$ iff either $\mathscr{B}$ has no cluster point in (Y, t') or $\mathscr{B}$ has a limit point in (Y, t') ($\mathscr{B}$ is from 2.3.3).*

B. *$h^* \mathscr{L}_Y^\infty$ is dense in $\mathscr{L}_X^\infty$ if $h^* \mathscr{L}_Y^\infty \subset \mathscr{L}_X^\infty$.*

Proof. A: Let $\mathscr{B}$ have no cluster point, $f \in \mathscr{L}_Y^\infty$, $a = \lim f$. We prove that h^*f has a limit at infinity. Given $\varepsilon > 0$ and a compact set $L \subset Y$ such that $|f - a| < \varepsilon$ on $Y - L$, we see by 2.3.3B that $h(X) \cap L = K$ is compact, thus also $h^{-1}(K)$ is compact. We have $|h^*f - a| < \varepsilon$ on $X - h^{-1}(K)$, hence $\lim h^*f = a$. If $\mathscr{B}$ has a limit l in (Y, t') then clearly $\lim h^*f = f(l)$ which proves the "if" part.

Let $\mathscr{B}$ have neither no cluster point in (Y, t'), nor a limit point. Thus there is a cluster point c of $\mathscr{B}$ which is not the limit of $\mathscr{B}$. Thus there is a t'-nbd K_c of c such that for any compact $K \subset X$ we have $h(X - K) \not\subset K_c$. Thus $h(X - K) - K_c \neq \emptyset$ for any compact $K \subset X$. Let (Z, u) be the one-point compactification of (Y, t'), $e : (Y, t') \to (Z, u)$ the homeomorphism into (Z, u), where $Z - e(Y)$ is a single point $\{p\}$. Then $\mathscr{W} = \{e(h(X - K) - K_c) \mid K \subset X \text{ compact}\}$ is a filter base in (Z, u) which has a cluster point $z \in Z$ as (Z, u) is compact. Clearly $e(c) \neq z$. We take $g \in C((Z, u) \to C)$ with $g(e(c)) = 0$, $g(z) = g(p) = 1$. Then $f = g \circ e \in \mathscr{L}_Y^\infty$ (see the end of the proof of 2.3.2) but $h^*f \notin \mathscr{L}_X^\infty$. Indeed, given a compact set $K \subset X$, then $U_z = \{t \in Z \mid |g(t)| > \frac{3}{4}\}$ is a u-nbd of z. As z is a cluster point of $\mathscr{W}$, there is $a \in U_z \cap e(h(X - K) - K_c)$. Thus $x = h^{-1} \circ e^{-1}(a) \in X - K$ and $h^*f(x) > \frac{3}{4}$. We may suppose $|g| < \frac{1}{4}$ on $e(K_c)$, otherwise we can take a smaller K_c. As c is a cluster point of $\mathscr{B}$, there is $b \in h(X - K) \cap K_c$. Then $y = h^{-1}(b) \in X - K$ and $h^* f(y) < \frac{1}{4}$. For any compact $K \subset X$ we have found two points $x, y \in X - K$ with $h^* f(x) > \frac{3}{4}$, $h^* f(y) < \frac{1}{4}$, hence $h^*f \notin \mathscr{L}_X^\infty$.

B: Let (R', u), (S, v) be the one-point compactifications of (X, t) and (Y, t'), respectively. As $h^* \mathscr{L}_Y^\infty \subset \mathscr{L}_X^\infty$, we get from 2.3.5A and 2.3.3 that there is a continuous extension $\hat{h} : (R', u) \to (S, v)$ of h which is 1-1. Clearly $M = \hat{h}^* C((S, v) \to C)$ is a symmetric subalgebra (see 2.2.2B) of $C = C((R', u) \to C)$ which separates points of R' and contains any constant function. By the Stone-Weierstrass theorem [5, Ch. 8, Sec. 3, p. 283] M is norm dense in C. Thus $h^* \mathscr{L}_Y^\infty$ is dense in $\mathscr{L}_X^\infty$ since $\mathscr{L}_X^\infty = \{f/X \mid f \in C\}$, $\mathscr{L}_Y^\infty = \{g/Y \mid g \in C((S, v) \to C)\}$, $h = \hat{h}/X$. The lemma is proven.

