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Jaroslav Nešetřil; Vojtěch Rödl

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RAMSEY THEOREM FOR CLASSES OF HYPERGRAPHS WITH FORBIDDEN COMPLETE SUBHYPERGRAPHS

JAROSLAV NEŠETŘIL, VOJTĚCH RÖDL, Praha

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INTRODUCTION

In this paper we prove what has been called the Galvin Ramsey property of hypergraphs:

For every (finite) k-uniform hypergraph $(X, \mathscr{M})$ and for every m natural there exists a k-uniform hypergraph $(Y, \mathscr{N})$ with the following property: for every partition

$$\mathscr{N} = A_1 \cup A_2 \cup \ldots \cup A_m$$

there exists a set $X' \subseteq Y$ such that $(X', \mathscr{N}|_{X'})$ is isomorphic to $(X, \mathscr{M})$ and $\mathscr{N}|_{X'} \subseteq A_i$ for a certain $i \in [1, n]$, where $\mathscr{N}|_{X'} = \{N \in \mathscr{N};\ N \subseteq X'\}$).

Moreover, in the case that the hypergraph $(X, \mathscr{M})$ does not contain the n-complete hypergraph (i.e. the hypergraph

$$(\{1, \ldots, n\}, \{M \subseteq [1, n];\ |M| = k\}))$$

then $(Y, \mathscr{N})$ can be chosen with the same property. This answers a problem of Erdös and others. This result was mentioned in [2]. See [1] for a survey of recent developments of this theory.

The theorem gives an essential strengthening of a classical Ramsey theorem [7]. Moreover, establishing the above theorems we have a perfect analogy with the graph-theoretical theorems proved in [3] and [4]. These theorems are generalized here, too. However, the case of hypergraphs seems to be much more difficult than the case of graphs and a new method of proof has to be used.

The method may be further strengthened and generalized for classes of hypergraphs and relational systems, see our forthcoming paper [6]. In these generalizations, more complex and symbolic (i.e. categorial) methods have to be used. The proof presented in this paper is chronologically the first one and in a way more direct and transparent than the methods of [6].

For $i \leqq j$ we put $[i, j] = \{i, i + 1, \ldots, j\}$. A hypergraph is *a couple* $\mathscr{H} = (X, \mathscr{M})$ where X is a finite set and $\mathscr{M} \subseteq P(X) = \{Y \subseteq X;\ Y \neq \emptyset\}$. A hypergraph $(X, \mathscr{M})$ is *k-uniform* (shortly k-hypergraph) if $M \in \mathscr{M} \Rightarrow |M| = k$.

An *embedding* f of a hypergraph $(X, \mathscr{M})$ into a hypergraph $(T, \mathscr{N})$ is a mapping $f : X \to Y$ which satisfies

(1) f is $1 - 1$,

(2) $M \in \mathscr{M} \Rightarrow \{f(m);\ m \in M\} = f(M) \in \mathscr{N}$,

(3) $f(M) \in \mathscr{N} \Rightarrow M \in \mathscr{M}$;

f is a *monomorphism* if f satisfies conditions (1) and (2). Let us remark that embeddings and monomorphisms are closed with respect to composition.

Denote by Mono $(\mathscr{H}, \mathscr{K})$ and Emb $(\mathscr{H}, \mathscr{K})$ the set of all monomorphisms and embeddings, from a hypergraph $\mathscr{H}$ into a hypergraph $\mathscr{K}$. Put Aut $(\mathscr{H}) =$ $=$ Mono $(\mathscr{H}, \mathscr{H})$ = Emb $(\mathscr{H}, \mathscr{H})$; Aut $(\mathscr{H})$ is a group.

We shall use the following convenient notation due to K. Leeb (see [1]):

$$\binom{\mathscr{K}}{\mathscr{H}} = \mathrm{Emb}\,(\mathscr{H}, \mathscr{K})/\mathrm{Aut}\,(\mathscr{H}) = \{[f];\ f \in \mathrm{Emb}\,(\mathscr{H}, \mathscr{K})\}$$

where $[f]$ is the equivalence class of the equivalence $\sim$ induced by Aut $(\mathscr{H})$, which contains f:

$$f \sim g \Leftrightarrow \exists h \in \mathrm{Aut}\,(\mathscr{H})\,(f = g \circ h)\,.$$

If $f : \mathscr{K} \to \mathscr{L}$ is an embedding and $\mathscr{H}$ is a hypergraph then $\binom{f}{\mathscr{H}} : \binom{\mathscr{K}}{\mathscr{H}} \to \binom{\mathscr{L}}{\mathscr{H}}$ is defined by $\binom{f}{\mathscr{H}}([g]) = [f \circ g]$. Using this notation one may restate the concept of *the Ramsey property of hypergraphs*: for every k-hypergraph $\mathscr{H}$ and for every m there exists a k-hypergraph $\mathscr{K}$ with the following property:

$$\text{for every mapping } c : \binom{\mathscr{K}}{\boldsymbol{k}} \to [1, m]$$

$$\text{there exists an embedding } f \in \mathrm{Emb}\,(\mathscr{H}, \mathscr{K})$$

such that the mapping $c \circ \binom{f}{\boldsymbol{k}}$ is constant. (If we do not need to specify the actual value of this constant we write $c \circ \binom{f}{\boldsymbol{k}} = \S.$) Here $\boldsymbol{k}$ is the k-hypergraph consisting of one edge only: $\boldsymbol{k} = ([1, k], \{[1, k]\})$.

To express briefly the above fact we write $\mathscr{H} \to_m^k \mathscr{K}$ (the *partition arrow* – see [2]).

Let us remark that

$$\mathscr{H} \to_m^k \mathscr{K} \to_n^k \mathscr{L} \Rightarrow \mathscr{H} \to_{mn}^k \mathscr{L} ;$$

hence the only essential arrow is $\mathscr{H} \to_2^k \mathscr{K}$ (of course, $\mathscr{H} \to_1^k \mathscr{K} \Leftrightarrow \mathrm{Emb}(\mathscr{H}, \mathscr{K}) \neq \emptyset$).

