Czechoslovak Mathematical Journal

Adolf Karger

Darboux motions in E_n

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 2, 303–317

Persistent URL: http://dml.cz/dmlcz/101607

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

DARBOUX MOTIONS IN E_n

ADOLF KARGER, Kuwait

(Received November 21, 1977)

Let two euclidean spaces E_n and $\bar{E}_n$ of dimension n be given. Let us fix an orthonormal frame $R_0 = \{A_0, f_1, \ldots, f_n\}$ in E_n and $\bar{R}_0 = \{\bar{A}_0, \bar{f}_1, \ldots, \bar{f}_n\}$ in $\bar{E}_n$. Let G be the Lie group of all euclidean transformations in the n-dimensional euclidean space regarded as the group of all square $(n + 1)$-matrices g of the form

$$g = \begin{pmatrix} 1, & 0 \\ t, & \gamma \end{pmatrix},$$

where $\gamma \in O(n)$ and t is a column with n entries. By a *frame* we shall mean in the following an orthonormal frame in E_n or $\bar{E}_n$. The group G acts naturally as the group of euclidean transformations from $\bar{E}_n$ to E_n by the rule $g(\bar{R}_0) = R_0 . g$ and every frame R or $\bar{R}$ in E_n or $\bar{E}_n$, respectively, can be written in the form $R = R_0 . g_1$ or $\bar{R} = \bar{R}_0 . g_2$, where $g_1, g_2 \in G$. A curve $g(t)$ on G regarded as a one-parametric system of euclidean transformations from $\bar{E}_n$ to E_n is called a *euclidean motion* in E_n.

Let now $g \in G$. Then by the fibre F_g of g we mean the set of all pairs $(R, \bar{R})$ of frames such that R is in E_n, $\bar{R}$ is in $\bar{E}_n$ and $g(\bar{R}) = R$. Let further $R = R_0 . g_1$, $\bar{R} = \bar{R}_0 . g_2$. Then $(R, \bar{R}) \in F_g$ iff $g_1 g_2^{-1} = g$. (Direct calculation gives $g(\bar{R}) = g(\bar{R}_0 . g_2) = g(\bar{R}_0) . g_2 = R_0 . g g_2 = R_0 . g_1$ and so $g = g_1 . g_2^{-1}$.)

Let us suppose we are given a motion $g(t)$ in E_n, defined on an interval I, which is differentiable sufficiently many times. By a lift of $g(t)$ we mean a set of pairs $(R(t), \bar{R}(t))$ of frames such that $(R(t), \bar{R}(t)) \in F_{g(t)}$ for each t in I, also with a sufficient degree of differentiability. Let such a lift of $g(t)$ be given. Then we denote $\bar{\vartheta} = g^{-1}\,\mathrm{d}g$, $\vartheta = \mathrm{d}g . g^{-1}$, $\mathrm{d}R = R\varphi$, $\mathrm{d}\bar{R} = \bar{R}\psi$, where $\bar{\vartheta}, \vartheta, \varphi, \psi$ are $\mathfrak{g}$-valued 1-forms on I, $\mathfrak{g}$ denotes the Lie algebra of G and $\varphi = g_1^{-1}\,\mathrm{d}g_1$, $\psi = g_2^{-1}\,\mathrm{d}g_2$. If $(R_1, \bar{R}_1)$ is another lift of $g(t)$, we have $R_1 = R_0 . \gamma_1$, $\bar{R}_1 = \bar{R}_0 . \gamma_2$, where $\gamma_1 \gamma_2^{-1} = g(t)$, $\gamma_1, \gamma_2 \in G$ and so $g_1 = \gamma_1 h$ and $g_2 = \gamma_2 h$ for some $h \in G$. Taking differentials, we get for the corresponding forms φ_1 and ψ_1

$$(1) \qquad \varphi - \psi = h^{-1}(\varphi_1 - \psi_1)\,h\,, \quad \varphi + \psi = h^{-1}(\varphi_1 + \psi_1)\,h + 2h^{-1}\,\mathrm{d}h\,.$$

So let us denote $\omega\,\mathrm{d}t = \frac{1}{2}(\varphi - \psi)$, $\eta\,\mathrm{d}t = \frac{1}{2}(\varphi + \psi)$. Then $\varphi = (\eta + \omega)\,\mathrm{d}t$, $\psi = (\eta - \omega)\,\mathrm{d}t$.

By a point or a vector in E_n we shall mean a column of $n + 1$ entries with the first one equal to one or zero, respectively. If $\bar{A}$ is a fixed point in $\bar{E}_n$, we have $\bar{A} = \bar{R}X$, where X is the column of coordinates of $\bar{A}$ in $\bar{R}$ and the trajectory $g(\bar{A})$ of $\bar{A}$ satisfies $g(\bar{A}) = RX$, because $g(\bar{R}) = R$. Taking derivatives we get

$$\bar{A}' = 0 = \bar{R}'X + \bar{R}X' = \bar{R}\left(\frac{\psi}{dt}X + X'\right) \quad \text{and so} \quad X' = -\frac{\psi}{dt}X .$$

Furthermore,

$$(g(\bar{A}))' = R'X + RX' = R\frac{\varphi}{dt} - R\frac{\psi}{dt} = 2R\omega X$$

is the expression for the tangent vector of the trajectory of $\bar{A}$ with respect to R.

Let us denote by ${}^k\Omega$ the operator of the k-th derivative of the trajectory of $\bar{A}$ with respect to the frame R, write $(g(\bar{A}))^{(k)} = 2R\,{}^k\Omega X$.

Then

$$(g(A))^{(k+1)} = 2R[(\omega + \eta)\,{}^k\Omega + {}^k\Omega(\omega - \eta) + ({}^k\Omega)']$$

and we get the following recurrent formulas

$$(2) \qquad {}^1\Omega = \omega\,, \quad {}^{k+1}\Omega = (\omega + \eta)\,{}^k\Omega + {}^k\Omega(\omega - \eta) + ({}^k\Omega)'\,.$$

Definition 1. A motion $g(t)$ in E_n will be called a *Darboux k-motion* iff

1. the trajectory of any point is contained in a subspace of E_n of dimension k and at least one of them is not contained in a subspace of dimension $k - 1$,
2. all trajectories are affine equivalent in kinematical sense, which means that there is a trajectory $X(t)$ such that to every other trajectory $Y(t)$ there is an affine transformation (not necessarily regular) $\mathscr{A}$ of E_n such that $Y(t) = \mathscr{A}X(t)$ for all $t \in I$.

