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NONUNIQUENESS FOR THE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Consider an initial value problem

$$x' = f(t, x)\,, \quad x(t_0) = x_0\,, \tag{1}$$

where x, f are n-dimensional vectors. While the uniqueness for (1) has been investigated in many papers, results dealing with nonuniqueness have been given only by a few authors. Necessary and sufficient conditions for uniqueness in the scalar case were shown by T. YOSIE [1]. A criterion for nonuniqueness in the scalar case was established in [2]. Recently, H. STETTNER [3] has studied the nonuniqueness of (1) in a general case but the proof of his theorem is not correct.

The aim of the present paper is to prove some new sufficient conditions for the nonuniqueness of (1). The results are presented in a general form employing Ljapunov-like functions (Theorem 1, Corollary 1). Corollary 2 is obtained by choosing a special Ljapunov-like function in Corollary 1. The applicability of the results is documented by Example 1 and Example 2.

Notation. Let R be the real number system and let R^n denote the Euclidean n-space. Define $R^+ = [0, \infty)$, $R^- = (-\infty, 0]$. By $\| \; \|$ we denote an arbitrary but fixed norm in R^n, $| \; |$ denotes the Euclidean norm in R^n. For the notation of the inner product in R^n we use the sign $\cdot$. Let $C[\Omega, \Lambda]$ be the class of all continuous functions $f : \Omega \to \Lambda$. By $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ we mean the class of all scalar functions $V(t, x)$ which are locally Lipschitzian in x for $(t, x) \in \Omega$, where $\Omega \subset R \times R^n$. Finally, if $t_0 < T$, $b > 0$, we set

$$\Gamma(T, b) = \{(t, x) : t_0 \leqq t \leqq T, \; \|x - x_0\| \leqq b\}\,,$$

$$\hat{\Gamma}(T, b) = \{(t, x) : t_0 \leqq t \leqq T, \; |x - x_0| \leqq b\}\,.$$

Definition 1. Let $a > 0$, $b > 0$ and $f(t, x) \in C[\Gamma(t_0 + a, b), R^n]$. We say that (1) *has at least two different solutions,* if there exists a $T \in (t_0, t_0 + a]$ such that (1) has solutions $x_1(t)$, $x_2(t)$ defined on $[t_0, T]$ and $x_1(t) \not\equiv x_2(t)$ on $[t_0, T]$. In this case we also say that (1) *has at least two different solutions on* $[t_0, T]$. The problem (1) is said to be *nonunique*, if there is a $T_0 \in (t_0, t_0 + a]$ such that for any $T \in (t_0, T_0]$ (1) has at least two different solutions on $[t_0, T]$.

Definition 2. Let $b > 0$, $t_0 < T$, $f(t, x) \in C[\Gamma(T, b), R^n]$ and $V(t, x) \in C[\Gamma(T, b), R]$. For $(t, x) \in \Gamma(T, b)$, $t < T$, $\|x - x_0\| < b$ we define $\dot{V}_f(t, x)$ by

$$\dot{V}_f(t, x) = \limsup_{h \to 0+} \frac{V(t + h, x + hf(t, x)) - V(t, x)}{h}.$$

Theorem 1. *Let* $a > 0$, $b > 0$, $f(t, x) \in C[\Gamma(t_0 + a, b), R^n]$, $\varkappa \in (0, 1)$, $K = \max\{\|f(t, x)\| : (t, x) \in \Gamma(t_0 + a, b)\}$, $\alpha = \min(a, \varkappa b/K)$, $t_1 \in (t_0, t_0 + \alpha)$ *and assume that*

(i) *there exist functions* $g(t, u)$, $h(t, u) \in C[(t_0, t_1] \times R, R]$ *nondecreasing in* u *and such that there are solutions* $\varphi(t)$, $t \in (t_0, t_1]$ *of*

(2) $$u' = g(t, u)$$

and $\psi(t)$, $t \in (t_0, t_1]$ *of*

(3) $$u' = h(t, u),$$

satisfying the conditions $\psi(t_1) < \varphi(t_1)$,

$$\lim_{t \to t_0+} \varphi(t) = \lim_{t \to t_0+} \psi(t) = 0;$$

(ii) *there exist a positive function* $\varepsilon(t) \in C[(t_0, t_1), R^+]$ *and a function* $V(t, x) \in C[\Gamma(t_1, b), R]$ *such that*