Let $k \leqq K$ be fixed. Denote by Hyp_k^K the class of all k-hypergraphs $\mathscr{K}$ with the property that $\mathrm{Emb}\left(\left([1,K]; \binom{[1,K]}{k}\right), \mathscr{K}\right) = \emptyset$ (k-hypergraphs without complete k-subhypergraphs with K vertices; for a set M, $\binom{M}{k}$ denotes the set of all k-element subsets of M).

SPECIAL CONCEPTS

The class of all finite k-hypergraphs together with the class of all embeddings between them form a category. To prove the Ramsey property of this category we need a "finer" structure:

Let $k \geqq 2$ (the arity of hypergraphs) be fixed from now on.

Let $0 \leqq a$ be a natural number. Denote by a Part (k) the class of all couples $((X_i;\, i \in [0, a]), \mathscr{M})$ where

a) $\bigcup_{i=0}^{a} X_i$ is an ordered set (the ordering will be denoted allways by $\leqq$, the "standard ordering");

b) $X_0 < X_a < X_{a-1} < \ldots < X_1$;

c) $X_i \neq \emptyset$, $i \in [1, a]$;

d) $\left(\bigcup_{i=0}^{a} X_i, \mathscr{M}\right)$ is a k-hypergraph;

e) $M \in \mathscr{M},\ i \in [1, a] \Rightarrow |M \cap X_i| \leqq 1$.

The family $(X_i;\, i \in [0, a])$ will be denoted briefly by $(X_i)_0^a$. Elements of the class a Part (k) will be called *a-parameter k-hypergraphs*. Let us observe that 0-parameter k-hypergraphs are just k-hypergraphs. Thus a-parameter k-hypergraphs are just k-hypergraphs with disjoint subsets of vertices used as parameters. The notion of an embedding may be generalized to the class a Part (k) as follows:

$$f \in a\ \mathrm{Emb}(\mathscr{H}, \mathscr{K}) \quad \text{for} \quad \mathscr{H} = ((X_i)_0^a, \mathscr{M}), \quad \mathscr{K} = ((Y_i)_0^a, \mathscr{N})$$

iff the following conditions are fulfilled:

a) $f : \bigcup_{i=0}^{a} X_i \to \bigcup_{i=0}^{a} Y_i$;

b) f is a monotone mapping (with respect to standard orderings);

c) $f \in \text{Mono}\left(\left(\bigcup_{i=0}^{a} X_i, \mathcal{M}\right)\left(\bigcup_{i=0}^{a} Y_i, \mathcal{N}\right)\right)$;

d) $f(X_i) \subset Y_i$; $i \in [0, a]$;

e) $f(M) \in \mathcal{N}, f(M) \cap Y_0 \neq \emptyset \Rightarrow M \in \mathcal{M}$.

f is called an *a-embedding*. An *a-monomorphism* is defined by the conditions a)–d).

Thus an a-embedding as a monomorphism which is an "embedding" for hyperedges which intersect X_0. As every a-embedding is a monotone maping, the set a Emb $(\mathcal{X}, \mathcal{X})$ consists of the identity mapping only and consequently, the equivalence induced by it on a Emb $(\mathcal{X}, \mathcal{Y})$ is the trivial equivalence.

As the notions introduced in the previous paragraph were categorial we may define for $\mathcal{X} = ((X_i)_0^a, \mathcal{M})$, $\mathcal{Y} = ((Y_i)_0^a, \mathcal{N})$,

$$\binom{((Y_i)_0^a, \mathcal{N})}{((X_i)_0^a, \mathcal{M})} = a \text{ Emb}(\mathcal{X}, \mathcal{Y}) .$$

Let $a \leqq k$. Put $\mathbf{k}_a = ((X_i)_0^a, \mathcal{M})$ where $X_0 = [1, k - a]$, $X_i = \{k - a + i\}$, $i \in [1, a]$, $\mathcal{M} = \{[1, k]\}$.

We write

$$\mathcal{X} \rightarrow_m^{k,a} \mathcal{Y}$$

iff for every mapping

$$c : \binom{\mathcal{Y}}{\mathbf{k}_a} \rightarrow [1, m]$$

there exists an a-embedding $f \in a$ Emb $(\mathcal{X}, \mathcal{Y})$ such that $c \circ \binom{f}{\mathbf{k}_a} = \S$ (= a constant maping); here $\binom{f}{\mathbf{k}_a}$ is defined by

$$\binom{f}{\mathbf{k}_a}(g) = f \circ g \quad \text{for} \quad g \in \binom{\mathcal{X}}{\mathbf{k}_a} .$$

Let us remark that the sets $\binom{((Y_i)_0^a, \mathcal{N})}{\mathbf{k}_a}$ and $\{N \in \mathcal{N};\ i \in [1, a] \Rightarrow N \cap Y_i \neq \emptyset\}$ are in a $1 - 1$ correspondence. We shall consider an $(a + 1)$-parameter k-hypergraph $((X_i)_0^{a+1}, \mathcal{M})$ sometimes as an a-parameter k-hypergraph $((\overline{X}_i)_0^a, \mathcal{M})$ where $\overline{X}_0 = X_0 \cup X_{a+1}$, $\overline{X}_i = X_i$ for $i \in [1, a]$.

The symbol $((X_i)_1^{a+1}, \mathcal{M})$ means the $(a + 1)$-parameter k-hypergraph $((\overline{X}_i)_0^{a+1}, \mathcal{M})$ where $\overline{X}_i = X_i$ for $i \in [1, a + 1]$ and $\overline{X}_0 = \emptyset$.

An embedding $f : ((X_i)_0^a, \mathcal{M}) \rightarrow ((Y_i)_0^{a+1}, \mathcal{N})$ means an a-embedding of $((X_i)_0^a, \mathcal{M})$ into $((\overline{Y}_i)_0^a, \mathcal{N})$, where $\overline{Y}_i = Y_i$ for $i \in [1, a]$, $\overline{Y}_0 = Y_0 \cup Y_{a+1}$.