Theorem 1. *A euclidean motion $g(t)$ in E_n is a Darboux k-motion iff there are functions $\alpha_1(t), \ldots, \alpha_k(t)$ such that*

$$(3) \qquad {}^{k+1}\Omega = \sum_{i=1}^{k} \alpha_i\,{}^i\Omega$$

and ${}^1\Omega, \ldots, {}^k\Omega$ are linearly independent.

Proof. Let $X(t)$ be the trajectory such that $Y(t) = \mathscr{A}X(t)$ for every other trajectory $Y(t)$, where of course $\mathscr{A}$ depends on Y. Taking derivatives with respect to t, we get $Y^{(i)}(t) = \mathscr{A}X^{(i)}(t)$. There is also a trajectory, say $Z(t)$, which is not contained in a subspace of dimension k. This means that there is t_0 such that the vectors $Z'(t_0), \ldots$ $\ldots, Z^{(k)}(t_0)$ are linearly independent. Because $Z^{(i)}(t_0) = \mathscr{A}X^{(i)}(t_0)$ for $i = 1, \ldots, k$, we see that the vectors $X'(t_0), \ldots, X^{(k)}(t_0)$ are also linearly independent; this means that $X(t)$ is not contained in any subspace of dimension $k - 1$. Because $Y^{(i)} = {}^i\Omega Y$ and ${}^i\Omega$ is a continuous operator (in fact it is linear), there is a neighbourhood U of

$X(t_0)$ such that $Y^{(i)}$ are linearly independent in U. Take Y in U. Then $Y^{(k+1)} = \sum_i \alpha_i(Y)\, Y^{(i)}$ and of course $X^{(k+1)} = \sum \alpha_i(X)\, X^{(i)}$ for all t in a certain interval around t_0. Then

$$Y^{(k+1)} = \mathscr{A}(Y)\, X^{(k+1)} = \sum \alpha_i(Y)\, Y^{(i)} = \mathscr{A}(Y) \sum \alpha_i(X)\, X^{(k+1)} = $$
$$= \sum \alpha_i(X)\, \mathscr{A}(Y)\, X^{(i)} = \sum \alpha_i(X)\, Y^{(i)}$$

and so $\sum (\alpha_i(X) - \alpha_i(Y))\, Y^{(i)} = 0$ in U. This gives $\alpha_i(X) = \alpha_i(Y)$ in U. So we have $Y^{(k+1)} = \sum \alpha_i Y^{(i)}$, where $\alpha_i = \text{const.}$ in U. Then ${}^{k+1}\Omega Y = \sum \alpha_i\, {}^{i}\Omega Y$ in U. If we now take the partial derivative with respect to the coordinate x_α $(\alpha = 1, \ldots, n)$ at X of the last equation, we get ${}^{k+1}\Omega Y = \sum \alpha_i\, {}^{i}\Omega Y$ for all Y and so ${}^{k+1}\Omega = \sum \alpha_i\, {}^{i}\Omega$ where α_i does not depend on X in a certain interval around t_0.

The proof that (3) is sufficient for a motion to be a Darboux k-motion is easy, because the solution of the differential equation $X^{(k+1)} = \sum \alpha_i X^{(i)}$ depends on a point and k vectors, which means that the solutions are affine equivalent and k-dimensional. Because ${}^{1}\Omega, \ldots, {}^{k}\Omega$ are linearly independent, not all of the solutions can lie in subspaces of dimensions less then k. This completes the proof.

From the definition of the Darboux k-motion we see that if the condition (3) is satisfied for one lift of $g(t)$, it is satisfied for all of them. Also if we change the parameter t, the validity of (3) will remain unchanged. It is also easy to check that there are no Darboux 1-motions in E_n apart from translations.

In what follows we shall classify all Darboux 2-motions in E_n. From the beginning let us exclude translations as a trivial case. Matrices ω and η can be written in the form

$$(4) \qquad \omega = \begin{pmatrix} 0, & 0 \\ \omega_0, & \omega_1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0, & 0 \\ \eta_0, & \eta_1 \end{pmatrix}, \quad \text{where } \omega_1 \neq 0\,,$$

ω_1, η_1 are skew-symmetric $n \times n$ matrices, ω_0, η_0 are columns. Equations (3) for the Darboux 2-motion can be written in the form

$$(5) \qquad {}^{3}\Omega = -4\alpha_1\, {}^{1}\Omega + \alpha_2\, {}^{2}\Omega\,.$$

Let us denote

$${}^{i}\Omega = \begin{pmatrix} 0, & 0 \\ {}^{i}\vartheta, & {}^{i}\Theta \end{pmatrix}.$$

(3) will then change to

$$(6) \qquad {}^{1}\vartheta = \omega_0\,, \quad {}^{i+1}\vartheta = (\omega_1 + \eta_1)\, {}^{i}\vartheta + {}^{i}\Theta(\omega_0 - \eta_0) + ({}^{i}\vartheta)'\,,$$
$$\qquad {}^{1}\Theta = \omega_1\,, \quad {}^{i+1}\Theta = (\omega_1 + \eta_1)\, {}^{i}\Theta + {}^{i}\Theta(\omega_1 - \eta_1) + ({}^{i}\Theta)'\,.$$

Changing the lift of $g(t)$, we can assume that there is a natural number k, $1 \leqq k \leqq \frac{1}{2}n$, such that ω_1 is of the form

$$\omega_1 = \begin{pmatrix} \omega_{11}, & 0 \\ 0, & 0 \end{pmatrix}$$

where ω_{11} is a regular $2k \times 2k$ matrix of the form

$$\omega_{11} = \begin{pmatrix} 0, & -D \\ D, & 0 \end{pmatrix}$$

with $D = \{\lambda_1, \ldots, \lambda_k\}$ diagonal and ω_0 is of the form

$$\omega_0 = \begin{pmatrix} \omega_{01} \\ \omega_{02} \end{pmatrix},$$

where $\omega_{01} = 0$ is a $2k \times 1$ matrix and ω_{02} is an $(n - 2k) \times 1$ matrix with all entries except possibly the first one equal to zero. Rearanging vectors in the frame and changing the parameter t we can achieve $1 = \lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_k > 0$. The group H_1 which preserves our specialization is the subgroup of all elements $g \in G$ of the form

$$g = \begin{pmatrix} 1, & 0, & 0 \\ 0, & \gamma_1, & 0 \\ t, & 0, & \gamma_2 \end{pmatrix}, \tag{7}$$

where $\gamma_1 \in O(2k)$, $\gamma_2 \in O(n - 2k)$, t is an $(n - 2k) \times 1$ matrix and

$$\gamma_1 \omega_{11} = \omega_{11} \gamma_1, \quad \gamma_2 \omega_{02} = \omega_{02}.$$