2.3.6. Theorem. *Given a locally compact presheaf* $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, \tau_\alpha) | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ (*see* 2.1.2A) *and* $B \subset A$ *such that* $\langle B \leqq \rangle$ *is well ordered, suppose that* $\mathscr{S}$ *is from* CLOS (*i.e.* $\varrho_{\alpha\beta} : \mathscr{X}_\alpha \to \mathscr{X}_\beta$ *is continuous for all* $\alpha, \beta \in A$, $\alpha \leqq \beta$) *and*

(1) *Either* B *is confinal in* $\langle A \leqq \rangle$ *or* $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ *well ordered and* $A - B \subset \mathscr{L}$.

(2) (a) *For every* $\alpha, \beta \in B$ *the map* $\varrho_{\alpha\beta}$ *is a homeomorphism of* $\mathscr{X}_\alpha$ *into* $\mathscr{X}_\beta$ *such that the filter base* $\mathscr{B}_{\alpha\beta} = \{\varrho_{\alpha\beta}(X_\alpha - K) \mid K \subset X_\alpha$ *compact*$\}$ *either has no cluster point or has a limit point in* $\mathscr{X}_\beta$.

(b) *The family* $\mathscr{E} = \{\mathscr{A}_\alpha = \mathscr{L}_{X_\alpha}^\infty \mid \alpha \in B\}$ *is connected.* (*This is always satisfied if the following holds*: *If* $\alpha \in B$ *is such that the predecessor* $\alpha - 1$ *of* α *in* $\langle B \leqq \rangle$ *does not exist, and if* $\lambda_\alpha : \mathscr{L}_\alpha = \varinjlim \mathscr{S}_{B[\alpha]} \to \mathscr{X}_\alpha$ *is the canonical map, then* λ_α^* *carries* $\mathscr{A}_\alpha$ *onto a norm dense subset of* $G_\alpha' = \{f \in C(\mathscr{L}_\alpha \to C) \mid \varrho_{\gamma\alpha}'^* f \in \mathscr{A}_\gamma$ *for all* $\gamma \in B[\alpha]\}$). *Then the* $\mathscr{E}^\beta$*-hull* $\mathscr{T}$ *of* $\mathscr{S}_B$ *is a compact hull of* $\mathscr{S}_B$ *and* $\mathscr{K} = \varinjlim \mathscr{T}$, $\mathscr{J} = \varinjlim \mathscr{S}_B$, $\mathscr{I} = \varinjlim \mathscr{S}$ *are f.s. The condition* (2b) *may be omitted if there is a countable confinal set in B.* (*Here* $\varrho_{\gamma\alpha}' : \mathscr{X}_\alpha \to \mathscr{L}_\alpha$ *are the canonical maps*).

Proof. By 2.3.2 and 2.3.5B, $\mathscr{A}_\alpha$ separates points and closed sets of $\mathscr{X}$ and $\varrho_{\alpha,\alpha+1}^* \mathscr{A}_{\alpha+1}$ is norm dense in $\mathscr{A}_\alpha$ for all $\alpha \in B$. By 2.3.5A, $\varrho_{\alpha\beta}^*$ maps $\mathscr{A}_\beta$ into $\mathscr{A}_\alpha$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$, thus the conditions of Th. 2.2.7 are fulfilled which yields our statement. The statement in the parentheses in (2b) follows from 2.2.5C and 2.2.4d.

2.3.7. Lemma. *Let* (X, t), (Y, t') *be l.c.,* $h : (X, t) \to (Y, t')$ *a homeomorphism into* (Y, t'). *Then* $\mathscr{L}_X^\infty \subset h^* \mathscr{L}_Y^\infty$.

Proof. Let $f \in \mathscr{L}_X^\infty$, $l = \lim f$. We denote by (Z, u) the onepoint compactification of (Y, t'). We may suppose $Y \subset Z$, $Z - Y = \{p\}$ – a single point, $t' = u/Y$ (see 0.14). Then the function $\tilde{f} = f \circ h^{-1}$ on $h(X)$, $\tilde{f} = l$ on $h(X)^- - h(X)$ is defined on $h(X)^-$ ($h(X)^-$ is the u – closure of $h(X)$). We show that $\tilde{f}$ is $u/h(X)^-$ continuous. To this end we prove this statement (S): If $z \in h(X)^- - h(X)$, $\varepsilon > 0$, then there is an open u-nbd U of z such that $|f \circ h^{-1} - l| < \varepsilon$ on $U \cap h(X)$. Indeed, as $f \in \mathscr{L}_X^\infty$, there is a compact $K \subset X$ such that $|f - l| < \varepsilon$ on $X - K$. Thus $|f \circ h^{-1} - l| < \varepsilon$ on $h(X) - h(K)$. Then $U = Z - h(K)$ has the desired property, which proves (S).