We need a suitable generalization of the property "without complete subhypergraphs". This can be achieved as follows. Let $2 \leqq k$, $0 \leqq a \leqq k$, $\omega \subseteq [1, a]$, $K \geqq 0$. Define a class of a-parameter k-hypergraphs

$$\frac{\omega}{a} \operatorname{Part} \binom{K}{k}$$

as follows:

$$\mathscr{X} = ((X_i)_0^a, \mathscr{M}) \in \frac{\omega}{a} \operatorname{Part} \binom{K}{k} \quad \text{if} \quad \mathscr{X} \in a \operatorname{Part}(k)$$

and there is no set $M \subseteq \bigcup_{i=0}^{a} X_i$, $|M| = K$ with the following properties:

i) $|M \cap X_i| = 1, i \geqq 1 \Leftrightarrow i \in \omega$,

ii) $\binom{M}{k} \subseteq \mathscr{M}$.

(Thus $\mathscr{X}$ does not contain K-complete k-subhypergraphs with the last $|\omega|$ vertices belonging precisely to the parameters from the set ω).

Clearly

$$\frac{\phi}{a} \operatorname{Part} \binom{0}{k} = a \operatorname{Part}(k).$$

We prove here:

Main theorem. *Let $k \geqq 2$, $K \geqq 0$, $0 \leqq a \leqq k$. Then the class*

$$\frac{\omega}{a} \operatorname{Part} \binom{K}{k}$$

has the k_a-partition property; i.e., for every

$$\mathscr{X} \in \frac{\omega}{a} \operatorname{Part} \binom{K}{k}$$

there exists

$$\mathscr{Y} \in \frac{\omega}{a} \operatorname{Part} \binom{K}{k}$$

such that

$$\mathscr{X} \to_m^{k,a} \mathscr{Y}.$$

As sketched above, this implies

Corollary. *The class* Hyp_k^K *has the k-partition property.*

The proof of the main theorem has the following scheme:

Theorem 1. *For every* $2 \leqq k$, $0 \leqq a \leqq k$ *the class* a Part (k) *has the* k_a*-partition property.*

Theorem 2. *For every* $2 \leqq k$, $0 \leqq a \leqq k$, $\omega \subseteq [1, a]$, $K \geqq 0$ *the class*

$$\frac{\omega'}{a+1} \text{Part} \left(\frac{K}{k} \right)$$

has the fraction partition property (see the definition below).

Theorem 3. *For every* $2 \leqq k$, $0 \leqq a \leqq k$, $\omega \subseteq [1, a]$, $K \geqq k$ *the class*

$$\frac{\omega}{a} \text{Part} \left(\frac{K}{k} \right)$$

has the $\boldsymbol{k}_a$*-partition property.*

Let $2 \leqq k$, $0 \leqq a \leqq k$, $\omega \subseteq [1, a]$, $K \geqq 0$. We put $\omega' = \omega \cup \{a + 1\}$. For $((X_i)_0^{a+1}, \mathcal{M}) \in (a + 1)$ Part (k) we put

$$\binom{((X_i)_0^{a+1}, \mathcal{M})}{\boldsymbol{k}_{a/a+1}} = \binom{((X_i)_0^{a+1}, \mathcal{M})}{\boldsymbol{k}_a} \setminus \binom{((X_i)_0^{a+1}, \mathcal{M})}{\boldsymbol{k}_{a+1}}$$

(see the convention about $((X_i)_0^{a+1}, \mathcal{M})$ considered as an a-parameter k-hypergraph).

We write

$$((X_i)_0^{a+1}, \mathcal{M}) \to_m^{k, a/a+1} ((Y_i)_0^{a+1}, \mathcal{N})$$

if the following statement is true:

For every colouring

$$c : \binom{((Y_i)_0^{a+1}, \mathcal{N})}{\boldsymbol{k}_{a/a+1}} \to [0, m]$$

there exists an $(a + 1)$-embedding $f : ((X_i)_0^{a+1}, \mathcal{M}) \to ((Y_i)_0^{a+1}, \mathcal{N})$ such that $c \circ f \circ g = \S$ for every

$$g \in \binom{((X_i)_0^{a+1}, \mathcal{N})}{\boldsymbol{k}_{a/a+1}}$$

(where $\S \in [0, m]$ is a constant).

Finally, the class

$$\frac{\omega'}{a+1} \text{Part} \left(\frac{K}{k} \right)$$

has the fraction partition property if for every

$$((X_i)_0^{a+1}, \mathcal{M}) \in \frac{\omega'}{a+1} \text{Part} \left(\frac{K}{k} \right)$$

there exists

$$((Y_i)_0^{a+1}, \mathcal{N}) \in \frac{\omega'}{a+1} \operatorname{Part}\left(\frac{K}{k}\right)$$

such that

$$((X_i)_0^{a+1}, \mathcal{M}) \to_2^{k,a/a+1} ((Y_i)_0^{a+1}, \mathcal{N}) .$$

The proof of Theorem 1 is crucial. Using its assertion one can easily prove Theorem 2 and then the proof of Theorem 3 is analogous to that of Theorem 1.

Proof of Theorem 1. Let $k \geqq 2$ be fixed. The proof will be done by induction on $k - a$.

I. The boundary case $k = a$: Let $((X_i)_0^k, \mathcal{M}) \in k \operatorname{Part}(k)$. Put $\mathcal{M}' = \{M \in \mathcal{M};\ M \subseteq \bigcup_{i=1}^k X_i\}$. It is $((X_i)_1^k, \mathcal{M}') \in k \operatorname{Part}(k)$. First we prove the existence of $((Y_i)_1^k, \mathcal{N}') \in k \operatorname{Part}(k)$ such that $((X_i)_1^k, \mathcal{M}') \to_2^{k,k} (Y_i)_1^k, \mathcal{N}')$. In this case k-embeddings of $((X_i)_1^k, \mathcal{M}')$ coincide with k-monomorphism and the existence of $((Y_i)_1^k, \mathcal{N}')$ is a straightforward application of the Dirichlet principle.