Denote

$$\eta_1 = \begin{pmatrix} \eta_{11}, & \eta_{12} \\ \eta_{21}, & \eta_{22} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} \eta_{01} \\ \eta_{02} \end{pmatrix},$$

where η_{11} is a skew-symmetric $2k \times 2k$ matrix, η_{22} is a skew-symmetric $(n - 2k) \times$ $\times (n - 2k)$ matrix, $\eta_{12} = -\eta_{21}^T$, where T denotes the transpose, η_{12} is a $2k \times$ $\times (n - 2k)$ matrix, η_{01} is a $2k \times 1$ and η_{02} is an $(n - 2k) \times 1$ matrix. Equations (6) will read as follows:

Write

$$^i\Theta = \begin{pmatrix} ^i\Theta_{11}, & ^i\Theta_{12} \\ ^i\Theta_{21}, & ^i\Theta_{22} \end{pmatrix}, \quad ^i\vartheta = \begin{pmatrix} ^i\vartheta_{01} \\ ^i\vartheta_{02} \end{pmatrix},$$

the size of matrices of the splitting is similar as above. Then

$$\begin{aligned} (8) \qquad & ^1\Theta_{11} = \omega_{11}, \quad ^1\Theta_{12} = {}^1\Theta_{21} = {}^1\Theta_{22} = 0, \\ & ^2\Theta_{11} = 2\omega_{11}^2 + \eta_{11}\omega_{11} - \omega_{11}\eta_{11} + \omega_{11}', \quad ^2\Theta_{12} = -\omega_{11}\eta_{12}, \\ & ^2\Theta_{21} = \eta_{21}\omega_{11}, \quad ^2\Theta_{22} = 0, \\ & ^3\Theta_{12} = -3\omega_{11}^2\eta_{12} - 2\eta_{11}\omega_{11}\eta_{12} + \omega_{11}\eta_{11}\eta_{12} + \omega_{11}\eta_{12}\eta_{22} - \\ & \qquad - \omega_{11}'\eta_{12} - (\omega_{11}\eta_{12})', \end{aligned}$$

$$^3\Theta_{21} = 3\eta_{21}\omega_{11}^2 - 2\eta_{21}\eta_{11}\omega_{11} + \eta_{21}\eta_{11}\omega_{11} + \eta_{22}\eta_{21}\omega_{11} + \\ + \eta_{21}\omega'_{11} + (\eta_{21}\omega_{11})' .$$

Because $^1\Theta_{12} + (^1\Theta_{21})^T = {}^2\Theta_{12} + (^2\Theta_{21})^T = 0$, we must have also $^3\Theta_{12} + (^3\Theta_{21})^T = 0$. But $^3\Theta_{12} + (^3\Theta_{21})^T = -6\omega_{11}^2\eta_{12}$ and so $\eta_{12} = 0$. Using this fact we get further

(8a) $$^2\Theta_{12} = {}^2\Theta_{21} = {}^3\Theta_{12} = {}^3\Theta_{21} = {}^3\Theta_{22} = 0 ,$$

$$^3\Theta_{11} = 3\eta_{11}\omega_{11}^2 - 3\omega_{11}^2\eta_{11} + 6\omega_{11}\omega'_{11} + 4\omega_{11}^3 + \eta_{11}^2\omega_{11} + \omega_{11}\eta_{11}^2 - \\ - 2\eta_{11}\omega_{11}\eta_{11} + 2\eta_{11}\omega'_{11} - 2\omega'_{11}\eta_{11} + \eta'_{11}\omega_{11} - \omega_{11}\eta'_{11} + \omega''_{11} ,$$

(9) $$^1\vartheta_{01} = 0 , \quad ^1\vartheta_{02} = \omega_{02} , \quad ^2\vartheta_{01} = -\omega_{11}\eta_{01} , \quad ^2\vartheta_{02} = \eta_{22}\omega_{02} + \omega'_{02} ,$$

$$^3\vartheta_{01} = -3\omega_{11}^2\eta_{01} - 2\eta_{11}\omega_{11}\eta_{01} + \omega_{11}\eta_{11}\eta_{01} - \omega'_{11}\eta_{01} - (\omega_{11}\eta_{01})' ,$$

$$^3\vartheta_{02} = \eta_{22}^2\omega_{02} + \eta_{22}\omega'_{02} + (\eta_{22}\omega_{02} + \omega'_{02})' .$$

Let us denote

$$\omega_{02} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad \eta_{22} = \begin{pmatrix} 0, & -b, & -m^T \\ b, & 0, & -n_1^T \\ m, & n_1, & n_2 \end{pmatrix},$$

where $a, b \in R$, m, n_1 are $(n - 2k - 2) \times 1$ matrices, n_2 is an $(n - 2k - 2) \times (n - 2k - 2)$ matrix. Equations (4) yield $-4\alpha_1\,{}^1\vartheta_{02} + \alpha_2\,{}^2\vartheta_{02} = {}^3\vartheta_{02}$. Substituting from (9) we get

(10) $$-4\alpha_1\omega_{02} + \alpha_2(\eta_{22}\omega_{02} + \omega'_{02}) = \eta_{22}^2\omega_{02} + \eta_{22}\omega'_{02} + (\eta_{22}\omega_{02} + \omega'_{02})' .$$

Writing (10) explicitly we get

(11) $$-4\alpha_1 \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} a' \\ ba \\ ma \end{pmatrix} = \begin{pmatrix} -(b^2 + m^Tm)\,a + a'' \\ ba' - n_1^Tma + (ba)' \\ ma' + (n_1b + n_2m)\,a + (ma)' \end{pmatrix}.$$