By (S), $\tilde{f}$ is continuous at the points of $h(X)^- - h(X)$. We prove the continuity of $\tilde{f}$ at the points of $h(X)$. If $z \in h(X)$, $\varepsilon > 0$ then there is u-nbd V of z such that $|\tilde{f}(y) - \tilde{f}(z)| < \varepsilon$ for all $y \in V \cap h(X)$. Let $y \in V \cap (h(X)^- - h(X))$. By (S) we can take a u-nbd U of y with $|\tilde{f} - l| < \varepsilon$ on $U \cap h(X)^-$. We may assume $U \subset V$ and take $x \in U \cap h(X)$. Then $|\tilde{f}(z) - \tilde{f}(y)| = |\tilde{f}(z) - l| \leqq |\tilde{f}(z) - \tilde{f}(x)| + |\tilde{f}(x) - l| <$

$< 2\varepsilon$ so $\tilde{f}$ is $u/h(X)^-$ – continuous. There is an extension $\tilde{g} \in C((Z, u) \to C)$ of $\tilde{f}$. Setting $g = \tilde{g}/Y$ we have $g \in \mathscr{L}_Y^\infty$ and $h^*g = f$. The proof is thereby finished.

2.3.8. Theorem. *Given a locally compact presheaf* $\mathscr{S} = \{\mathscr{X}_\alpha = (X_\alpha, \tau_\alpha) \,|\varrho_{\alpha\beta}|\, \langle A \leqq \rangle\}$ *from* CLOS *and a set* $B \subset A$ *such that the condition* (1) *of Th.* 2.3.6 *holds, assume that*

(a) *For every* $\alpha \in B$ *the map* $\varrho_{\alpha\alpha+1}$ *is a homeomorphism into* $\mathscr{X}_{\alpha+1}$.

(b) *The family* $\mathscr{E} = \{\mathscr{A}_\alpha = \mathscr{L}^\infty_{X_\alpha} \mid \alpha \in B\}$ *is connected (this is satisfied namely if the following holds: If* $\alpha \in B$ *is such that the predecessor* $\alpha - 1$ *of* α *in* $\langle B \leqq \rangle$ *does not exist,* $\beta \in B[\alpha]$ *and* $G_{\beta\alpha} = \{f \in C(\mathscr{L}_\alpha = \underrightarrow{\lim}\, \mathscr{S}_{B[\alpha]} \to R) \mid \varrho'^*_{\gamma\alpha} f \in \mathscr{A}_\gamma$ *for all* $\gamma \in \langle \beta\alpha) \cap B\}$, *then* $G_{\beta\alpha} \subset \lambda^*_\alpha \mathscr{A}_\alpha$ $(\lambda_\alpha : \mathscr{L}_\alpha \to \mathscr{X}_\alpha$, $\varrho'_{\gamma\alpha} : \mathscr{X}_\gamma \to \mathscr{L}_\alpha$, $\gamma \in B[\alpha]$ *are the canonical maps)). Then there is a compact hull* $\mathscr{T}$ *of* $\mathscr{S}_B$ *such that* $\underrightarrow{\lim}\, \mathscr{T}$, $\mathscr{J} = \underrightarrow{\lim}\, \mathscr{S}_B$ *and* $\underrightarrow{\lim}\, \mathscr{S}$ *are f.s. The condition* (2b) *may be omitted if there is a countable confinal set in* B *and if* $\varrho_{\alpha\beta}$ *is a homeomorphism of* $\mathscr{X}_\alpha$ *into* $\mathscr{X}_\beta$ *for all* $\alpha, \beta \in B$, $\alpha \leqq \beta$.

Proof. By 2.3.7 and Th. 1.5.1, $\mathscr{J}$ is f.s. By 2.2.9, there is a compact hull $\mathscr{T}$ of $\mathscr{S}_B$ such that $\underrightarrow{\lim}\, \mathscr{T}$ is f.s. because $\mathscr{E}$ is strongly separating.

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