Let

$$\binom{((Y_i)_1^k, \mathcal{N}')}{((X_i)_1^k, \mathcal{M}')} = \{f_j; j \in [1, r]\}$$

(see the remarks concerning the definition of a-embeddings). Put $Y_0 = X_0 \times [1, r]$ and define the ordering of $\bigcup_{i=0}^k Y_i$ in such a way that

$$x \in X_0\,,\quad j \in [1, r] \Rightarrow (x, j) < Y_k\,,$$

$$(x, j) < (x', j') \Leftrightarrow \text{either } j < j' \text{ or } j = j' \text{ and } x < x'\,.$$

Furthermore, let $\bar{f}_j : X_0 \to Y_0$, $j \in [1, r]$ be monotone $1 - 1$ mappings which satisfy $\bar{f}_j(X_0) < \bar{f}_{j'}(X_0)$ for $j < j'$ (this is possible by the above choice of Y_0). We may define $\mathcal{N}$ by

$$N \in \mathcal{N} \Leftrightarrow \text{either } N \in \mathcal{N}' \text{ or } N = f_j(M \cap (\bigcup_{i=1}^k X_i)) \cup \bar{f}_j(M \cap X_0)$$

for a certain $j \in [1, r]$ and $M \in \mathcal{M}$.

From the definition of $\mathcal{N}$ and by the fact

$$((X_i)_1^k, \mathcal{M}') \to_2^{k,k} ((Y_i)_1^k, \mathcal{N}')$$

we get immediately

$$((X_i)_0^k, \mathcal{M}) \to_2^{k,k} ((Y_i)_0^k, \mathcal{N}) .$$

This completes the proof of the boundary case $k = a$ and I. Suppose that the assertion of Theorem 1 is valid for all a', $a < a' \leqq k$ and let $a < k$. In this situation we need another simplification which is crucial to our method:

II. Reduction to induced colourings.

Lemma. *Let $a < k$ be fixed. Then the following two statements are equivalent.*
1) *For every* $((X_i)_0^a, \mathscr{M}) \in a$ Part (k) *there exists* $((Y_i)_0^a, \mathscr{N}) \in a$ Part (k) *such that*

$$((X_i)_0^a, \mathscr{M}) \to_2^{k,a} ((Y_i)_0^a, \mathscr{N}) .$$

2) *For every* $((X_i)_0^a, \mathscr{M}) \in a$ Part (k) *there exists* $((Y_i)_0^a, \mathscr{N}) \in a$ Part (k) *such that*

$$((X_i)_0^a, \mathscr{M}) \to_2^{k,a,\mathrm{good}} ((Y_i)_0^a, \mathscr{N}) .$$

Here the only undefined symbol $\to_2^{k,a,\mathrm{good}}$ means the following:

$$\text{for every } c : \binom{((Y_i)_0^a, \mathscr{N})}{k_a} \to [0, 1]$$

there exists $f \in a$ Emb $(((X_i)_0^a, \mathscr{M}), ((Y_i)_0^a, \mathscr{N}))$ and a mapping $c' : Y_0 \to [0, 1]$ such that it holds for every $M \in \mathscr{M}$ satisfying $M \cap X_i \neq \emptyset$ for $i \in [0, a]$: $c(f(M)) = c'(f(m_M))$, where m_M is the last element of the set $M \cap X_0$).

Proof of Lemma. Obviously 1) $\Rightarrow$ 2).

Let 2) be true and let $((X_i)_0^a, \mathscr{M})$ be given. Consider $(X_0, \mathscr{M}_0)$ where $\mathscr{M}_0 = \{M \in \mathscr{M};\ M \subseteq X_0\}$. Let $(X'_0, \mathscr{M}'_0)$ be a k-hypergraph with the following property: for every partition $X'_0 = X' \cup X''$ there exists an embedding $f : (X_0, \mathscr{M}_0) \to (X'_0, \mathscr{M}'_0)$ such that either $f(X_0) \subseteq X'$ or $f(X_0) \subseteq X''$ (this fact is in [6] denoted by $(X_0, \mathscr{M}_0) \to_2^1$ $\to_2^1 (X'_0, \mathscr{M}'_0)$). The existence of such a k-hypergraph can be proved by various means: either similarly to Folkman's method [0], or (less elementarily) by a type representation of hypergraphs (see [2]) or (most quickly) using the Erdös-Hajnal Theorem (see [5], where the result needed here is explicitly proved).

Let Emb $((X_0, \mathscr{M}_0), (X'_0, \mathscr{M}'_0)) = \{f_j;\ j \in [1, r]\}$. Let $((X'_i)_0^a, \mathscr{M}')$ be an a-parameter k-hypergraph which satisfies: for every f_j, $j \in [1, r]$, there exists an a-embedding $\bar{f}_j : ((X_i)_0^a, \mathscr{M}) \to ((X'_i)_0^a, \mathscr{M}')$ such that $\bar{f}_j/X_0 = f_j$. (This fact may be established quite similarly as in I by suitably enlarging the sets X_i, $i > 0$.)

Now

$$((X'_i)_0^a, \mathscr{M}') \to_2^{k,a,\mathrm{good}} ((Y_i)_0^a, \mathscr{N})$$

implies

$$((X_i)_0^a, \mathscr{M}) \to_2^{k,a} ((Y_i)_0^a, \mathscr{N})$$

by putting together the definitions of

$$\to_2^{k,a,\mathrm{good}} \quad \text{and} \quad ((X'_i)_0^a, \mathscr{M}') .$$

This proves Lemma.

Let $((X_i)_0^a, \mathscr{M}) \in a\ \mathrm{Part}\,(k)$, $a < k$, be fixed. Assume that Theorem 1 is valid for all a', $k \geqq a' > a$. In this situation we prove the existence of

$$((Y_i)_0^a, \mathscr{N}) \in a\ \mathrm{Part}\,(k)$$

such that

$$((X_i)_0^a, \mathscr{M}) \to_2^{k,a,\mathrm{good}} ((Y_i)_0^a, \mathscr{N}).$$

By virtue of the above Lemma this implies Theorem 1. This will be proved by induction on $|X_0|$. The boundary case $X_0 = \emptyset$ is trivial. (In this case $\mathscr{M} = \emptyset$ by $k > a$.)