Now we have to distinguish between three cases:

a) $a \neq 0$. Then the matrices γ_2 from (7) form the group $O(n - 2k - 1)$,

$$\gamma_2 = \begin{pmatrix} 1, & 0 \\ 0, & \gamma_3 \end{pmatrix},$$

where $\gamma_3 \in O(n - 2k - 1)$ and γ_2 acts on the first column v of η_{22} in the natural way, $\tilde{v} = \gamma_{22}v$, where

$$v = \begin{pmatrix} 0 \\ b \\ m \end{pmatrix}.$$

As the orthogonal group is transitive on directions, we can change the lift of $g(t)$

in such a way that $m = 0$ and $b \geqq 0$. Then (11) will change to

$$(12) \qquad -4\alpha_1 \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} a' \\ ba \\ 0 \end{pmatrix} = \begin{pmatrix} -b^2 a + a'' \\ ba' + (ba)' \\ n_1 ba \end{pmatrix}.$$

a1) Let $b \neq 0$. Then $n_1 = 0$. Using a suitable lift of $g(t)$ we can achieve $n_2 = 0$, $\eta_{02} = 0$. So in this case we can restrict ourselves to the dimension $2k + 2$.

a2) Let now $b = 0$. Then there is no restriction on $O(n - 2k - 1)$; choosing a suitable lift we get $n_1 = 0$, $n_2 = 0$, $\eta_{02} = 0$. We can restrict ourselves to the case of the dimension $2k + 1$.

b) $a = 0$. Then (12) is automatically satisfied and there is no restriction on γ_2, $\gamma_2 \in O(n - 2k)$. Using a suitable lift, we get $\eta_{02} = 0$, $\eta_{22} = 0$ and the dimension can be restricted to $2k$. For ω and η we get in the above mentioned three cases:

a1)
$$\omega = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & \omega_{11}, & 0, & 0 \\ a, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0, & 0, & 0, & 0 \\ \eta_{01}, & \eta_{11}, & 0, & 0 \\ 0, & 0, & 0, & -b \\ 0, & 0, & b, & 0 \end{pmatrix}$$

a2)
$$\omega = \begin{pmatrix} 0, & 0, & 0 \\ 0, & \omega_{11}, & 0 \\ a, & 0, & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0, & 0, & 0 \\ \eta_{01}, & \eta_{11}, & 0 \\ 0, & 0, & 0 \end{pmatrix}$$

b)
$$\omega = \begin{pmatrix} 0, & 0 \\ 0, & \omega_{11} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0, & 0 \\ \eta_{01}, & \eta_{11} \end{pmatrix}.$$

Let us suppose that we have removed from the list of Darboux 2-motions in the euclidean space of a dimension n all such Darboux 2-motions, which are already listed under a certain dimension less then n. By this we mean that our motion in E_n is not a product of a Darboux motion in a proper subspace of E_n and the identity in the direction of the orthogonal complement. This allows us to treat all three cases a1, a2, b as the case a1 only, provided that the last one or two zero rows and columns are removed if necessary. Equations (12) will then reduce to

$$(13) \qquad -4\alpha_1 a + \alpha_2 a' = -ab^2 + a'', \quad \alpha_2 ab = ab' + 2ba',$$

where we can suppose $a \geqq 0$, $b \geqq 0$.

The remaining isotropy group is H_1', where $g \in H_1'$ is of the form

$$g = \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & \gamma_1, & 0, & 0 \\ t_1, & 0, & 1, & 0 \\ t_2, & 0, & 0, & 1 \end{pmatrix},$$

where $t_1, t_2 \in R$ are constants and $\gamma_1 \omega_{11} = \omega_{11} \gamma_1$.

This means that the cases a1) and a2) are singular in the sense that the Frenet frame is not uniquely determined. It remains now to solve equations (5) for ϑ_{01} and Θ_{11}. Write

$$\omega_{11} = \begin{pmatrix} 0, & -D \\ D, & 0 \end{pmatrix}$$

as before and

$$\eta_{11} = \begin{pmatrix} N + X, & -M + Y \\ M + Y, & N - X \end{pmatrix},$$

where N, X, Y are skew-symmetric, M is symmetric and all of them are $k \times k$ matrices. Let us denote by ${}^iS$ the symmetric part of ${}^i\Theta_{11}$, $i = 1, 2, 3$. Then from (5) we have

$$-4\alpha_1\, {}^1S + \alpha_2\, {}^2S = {}^3S\,.$$

After substitution we get

$${}^1S = 0\,, \quad {}^2S = 2\omega_{11}^2\,, \quad {}^3S = 3\eta_{11}\omega_{11}^2 - 3\omega_{11}^2\eta_{11} + 6\omega_{11}\omega_{11}'$$

and finally

(14)
$$2\alpha_2 \begin{pmatrix} -D^2, & 0 \\ 0, & -D^2 \end{pmatrix} =$$

$$= 3\begin{pmatrix} D^2N - ND^2 + D^2X - XD^2, & MD^2 - D^2M + D^2Y - YD^2 \\ D^2M - MD^2 + D^2Y - YD^2, & D^2N - ND^2 + XD^2 - D^2X \end{pmatrix} -$$

$$- 6\begin{pmatrix} DD', & 0 \\ 0, & DD' \end{pmatrix}.$$

Taking elements on the main diagonal only, we get

$$-2\alpha_2 D^2 = -6DD'\,.$$

Because $\lambda_1 = 1$, we have $\lambda_1' = 0$ and so $\alpha_2 = 0$ and $D' = 0$. From (14) we now have

(15) $$ND^2 - D^2N = MD^2 - D^2M = XD^2 - D^2X = YD^2 - D^2Y = 0\,.$$

Let $\lambda_1, \ldots, \lambda_m$ be all different elements from D, the multiplicity of λ_α in D being k_α, $\alpha = 1, \ldots, m$, where $\sum_{\alpha=1}^{m} k_\alpha = k$. Let further $M_{\alpha\beta}$, $N_{\alpha\beta}$, $X_{\alpha\beta}$, $Y_{\alpha\beta}$ be appropriate splittings of M, N, X, Y into block matrices of size $k_\alpha \times k_\beta$. Equations (15) show that

$$M_{\alpha\beta} = N_{\alpha\beta} = X_{\alpha\beta} = Y_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta\,.$$