Let $|X_0| > 0$ and let x be the last element of X_0 in the standard ordering of X_0. Put $X'_0 = X_0 \setminus \{x\}$, $X'_i = X_i$ for $a \geqq i > 0$, $X'_{a+1} = \{x\}$. Put $\mathscr{M}' = \{M \in \mathscr{M};\ x \in M\}$.

By the induction hypothesis there exists $((Y'_i)_0^a, \mathscr{N}') \in a\ \mathrm{Part}\,(k)$ such that

$$((X'_i)_0^a, \mathscr{M}') \to_2^{k,a,\mathrm{good}} ((Y'_i)_0^a, \mathscr{N}').$$

Note that $((X'_i)_0^{a+1}, \mathscr{M}) \in (a+1)\ \mathrm{Part}\,(k)$. Write two lines of the Ramsey arrows

$$\begin{aligned}
\text{LA: } ((X'_i)_0^a, \mathscr{M}') &\to_2^{k,a,\mathrm{good}} ((Y'_i)_0^a, \mathscr{N}') \\
&\qquad\qquad\downarrow \varepsilon \\
&\qquad ((Y_i^*)_0^{a+1}, \mathscr{N}^*) \to_m^{k,a+1} ((Y''_i)_0^{a+1}, \mathscr{N}''),\\
\text{LB: } ((X'_i)_0^{a+1}, \mathscr{M}) &\to_2^{k,a+1} ((Z'_i)_0^{a+1}, \mathscr{P}') \\
&\qquad\qquad\downarrow \iota \\
&\qquad ((Z_i^*)_0^{a+1}, \mathscr{P}^*) \to_n^{k,a/a+1} ((Z''_i)_0^{a+1}, \mathscr{P}'').
\end{aligned}$$

This is the basic part of the proof and the not yet defined symbols have the following meaning:

i) $((Y_i^*)_0^{a+1}, \mathscr{N}^*)$ is a modification of the hypergraph $((Y'_i)_0^a, \mathscr{N}')$ obtained as follows:

We put $Y_i^* = Y'_i$ for $i \in [0, a]$, $Y_{a+1}^* = \{x^*\}$ where $x^* \notin \bigcup_{i=0}^{a} Y_i^*$ and the standard ordering of $\bigcup_{i=0}^{a+1} Y_i^*$ is defined by the standard ordering of Y together with $Y'_0 < x^* < Y'_a$;

$$\mathscr{N}^* = \mathscr{N}' \cup \mathscr{N}'^*$$

where $N \in \mathscr{N}'^* \Leftrightarrow |N \cap Y'_i| \leqq 1$, $i \in [1, a]$, $|N| = k$ and $x^* \in N$. Clearly $((Y_i^*)_0^{a+1}, \mathscr{N}^*) \in (a+1)\ \mathrm{Part}\,(k)$.

ε denotes the just described inclusion (which is in fact, an a-monomorphism).

ii) $((Y''_i)_0^{a+1}, \mathscr{N}'') \in (a+1)\ \mathrm{Part}\,(k)$ is an $(a+1)$-parameter k-hypergraph whose existence is guaranted by the induction hypothesis, m is a parameter whose value will be discussed later in the proof.

iii) The existence of $((Z'_i)_0^{a+1}, \mathscr{P}') \in (a+1)\ \mathrm{Part}\,(k)$ follows again by the induction hypothesis.

iv) $((Z_i^*)_0^{a+1}, \mathscr{P}^*) \in (a+1)$ Part (k) is a modification of $((Z_i')_0^{a+1}, \mathscr{P}')$ which we get as follows:

$$Z_i' = Z_i^*, \quad i \in [0, a+1],$$

$$\mathscr{P}^* = \mathscr{P}' \cup \mathscr{P}'^*$$

where

$$M \in \mathscr{P}'^* \Leftrightarrow |M \cap Z_i'| = 1, \quad i \in [1, a]$$

and

$$|M \cap Z_0'| = k - a.$$

ι denotes the just described inclusion, it is an $(a+1)$-monomorphism.

v) The arrow symbol $\to_n^{k,a,a/a+1}$, the fraction Ramsey arrow, was defined above. The value of the parameter n will be specified later in the proof.

This explains all the necessary symbols. All objects are properly defined either directly or by induction hypothesis. Only the existence of $((Z_i'')_0^{a+1}, \mathscr{P}'') \in (a+1)$ Part (k) with the property given by the fraction Ramsey arrow has to be proved. Let us postpone this to the end of the proof.

Define $((Y_i)_0^{a+1}, \mathscr{N}) \in (a+1)$ Part (k) by $Y_i = Y_i'' \times Z_i''$ for $i \in [0, a+1]$ and let the standard ordering of $\bigcup_{i=0}^{a+1} Y_i$ be defined lexicographically by standard orderings;

$$N \in \mathscr{N} \Leftrightarrow N = \{(x_i, y_i),\ i \in [1, k]\},$$

where

$$x_1 < x_2 < \ldots < x_k, \quad y_1 < y_2 < \ldots < y_k,$$

$$N'' = \{x_i;\ i \in [1, k]\} \in \mathscr{N}'', \quad P'' = \{y_i;\ i \in [1, k]\} \in \mathscr{P}''$$

and

$$N'' \cap Y_i'' \neq \emptyset \Leftrightarrow P'' \cap Z_i'' \neq \emptyset.$$

Proposition. *There are m, n such that*

$$((X_i')_0^{a+1}, \mathscr{M}) \to_2^{k,a,\text{good}} ((Y_i)_0^{a+1}, \mathscr{N}).$$

Proof. Let

$$c : \binom{((Y_i)_0^{a+1}, \mathscr{N})}{k_a} \to [0, 1]$$

be a fixed colouring.