Similarly, if we write γ_1 from $g \in H_1'$ in the form

$$\gamma_1 = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$$

and split $\alpha, \beta, \gamma, \delta$ accordingly into $\alpha_{\alpha\beta}, \beta_{\alpha\beta}, \gamma_{\alpha\beta}, \delta_{\alpha\beta}$, we get also $\alpha_{\alpha\beta} = \beta_{\alpha\beta} = \gamma_{\alpha\beta} = \delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$. This means that the isotropy group H'_1 splits into the direct product of subgroups of $O(2k_\alpha)$. So we can solve equations (5) separately for each index α. This also means that every Darboux 2-motion can be written as a product of Darboux 2-motions having only one characteristic value of the tangent operator, multiplied possibly by a certain translation or by a certain plane motion. For the sake of simplicity let us deal from now on with one block only and let us drop indices α and β. Equations (14) are automatically satisfied, because now $D = \lambda E$.

Denote ${}^iA$ the skew-symmetric part of ${}^i\Theta_{11}$, $i = 1, 2, 3$. Then

$$ {}^1A = \omega_{11}\,, \quad {}^2A = \eta_{11}\omega_{11} - \omega_{11}\eta_{11}\,, $$

$$ {}^3A = 4\omega_{11}^3 + \eta_{11}^2\omega_{11} + \omega_{11}\eta_{11}^2 - 2\eta_{11}\omega_{11}\eta_{11} + \eta'_{11}\omega_{11} - \omega_{11}\eta'_{11}\,. $$

For the skew-symmetric part of (5) we get $-4\alpha_1\,{}^1A = {}^3A$ and the substitution yields

$$ (16) \qquad -4\alpha_1\lambda\begin{pmatrix}0, & -E\\ E, & 0\end{pmatrix} = 4\lambda^3\begin{pmatrix}0, & E\\ -E, & 0\end{pmatrix} + 2\lambda\begin{pmatrix}Y', & -X'\\ -X', & -Y'\end{pmatrix} + $$

$$ + 2\lambda\begin{pmatrix} NY - YN + MX + XM + 2(XY - YX), & MY + YM + XN - NX - 2(X^2 + Y^2) \\ MY + YM + XN - NX + 2(X^2 + Y^2), & -MX - XM + YN - NY + 2(XY - YX) \end{pmatrix}. $$

After taking linear combinations in (16) we get

$$ (17) \qquad XY - YX = 0\,, \quad X^2 + Y^2 = (\lambda^2 - \alpha_1)\,E\,, \quad \text{where} \quad \alpha_1 - \lambda^2 \geqq 0\,, $$

$$ MX + XM + NY - YN + Y' = MY + YM + XN - NX - X' = 0\,. $$

Denoting $X + iY = Z$, $N + iM = F$, $\sqrt{(\alpha_1 - \lambda^2)} = \mu$, $i = \sqrt{-1}$, we get

$$ (18) \qquad Z\,.\,\bar{Z}^T = \mu^2 E \quad \text{and} \quad Z\,.\,\bar{F} - F\,.\,Z = Z'\,, $$

where $Z^T = -Z$, $\bar{F}^T = -F$ and the bar denotes the conjugate.

Elements γ_1 from the group H'_1 can be written in the form

$$ \gamma_1 = \begin{pmatrix}\alpha, & -\beta\\ \beta, & \alpha\end{pmatrix}, \quad \text{where} \quad \gamma_1\gamma_1^T = E\,. $$

If we write $\gamma = \alpha + i\beta$, we have $\gamma\bar{\gamma}^T = E$ so H'_1 becomes the unitary group $U(k)$ and its Lie algebra consists of all matrices of the form $L = r + is$, where $L + \bar{L}^T = 0$. So F belongs to the Lie algebra of H'_1 and the action of H'_1 on Z is $\tilde{Z} = \gamma Z \gamma^T$.

Now we have to consider two cases:

α) $\mu = 0$. Then $Z\,.\,\bar{Z}^T = 0$ and so $Z = 0$. Equations (18) are automatically satisfied.

Because F is in the Lie algebra of the isotropy group H'_1, we can change the lift of $g(t)$ (up to a constant matrix) so as to make F vanish.

β) $\mu > 0$. Let now the unitary group $U(k)$ act on a unitary vector space of dimension k with an orthonormal base e_i, $i = 1, \ldots, k$. Then for the unitary scalar product (u, v) of vectors u and v we get $(u, v) = u^T . \bar{v}$, where u and v are columns of coordinates of the vectors u and v. Let us define a new form $[u, v] = u^T Z v$. We get

$$[u, v] = (u, \bar{Z}\bar{v}) = (Zv, \bar{u}) = -(Z^T v, \bar{u}) = -v^T Z u = -[v, u],$$

and so $[u, v]$ is skew-symmetric. Now we can require the first equation of (18) to be satisfied. Having in mind that the unitary group acts transitively on directions, we can change the orthonormal base e_i in such a way that Z assumes the form

$$Z = \begin{pmatrix} 0, & -\mu E \\ \mu E, & 0 \end{pmatrix}, \tag{19}$$

where E is the $k/2 \times k/2$ unit matrix and so k must be an even number. (This means that the multiplicity of λ is even.)

We can write F similarly as in (19) in the form

$$F = \begin{pmatrix} A, & B \\ -\bar{B}^T, & C \end{pmatrix},$$

where A, B, C are complex $k/2 \times k/2$ matrices with $A + \bar{A}^T = C + \bar{C}^T = 0$. The second equation from (18) gives

$$\mu \begin{pmatrix} B^T - B, & A - \bar{C} \\ \bar{A} - C, & \bar{B} - \bar{B}^T \end{pmatrix} = \begin{pmatrix} 0, & -\mu' E \\ \mu' E, & 0 \end{pmatrix}. \tag{20}$$

Equations $A + \bar{A}^T = C + \bar{C}^T = 0$ say that the main diagonal of $A - \bar{C}$ is pure imaginary. This gives $\mu' = 0$ and so $\mu = \text{const.}$ and $\alpha_1 = \text{const.}$ Further we have $B^T = B$, $C = \bar{A}$ and so

$$F = \begin{pmatrix} A, & B \\ -\bar{B}, & \bar{A} \end{pmatrix} \quad \text{with} \quad A + \bar{A}^T = 0 \quad \text{and} \quad B^T = B. \tag{21}$$

The new isotropy group H_2 is a subgroup of $U(k)$ which preserves Z, so it is the group of all unitary matrices of the form

$$h = \begin{pmatrix} r, & s \\ -\bar{s}, & \bar{r} \end{pmatrix}$$

and so F belongs to the Lie algebra of H_2. At this moment we again have to distinguish two cases:

α) $\eta_{01} = 0$. In this case there are no main components (η cannot be changed by action of H_2 and so we can only change the lift of $g(t)$ in such a way that F be zero). The Frenet frame (the canonical lift of $g(t)$) will be fixed up to a constant matrix from H_2. We see also immediately that ${}^3\vartheta = 0$ and we have to solve (13) only to get the invariants of the motion.

β) $\eta_{01} \neq 0$. It is easy to realize that H_2 is the symplectic group of order $k/2$ over quaternions, so $H_2 = \mathrm{Sp}\,(k/2)$. The symplectic group is the group of all matrices q with quaternion elements such that $q\,.\,(q^T)^\iota = E$, where iota denotes the conjugate quaternion.

Sp $(k/2)$ acts in a natural way on a vector space of dimension $k/2$ over quaterions and it is known (see [1]) that this action is transitive on directions, so we can change the lift of $g(t)$ in such a way that

$$\eta_{01} = \begin{pmatrix} v \\ 0 \end{pmatrix},$$

where $v > 0$ is a real number. We must now solve (9) for ${}^i\vartheta_{01}$; this yields ${}^3\vartheta_{01} = 0$. Les us write

$$\eta_{01} = \begin{pmatrix} p \\ 0 \end{pmatrix},$$

where p is a column with k entries, the first one equal to v, the others equal to zero. After a substitution we get

$$(3\lambda E + M)\,p = 0\,, \quad (3X - N)\,p = p'\,. \tag{22}$$

Since $3X - N$ is skew-symmetric, we get $p' = 0$ and so $v = \text{const}$. Taking into acount that p has only the first element different from zero, we see that the first columns of $3\lambda E + M$ and of $3X - N$ are equal to zero. This shows that the new isotropy group H_3 will be Sp $(k/2 - 1)$; there are no main components, because the remaining components are in the Lie algebra of H_3. So we can put the remaining components equal to zero and the Frenet frame is fixed up to a constant element from Sp $(k/2 - 1)$.

As a result we get

$$\eta_{11} = \begin{bmatrix} 0, & 0, & -4\mu, & 0, & 3\lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & -\mu E, & 0, & 0, & 0, & 0 \\ 4\mu, & 0, & 0, & 0, & 0, & 0, & -3\lambda, & 0 \\ 0, & \mu E, & 0, & 0, & 0, & 0, & 0, & 0 \\ -3\lambda, & 0, & 0, & 0, & 0, & 0, & -2\mu, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & \mu E \\ 0, & 0, & 3\lambda, & 0, & 2\mu, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & -\mu E, & 0, & 0 \end{bmatrix} \tag{23}$$

where each even row or column represents $(k/2 - 1)$ rows or columns. ω_{11} written in a similar way will be

$$(24) \qquad \omega_{11} = \begin{bmatrix} 0, & 0, & 0, & 0, & -\lambda, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & -\lambda E, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & -\lambda, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & 0, & -\lambda E \\ \lambda, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & \lambda E, & 0, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & \lambda, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & \lambda E, & 0, & 0, & 0, & 0 \end{bmatrix}.$$

From (23) and (24) we see that for the Frenet lift $(F, \bar{F})$ of $g(t)$ we get two sets of independent differential equations (apart from the last two rows, which will be discussed separately). After a suitable permutation of vectors in $F = \{A, e_1, \ldots, e_n\}$ and $\bar{F} = \{\bar{A}, \bar{e}_1, \ldots, \bar{e}_n\}$ we can write them in the following form:

Equations for F:

$$(25) \quad \text{a)} \quad A' = ve_1,\ e_1' = 4\mu e_2 - 2\lambda e_3, \qquad \text{b)} \quad e_{1,i}' = \mu e_{2,i} + \lambda e_{3,i},$$
$$e_2' = -4\mu e_1 + 4\lambda e_4, \qquad e_{2,i}' = -\mu e_{1,i} + \lambda e_{4,i},$$
$$e_3' = 2\lambda e_1 + 2\mu e_4, \qquad e_{3,i}' = -\lambda e_{1,i} - \mu e_{4,i},$$
$$e_4' = -4\lambda e_2 - 2\mu e_3. \qquad e_{4,i}' = -\lambda e_{2,i} + \mu e_{3,i},$$

where $i = 1, \ldots, k/2 - 1$.

Equations for $\bar{F}$ (let us leave out the bars):

$$\text{c)} \quad A' = ve_1,\ e_1' = 4\mu e_2 - 4\lambda e_3, \qquad \text{d)} \quad e_{1,i}' = \mu e_{2,i} - \lambda e_{3,i},$$
$$e_2' = -4\mu e_1 + 2\lambda e_4, \qquad e_{2,i}' = -\mu e_{1,i} - \lambda e_{4,i},$$
$$e_3' = 4\lambda e_1 + 2\mu e_4, \qquad e_{3,i}' = \lambda e_{1,i} - \mu e_{4,i},$$
$$e_4' = -2\lambda e_2 - 2\mu e_3. \qquad e_{4,i}' = \lambda e_{2,i} + \mu e_{3,i},$$

where again $i = 1, \ldots, k/2 - 1$.