The proof will be divided in to five steps denoted by c(1) – c(5).

c(1): Put

$$\binom{((Z_i'')_0^{a+1}, \mathscr{P}'')}{k_{a+1}} = \mathfrak{A}$$

and define the colouring

$$c'' : \binom{((Y_i'')_0^{a+1}, \mathcal{N}'')}{\boldsymbol{k}_{a+1}} \to [0, 1]^{\mathfrak{A}}$$

by $c''(f) = (c(f, g);\ g \in \mathfrak{A})$. For

$$f : \boldsymbol{k}_{a+1} \to ((Y_i'')_0^{a+1}, \mathcal{N}''),$$
$$g : \boldsymbol{k}_{a+1} \to ((Z_i'')_0^{a+1}, \mathcal{P}''),$$

$(f, g) = f \times g$ is the unique mapping $\boldsymbol{k}_{a+1} \to ((Y_i)_0^{a+1}, \mathcal{N})$ induced by f and g.

If we choose $m \geqq 2^{|\mathfrak{A}|}$, the line LA implies the existence of an $(a + 1)$-embedding $\varphi'' : ((Y_i^*)_0^{a+1}, \mathcal{N}^*) \to ((Y_i'')_0^{a+1}, \mathcal{N}'')$ with the property

$$c'' \circ \binom{\varphi''}{\boldsymbol{k}_{a+1}} = \S\,.$$

c(2): Put

$$\mathcal{B} = \binom{((Y_i^*)_0^{a+1}, \mathcal{N}^*)}{\boldsymbol{k}_{a/a+1}},$$

define the colouring

$$d'' : \binom{(Z_i'')_0^{a+1}, \mathcal{P}'')}{\boldsymbol{k}_{a/a+1}} \to [0, 1]^{|\mathcal{B}|}$$

by $d''(g) = (c(\varphi'' \circ f, g);\ f \in \mathcal{B})$.

If we choose $n \geqq 2^{|\mathcal{B}|}$ then the line LB implies the existence of an $(a + 1)$-embedding $\psi'' : ((Z_i^*)_0^{a+1}, \mathcal{P}^*) \to ((Z_i'')_0^{a+1}, \mathcal{P}'')$ such that ψ'' satisfies the conditions from the definition of the fraction arrow $\to_n^{k,a/a+1}$.

Let us remark that the above choice of m and n is consistent: given $((Y_i^*)_0^{a+1}, \mathcal{N}^*)$ and $((Z_i^*)_0^{a+1}, \mathcal{P}^*)$, we choose n first and after defining $((Z_i'')_0^{a+1}, \mathcal{P}'')$ we choose m.

c(3): Define the colouring

$$d' : \binom{((Z_i')_0^{a+1}, \mathcal{P}')}{\boldsymbol{k}_{a+1}} \to [0, 1]$$

by $d'(g) = i \Leftrightarrow c(\varphi'' \circ f, \psi'' \circ \iota \circ g) = i$ for every

$$f \in \binom{((Y_i^*)_0^{a+1}, \mathcal{N}^*)}{\boldsymbol{k}_{a+1}}.$$

(By c(1) this definition is consistent.) By the line LB there exists an $(a + 1)$-embedding

$$\psi' : ((X_i')_0^{a+1}, \mathcal{M}) \to ((Z_i')_0^{a+1}, \mathcal{P}')$$

such that

$$d' \circ \binom{\psi'}{\boldsymbol{k}_{a+1}} = \S\,.$$

c(4): Define the colouring

$$c' : \binom{((Y'_i)_0^a, \mathcal{N}')}{k_a} \to [0, 1]$$

by $c'(f) = i \Leftrightarrow c(\varphi'' \circ \varepsilon \circ f, \psi \circ \iota \circ g) = i$ for every

$$g \in \binom{((Z'_i)_0^{a+1}, \mathcal{P}')}{k_{a/a+1}}.$$

(By c(2) this definition is consistent.) By the line LA there exists an a-embedding

$$\varphi' : ((X'_i)_0^a, \mathcal{M}') \to ((Y'_i)_0^a, \mathcal{N}')$$

and a mapping $c^v : Y'_0 \to [0, 1]$ such that $c'(\varphi' \circ f) = c^v(\varphi' \circ f(*))$ for every

$$f \in \binom{((X'_i)_0^a, \mathcal{M}')}{k_a}$$

(here $*$ is the $(k - a)$-th vertex in the standard ordering of k_a – see the definition of $\to_2^{k,a,\text{good}}$).

c(5): Let us define the mapping $\chi : \bigcup_{i=0}^{a} X_i \to \bigcup_{i=0}^{a+1} Y_i$ by $\chi(y) = (\varphi'' \circ \varepsilon \circ \varphi'(y), \psi'' \circ \iota \circ \psi'(y))$ for $y \neq x$ and $\chi(x) = (x^*, \psi'' \circ \iota \circ \psi'(x))$.

We have to prove that χ is an a-embedding and that it satisfies the condition given by the definition $\to_2^{k,a,\text{good}}$. Clearly χ is an a-monomorphism.

Let $\chi(M) \in \mathcal{N}$, $\chi(M) \cap Y_0) \neq \emptyset$. Then there are two possibilities:

either (i) $\chi(M) \cap Y_{a+1} \neq \emptyset$

or (ii) $\chi(M) \cap Y_{a+1} = \emptyset$.

In the case (i) necessarily $|\chi(M) \cap Y_{a+1}| = 1$ (by the definition of $((Y_i)_0^{a+1}, \mathcal{N})$, ψ' and ψ'' are $(a + 1)$-embeddings), and as $\iota(A) \in \mathcal{P}^* \Rightarrow A \in \mathcal{P}'$ whenever $\iota(A) \cap Z^*_{a+1} \neq \emptyset$ we get $M \in \mathcal{M}$.

In the case (ii) we use similarly the a-embeddings φ'' and φ' and the fact that $\varepsilon(A) \in \mathcal{N}^* \Rightarrow A \in \mathcal{N}'$ whenever $\varepsilon(A) \cap Y^*_{a+1} = \emptyset$. Consequently, χ is an a-embedding.

To prove the "goodness" of χ with respect to the colouring c let us define $\bar{c} : Y_0 \to [0, 1]$ by $\bar{c}|\chi(X'_0) = c^v$ (see c(4)) and $\bar{c}(x) = i$ where

$$d' \circ \binom{\psi'}{k_{a+1}} \equiv i$$

(see c(3)).

As

$$\binom{((X_i)_0^a, \mathcal{M})}{k_a} = \binom{((X'_i)_0^{a+1}, \mathcal{M})}{k_{a+1}} \cup \binom{((X'_i)_0^{a+1}, \mathcal{M})}{k_{a/a+1}}$$

and the sets on the right hand side are disjoint we have two possibilities:

(i) $$f \in \binom{((X_i')_0^{a+1}, \mathscr{M})}{\boldsymbol{k}_{a+1}} \Rightarrow c(\chi \circ f) = d'(\psi' \circ f) = i$$

by c(1) and c(3);

(ii) $$f \in \binom{((X_i')_0^{a+1}, \mathscr{M})}{\boldsymbol{k}_{a/a+1}} \Rightarrow c(\chi \circ f) = c'(\varphi' \circ f) = c^v(\varphi' \circ f(*))$$

where $*$ is the $(k - a)$-th point of the set $[1, k]$ in the standard ordering.

This follows by c(2) and c(4).

Thus we proved that for every colouring c there exists an a-embedding $\chi : ((X_i)_0^a, \mathscr{M}) \to ((Y_i)_0^{a+1}, \mathscr{N})$ and a mapping $\bar{c} : Y_0 \to [0, 1]$ with the properties given by the arrow $\to_2^{k,a,\text{good}}$. This completes the proof of Proposition.

To complete the proof of Theorem 1 it remains to prove the existence of $((Z_i'')_0^{a+1}, \mathscr{P}'')$ such that

$$((Z_i^*)_0^{a+1}, \mathscr{P}^*) \to_n^{k,a/a+1} ((Z_i'')_0^{a+1}, \mathscr{P}'') .$$

Let us remark that $((Z_i^*)_0^{a+1}, \mathscr{P}^*)$ has the following special property (which is guaranteed by the monomorphism ι):

$$\mathscr{P}^* \supseteq \mathscr{P}_0^* = \{M \subseteq \bigcup_{i=0}^{a} Z_i;\ i \in [1, a] \Rightarrow |M \cap Z_i| = 1\} .$$

The existence of $((Z_i'')_0^{a+1}, \mathscr{P}'')$ may be seen as follows:

First, let

$$((Z_i^*)_0^a, \mathscr{P}_0^*) \to_n^{k,a} ((Z_i'')_0^a, \mathscr{P}_0'') .$$

(This may be established by virtue of the Ramsey theorem similarly as in I above. One uses the fact that each member of $\mathscr{P}_0^*$ has an intersection with *all* the sets Z_i, $i \in [1, a]$.) Put

$$\binom{((Z_i'')_0^a, \mathscr{P}_0'')}{((Z_i^*)_0^a, \mathscr{P}_0^*)} = \{f_j;\ j \in [1, r]\} .$$

Again, it is simple to find an $(a + 1)$-parameter k-hypergraph $((Z_i'')_0^{a+1}, \mathscr{P}'')$ such that every embedding f_j, $j \in [1, r]$, may be extended to an $(a + 1)$-embedding $\bar{f}_j : ((Z_i^*)_0^{a+1}, \mathscr{P}^*) \to ((Z_i'')_0^{a+1}, \mathscr{P}'')$. Finally,

$$((Z_i^*)_0^{a+1}, \mathscr{P}^*) \to_n^{k,a/a+1} ((Z_i'')_0^{a+1}, \mathscr{P}'')$$

follows by checking the definitions.

This is the end of the proof of Theorem 1.

Proof of Theorem 2 uses Theorem 1.

Let

$$((X_i)_0^{a+1}, \mathcal{M}) \in \frac{\omega'}{a+1} \operatorname{Part}\left(\frac{K}{k}\right),$$

$$\omega' = \omega \cup \{a+1\}.$$

Consider $((X_i)_0^a, \mathcal{M}')$ where $\mathcal{M}' = \{M \in \mathcal{M};\ M \cap X_{a+1} = \emptyset\}$. By Theorem 1, there exists $((Y_i)_0^a, \mathcal{N}') \in a \operatorname{Part}(k)$ such that

$$((X_i)_0^a, \mathcal{M}') \to_2^{k,a} ((Y_i)_0^a, \mathcal{N}').$$

Let

$$\binom{((Y_i)_0^a, \mathcal{N}')}{((X_i)_0^a, \mathcal{M}')} = \{f_j;\ j \in [1, r]\}.$$

Then there exists $((Y_i)_0^{a+1}, \mathcal{N}) \in (a+1) \operatorname{Part}(k)$ such that

(i) the inclusion $((Y_i)_0^a, \mathcal{N}') \to ((Y_i)_0^{a+1}, \mathcal{N})$ is an a-embedding;

(ii) for every $j \in [1, r]$ there exists an $(a+1)$-embedding $\bar{f}_j : ((X_i)_0^{a+1}, \mathcal{M}) \to \to ((Y_i)_0^{a+1}, \mathcal{N})$ such that $\bar{f}_j(x) = f_j(x)$ for $x \notin X_{a+1}$ and $\bar{f}_j(X_{a+1}) \cap \bar{f}_{j'}(X_{a+1}) = \emptyset$ whenever $j \neq j'$;

(iii) for every $N \in \mathcal{N} \setminus \mathcal{N}'$ there exists $j \in [1, r]$ such that $\bar{f}_j(M) = N$ for an $M \in \mathcal{M}$.