Denote $\varepsilon = \sqrt{(\lambda^2 + \mu^2)}$. It is convenient to write solutions of (25) in a matrix form. So denote by $g_1, g_{2,i}, g_3, g_{4,i}$ the solution of a), b), c), d), respectively. Initial conditions are chosen in such a way that all these matrices are equal to the identity matrix at $t = 0$. Then

$$
g_1 = \frac{1}{\varepsilon^2}\begin{bmatrix}
1, & 0, & 0, & 0, & 0 \\
\frac{\nu}{4\varepsilon}(\mu^2 \sin 4\varepsilon t + 2\lambda^2 \sin 2\varepsilon t), & \mu^2 \cos 4\varepsilon t + \lambda^2 \cos 2\varepsilon t, & -\mu\varepsilon \sin 4\varepsilon t, & \lambda\varepsilon \sin 2\varepsilon t, & \lambda\mu\,(\cos 2\varepsilon t - \cos 4\varepsilon t) \\
\frac{\nu\mu}{4}(1 - \cos 4\varepsilon t), & \mu\varepsilon \sin 4\varepsilon t, & \varepsilon^2 \cos 4\varepsilon t, & 0, & -\lambda\varepsilon \sin 4\varepsilon t \\
\frac{\nu\lambda}{2}(\cos 2\varepsilon t - 1), & -\lambda\varepsilon \sin 2\varepsilon t, & 0, & \varepsilon^2 \cos 2\varepsilon t, & -\mu\varepsilon \sin 2\varepsilon t \\
\frac{\nu\lambda\mu}{4\varepsilon}(2 \sin 2\varepsilon t - \sin 4\varepsilon t), & \lambda\mu(\cos 2\varepsilon t - \cos 4\varepsilon t), & \lambda\varepsilon \sin 4\varepsilon t, & \mu\varepsilon \sin 2\varepsilon t, & \lambda^2 \cos 4\varepsilon t + \mu^2 \cos 2\varepsilon t
\end{bmatrix},
$$

$$
g_3 = \frac{1}{\varepsilon^2}\begin{bmatrix}
1, & 0, & 0, & 0, & 0 \\
\frac{\nu\varepsilon}{4} \sin 4\varepsilon t, & \varepsilon^2 \cos 4\varepsilon t, & -\mu\varepsilon \sin 4\varepsilon t, & \lambda\varepsilon \sin 4\varepsilon t, & 0 \\
\frac{\nu\mu}{4}(1 - \cos 4\varepsilon t), & \mu\varepsilon \sin 4\varepsilon t, & \mu^2 \cos 4\varepsilon t + \lambda^2 \cos 2\varepsilon t, & \lambda\mu(\cos 2\varepsilon t - \cos 4\varepsilon t), & -\lambda\varepsilon \sin 2\varepsilon t \\
\frac{\nu\lambda}{4}(\cos 4\varepsilon t - 1), & -\lambda\varepsilon \sin 4\varepsilon t, & \lambda\mu(\cos 2\varepsilon t - \cos 4\varepsilon t), & \lambda^2 \cos 4\varepsilon t + \mu^2 \cos 2\varepsilon t, & -\mu\varepsilon \sin 2\varepsilon t \\
0, & 0, & \lambda\varepsilon \sin 2\varepsilon t, & \mu\varepsilon \sin 2\varepsilon t, & \cos 2\varepsilon t
\end{bmatrix},
$$

$$g_{2,i}, g_{4,i} = \frac{1}{\varepsilon}\begin{bmatrix} \varepsilon\cos\varepsilon t, & -\mu\sin\varepsilon t, & -\delta\sin\varepsilon t, & 0 \\ \mu\sin\varepsilon t, & \varepsilon\cos\varepsilon t, & 0 \quad , & -\delta\sin\varepsilon t \\ \delta\sin\varepsilon t, & 0 \quad , & \varepsilon\cos\varepsilon t, & \mu\sin\varepsilon t \\ 0 \quad , & \delta\sin\varepsilon t, & -\mu\sin\varepsilon t, & \varepsilon\cos\varepsilon t \end{bmatrix},$$

where $\delta = \lambda$ for $g_{2,i}$ and $\delta = -\lambda$ for $g_{4,i}$.

Denote $G_1 = g_1 g_3^{-1}$, $G = g_{2,i} g_{4,i}^{-1}$. Then

$$G_1 = \begin{pmatrix} 1, & 0 \\ T, & G \end{pmatrix}$$

and

(26)

$$G = \frac{1}{\varepsilon^2}\begin{bmatrix} \mu^2 + \lambda^2\cos 2\varepsilon t, & 0 \quad , & -\lambda\varepsilon\sin 2\varepsilon t \quad , & \lambda\mu(1 - \cos 2\varepsilon t) \\ 0 \quad , & \mu^2 + \lambda^2\cos 2\varepsilon t, & \lambda\mu(\cos 2\varepsilon t - 1), & -\lambda\varepsilon\sin 2\varepsilon t \\ \lambda\varepsilon\sin 2\varepsilon t \quad , & \lambda\mu(1 - \cos 2\varepsilon t), & \mu^2 + \lambda^2\cos 2\varepsilon t, & 0 \\ \lambda\mu(\cos 2\varepsilon t - 1), & \lambda\varepsilon\sin 2\varepsilon t \quad , & 0 \quad , & \mu^2 + \lambda^2\cos 2\varepsilon t \end{bmatrix},$$

$$T = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\nu}{2\varepsilon^2}(\cos 2\varepsilon t - 1) \\ 0 \end{bmatrix}. \tag{27}$$

Let us return again to the original situation of more characteristic values λ_α of ω_{11}. For G from (26) let us write $G(\lambda_\alpha)$ and for T from (27) let us write $T(\lambda_\alpha)$ in case that it corresponds to a characteristic value λ_α of ω_{11} with a multiplicity k_α.

Finally, we write

$$t_\alpha = \begin{bmatrix} T(\lambda_\alpha) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad g_\alpha = \begin{bmatrix} G(\lambda_\alpha), & 0, & \dots, & 0 \\ 0, & G(\lambda_\alpha), & \dots, & 0 \\ \vdots & \vdots & & \vdots \\ 0, & 0, & \dots, & G(\lambda_\alpha) \end{bmatrix}, \tag{28}$$

where t_α is a $k_\alpha \times 1$ matrix which has 4×1 matrices as elements, g_α is a $k_\alpha \times k_\alpha$ matrix which has 4×4 matrices as elements.