These properties may be taken as the definition of $((Y_i)_0^{a+1}, \mathcal{N})$. As

$$((X_i)_0^{a+1}, \mathcal{M}) \in \frac{\omega'}{a+1} \operatorname{Part}\left(\frac{K}{k}\right),$$

it is easy to see (from the definition) that

$$((Y_i)_0^{a+1}, \mathcal{N}) \in \frac{\omega'}{a+1} \operatorname{Part}\left(\frac{K}{k}\right).$$

Moreover,

$$((X_i)_0^{a+1}, \mathcal{M}) \to_2^{k,a/a+1} ((Y_i)_0^{a+1}, \mathcal{N}).$$

Proof of Theorem 3 is quite analogous to the proof of Theorem 1 with only one modification:

One has to prove that all constructed hypergraphs belong to the class

$$\frac{\omega}{a} \operatorname{Part}\left(\frac{K}{k}\right).$$

This is true by the following argument (we refer to the above proof of Theorem 1):

Let $\omega \subseteq [1, a]$, $K > k \geqq a$ be fixed (the case $K = k$ for

$$\frac{\omega}{a} \operatorname{Part}\left(\frac{K}{k}\right)$$

involves only hypergraphs without any hyperedges). Given

$$((X_i)_0^a, \mathscr{M}) \in \frac{\omega}{a}\,\mathrm{Part}\left(\frac{K}{k}\right),$$

we prove by induction on $k - a$ the existence of

$$((Y_i)_0^a, \mathscr{N}) \in \frac{\omega}{a}\,\mathrm{Part}\left(\frac{K}{k}\right)$$

such that

$$((X_i)_0^a, \mathscr{M}) \to_2^{k,a} ((Y_i)_0^a, \mathscr{N}).$$

ad I) (we follow the proof of Theorem 1): The boundary case $k = a$ can be handled exactly in the same way as

$$\frac{\omega}{k}\,\mathrm{Part}\left(\frac{K}{k}\right) = k\,\mathrm{Part}\,(k).$$

ad II): Lemma remains valid if we write everywhere

$$\frac{\omega}{a}\,\mathrm{Part}\,\frac{K}{k}$$

instead of a Part (k).

The proof of Lemma does not change, we have to prove only that $((X'_i)_0^a, \mathscr{N}')$ may be chosen such that

$$((X'_i)_0^a, \mathscr{N}') \in \frac{\omega}{a}\,\mathrm{Part}\left(\frac{K}{k}\right)$$

by amalgamation. The following one is the basic fact which makes it possible to translate the proof for the class a Part (k) into the proof for the class

$$\frac{\omega}{a}\,\mathrm{Part}\,\frac{K}{k}:$$

if

$$((X_i)_0^a, \mathscr{M}) \in \frac{\omega}{a}\,\mathrm{Part}\left(\frac{K}{k}\right), \quad ((X'_i)_0^a, \mathscr{M}') \in \frac{\omega}{a}\,\mathrm{Part}\left(\frac{K}{k}\right)$$

and if

$$\{M \in \mathscr{M};\ M \subseteq \bigcup_{i=0}^{a} X_i \cap \bigcup_{i=0}^{a} X'_i\} = \{M \in \mathscr{M}';\ M \subseteq \bigcup_{i=0}^{a} X_i \cap \bigcup_{i=0}^{a} X'_i\}$$

then

$$((X_i \cup X'_i)_0^a, \mathscr{M} \cup \mathscr{M}') \in \frac{\omega}{a}\,\mathrm{Part}\left(\frac{K}{k}\right)$$

("amalgamation property").

ad III): We may choose

$$((Y_i)_0^a, \mathcal{N}) \in \frac{\omega}{a} \operatorname{Part}\binom{K}{k}$$

by the amalgamation property.

ad IV): It is

$$((X_i')_0^a, \mathcal{M}') \in \frac{\omega}{a} \operatorname{Part}\binom{K}{k}$$

and

$$((X_i'')_0^a, \mathcal{M}) \in \frac{\omega'}{a+1} \operatorname{Part}\binom{K}{k},$$

where $\omega' = \omega \cup \{a + 1)$. The proof follows the lines LA and LB in this way:

LA: $$((Y_i')_0^a, \mathcal{N}') \in \frac{\omega}{a} \operatorname{Part}\binom{K}{k} \text{ (by the induction hypothesis),}$$

$$((Y_i^*)_0^{a+1}, \mathcal{N}^*) \in \frac{\omega}{a+1} \operatorname{Part}\binom{K}{k} \text{ (by the construction),}$$

$$((Y_i'')_0^{a+1}, \mathcal{N}'') \in \frac{\omega}{a+1} \operatorname{Part}\binom{K}{k} \text{ (by the induction hypothesis);}$$

LB: $$((Z_i')_0^{a+1}, \mathcal{P}') \in \frac{\omega'}{a+1} \operatorname{Part}\binom{K}{k} \text{ (by the induction hypotheis).}$$

We put

$$((Z_i')_0^{a+1}, \mathcal{P}') = ((Z_i^*)_0^{a+1}, \mathcal{P}^*) \in \frac{\omega'}{a+1} \operatorname{Part}\binom{K}{k},$$

$$((Z_i'')_0^{a+1}, \mathcal{P}'') \in \frac{\omega'}{a+1} \operatorname{Part}\binom{K}{k} \text{ (by Theorem 2)}.$$

It remains to prove

$$((Y_i)_0^a, \mathcal{N}) \in \frac{\omega}{a} \operatorname{Part}\binom{K}{k}.$$

Suppose, on the contrary, that there exists a set $N \subset \bigcup_{i=0}^{a} Y_i$, $|N| = K$, such that

i) $|N \cap Y_i| \neq \emptyset,\ i > 0 \Leftrightarrow i \in \omega$,

ii) $\binom{N}{k} \subseteq \mathcal{N}$.

Then there are two possibilities: either $N \cap Y_{a+1} = \emptyset$ and in this case we get a contradiction with

$$((Y_i'')_0^a, \mathcal{N}'') \in \frac{\omega}{a} \operatorname{Part}\binom{K}{k}$$

or $N \cap Y_{a+1} \neq \emptyset$, consequently $|N \cap Y_{a+1}| = 1$ and we get a contradiction with

$$((Z_i'')_0^a, \mathcal{P}'') \in \frac{\omega'}{a+1} \operatorname{Part}\binom{K}{k}.$$

(In both cases the construction of $((Y_i)_0^a, \mathcal{N})$ is essentially used.)

This completes the proof of Theorem 3.

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Author's address: Jaroslav Nešetřil, 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK), Vojtěch Rödl, 110 00 Praha 1, Husova 5, ČSSR (FJFI ČVUT).