If $\mu = 0$ for some λ_α, then the multiplicity of λ_α need not be an even number and the expressions for t_α and g_α will become simpler. In this case

$$t_\alpha = \begin{bmatrix} 0 \\ \frac{1}{2}\nu(\cos 2t - 1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad g_\alpha = \begin{bmatrix} \cos 2t, & -\sin 2t, & & & \\ \sin 2t, & \cos 2t, & 0 & & \\ & & \ddots & & \\ 0 & & & \cos 2t, & -\sin 2t \\ & & & \sin 2t, & \cos 2t \end{bmatrix} \tag{29}$$

where t_α is a column with $2k_\alpha$ elements and g_α is a $2k_\alpha \times 2k_\alpha$ matrix; it expresses the direct product of k_α plane rotations.

Finally, we must solve the differential equation for the last two rows. From (8a) we know that this system of differential equations can be solved independently of those above. So let $\{A, e_1, e_2\}$, $\{\bar{A}, \bar{e}_1, \bar{e}_2\}$ be a frame in E_2 and in $\bar{E}_2$, respectively. The differential equations for them will be

$$(30)\qquad \begin{aligned} A' &= \quad a e_1\,, \quad e_1' = b e_2\,, \quad e_2' = -b e_1\,, \\ \bar{A}' &= -a\bar{e}_1\,, \quad \bar{e}_1' = b\bar{e}_2\,, \quad \bar{e}_2' = -b\bar{e}_2\,, \end{aligned}$$

where a and b satisfy (13). The solution of (13) is

$$a = \beta(\gamma + \sin 4\varepsilon_0 t)^{1/2}\,, \quad b = 2\varepsilon_0 \sqrt{(\gamma^2 - 1)}\,(\gamma + \sin 4\varepsilon_0 t)^{-1}\,,$$

where $\gamma \geqq 1$, $\beta \geqq 0$ are constants and $\varepsilon_0 = \sqrt{\alpha_1}$. The solution of (30) are two matrices, say g_5 for the first, g_6 for the second equation. Then

$$g_5 = \begin{pmatrix} 1, & 0 \\ T_0, & G_0 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1, & 0 \\ -T_0, & G_0 \end{pmatrix}$$

and

$$T_0 = \frac{\beta}{2\varepsilon_0\sqrt{\gamma}} \begin{pmatrix} \gamma \sin 2\varepsilon_0 t + 1 - \cos 2\varepsilon_0 t \\ \sqrt{(\gamma^2 - 1)}\,(1 - \cos 2t) \end{pmatrix},$$

$$G_0 = (\gamma^2 + \gamma \sin 4\varepsilon_0 t)^{-1/2} \begin{pmatrix} \gamma \cos 2\varepsilon_0 t + \sin 2\varepsilon_0 t, & -\sqrt{(\gamma^2 - 1)} \sin 2\varepsilon_0 t \\ \sqrt{(\gamma^2 - 1)} \sin 2\varepsilon_0 t\;, & \gamma \cos 2\varepsilon_0 t + \sin 2\varepsilon_0 t \end{pmatrix}$$

Further we get

$$(31)\qquad g_5 g_6^{-1} = \begin{pmatrix} 1, & 0 \\ 2T_0, & E \end{pmatrix};$$

denote $t_0 = 2T_0$.

Theorem 2. *The matrix of any Darboux 2-motion in E_n can be written in the following form:*

$$(32)\qquad g(t) = \begin{bmatrix} 1, & 0, & \dots, & 0, & 0 \\ t_1, & g_1, & \dots, & 0, & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ t_m, & 0, & \dots, & g_m, & 0 \\ t_0, & 0, & \dots, & 0, & E \end{bmatrix},$$

where $t_1, \dots, t_m, g_1, \dots, g_m$ are given by (28) or (29), t_0 is given by (31), E is the 2×2 identity matrix, m is the number of different characteristic roots of ω_{11}.

To simplify the final form of our result, let us change the parameter t of the motion in such a way that $\alpha_1 = 1$. Then of course the condition $\lambda_1 = 1$ must be omitted. Let us remark that in that case $\mu_\alpha = 0$ means $\lambda_\alpha = 1$.

Theorem 3. *Let the following numbers* $\lambda_1, \ldots, \lambda_m, \gamma_1, \ldots, \gamma_m, \beta, \gamma, k_1, \ldots, k_m, \delta_1, \delta_2$ *be given in such a way that*

i) $\lambda_i, \nu_i, \beta, \gamma$ *are real numbers,* k_i *are natural numbers,* $i = 1, \ldots, m$,

ii) $\lambda_1 > \lambda_2 > \ldots > \lambda_m > 0$, $\nu_i \geqq 0$, $\beta \geqq 0$, $\gamma \geqq 1$, δ_1 *and* δ_2 *are equal to zero or one,*

iii) $2\sum_{i=1}^{m} k_i + \delta_1 + \delta_2 = n$,

iv) $\lambda_i \neq 1$ *implies* k_i *even,*

v) $\delta_1 = 0$ *iff* $\beta = 0$; $\delta_2 = 0$ *iff* $\gamma = 1$; $\delta_2 = 1$ *implies* $\delta_1 = 1$. *Then there is exactly one Darboux 2-motion in* E_n *which has given numbers as its invariants in the sense described above,* δ_1, δ_2 *are connected with the appearance of the last two rows and columns. All Darboux 2-motions in* E_n *which are not motions in any* E_{n-1} *are those described above.*

Theorem 4. *All trajectories of any Darboux 2-motion in* E_n *are either ellipses or direct line segments.*

Proof. For any trajectory we have $X''' + X' = 0$. Integration yields the desired result.

Remark. In E_2 there is only one Darboux 2-motion, namely the elliptical motion, in E_3 we have also only one Darboux 2-motion, namely that originally described by Darboux (see [2]). In E_4 we have two Darboux 2-motions, either $k_1 = 2$, $\delta_1 = \delta_2 = 0$ (with $\lambda_1 \neq 1$ and $\lambda_1 = 1$ as subcases) or $k_1 = 1$, $\delta_1 = \delta_2 = 1$.

References

[1] *Chevalley C.:* Theory of Lie groups I, Princeton Univ. Press, Princeton, New Jersey, 1946.

[2] *Blaschke W.:* Zur Kinematik. Abh. math. Sem. Univ. Hamburg, *22*, (1958), 171—175.

[3] *Karger A.:* Kinematic geometry in *n*-dimensional Euclidean and spherical space. Czech. Math. Journ., *22* (*97*) (1972), 83—107.

Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).