(4) $$V(t_0, x) = 0 \quad \textit{iff} \quad x = x_0,$$

(5) $$\psi(t_1) < V(t_1, y_0) < \varphi(t_1) \quad \textit{for some} \quad y_0, \quad \|y_0 - x_0\| < (1 - \varkappa)\, b,$$

and the sets

(6) $$\Omega_\varphi = \{(t, x) : \varphi(t) < V(t, x) < \varphi(t) + \varepsilon(t),\ t_0 < t < t_1,\ \|x - x_0\| < b\},$$

(7) $$\Omega_\psi = \{(t, x) : \psi(t) - \varepsilon(t) < V(t, x) < \psi(t),\ t_0 < t < t_1,\ \|x - x_0\| < b\}$$

fulfil the following conditions:

(8) $$V(t, x) \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_\varphi \cup \Omega_\psi),$$

(9) $$\dot{V}_f(t, x) \geqq g(t, V(t, x)) \quad \textit{on } \Omega_\varphi, \quad \textit{if} \quad \Omega_\varphi \neq \emptyset,$$

(10) $$\dot{V}_f(t, x) \leqq h(t, V(t, x)) \quad \textit{on } \Omega_\psi, \quad \textit{if} \quad \Omega_\psi \neq \emptyset.$$

Then the problem (1) *has at least two different solutions on* $[t_0, t_1]$.

Proof. In virtue of the continuity of $V(t, x)$ and of (5), there exists a δ, $0 < \delta < (1 - \varkappa) b - \|y_0 - x_0\|$ satisfying the condition:

$$\psi(t_1) < V(t_1, x) < \varphi(t_1) \quad \text{for} \quad \|x - y_0\| < \delta .$$

Choose $x_1, x_2 \in \{x : \|x - y_0\| < \delta\}$, $x_1 \neq x_2$ and consider solutions $x_i(t)$ of

$$(11_i) \qquad x' = f(t, x), \quad x_i(t_1) = x_i ,$$

$i = 1, 2$. Since $\|x_i - x_0\| < (1 - \varkappa) b$ for $i = 1, 2$, the solutions $x_1(t), x_2(t)$ are defined for $t \in [t_0, t_1]$. If we prove $x_1(t_0) = x_2(t_0) = x_0$, the proof of the theorem will be complete.

Set

$$(12) \qquad x(t) = x_i(t), \quad m(t) = V(t, x_i(t))$$

for any $i \in \{1, 2\}$. Clearly,

$$\psi(t_1) < m(t_1) < \varphi(t_1) .$$

We shall show that $\{t \in (t_0, t_1) : (t, x(t)) \in \Omega_\varphi\} = \emptyset$. If this is not the case, let $s \in (t_0, t_1)$, $(s, x(s)) \in \Omega_\varphi$. For sufficiently small $h > 0$ we obtain

$$(14) \qquad m(s + h) - m(s) = V(s + h, x(s + h)) - V(s, x(s)) =$$

$$= V(s + h, x(s) + h f(s, x(s)) + h R(h)) - V(s, x(s)) ,$$

where

$$(15) \qquad \lim_{h \to 0+} \|R(h)\| = 0 .$$

Since $V(t, x) \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_\varphi \cup \Omega_\psi)$, we have

$$(16) \qquad |m(s + h) - m(s) - V(s + h, x(s) + hf(s, x(s))) +$$

$$+ V(s, x(s))| \leqq Lh\|R(h)\|$$

for $h > 0$ small enough and for some $L > 0$. The conditions (15), (16) together with the definition of $\dot{V}_f(t, x)$ yield

$$(17) \qquad D^+ m(s) = \limsup_{h \to 0+} \frac{m(s + h) - m(s)}{h} = \dot{V}_f(s, x(s)) .$$

It is easy to see that there exists an interval $I = (t_2, t_3)$ such that $t_0 < t_2 < t_3 < t_1$,

$$(18) \qquad m(t_3) = \varphi(t_3)$$

and

$$(19) \qquad \varphi(s) < m(s) < \varphi(s) + \varepsilon(s) \quad \text{for} \quad s \in I .$$

Thus $(s, x(s)) \in \Omega_\varphi$ for $s \in I$. Using (9) and (17) we deduce

$$D^+[m(s) - \varphi(s)] = D^+ m(s) - \varphi'(s) \geqq g(s, m(s)) - \varphi'(s), \quad s \in I.$$

The nondecreasing character of $g(s, u)$ implies

$$D^+[m(s) - \varphi(s)] \geqq g(s, \varphi(s)) - \varphi'(s) \equiv 0, \quad s \in I.$$

Consequently, the function $m(s) - \varphi(s)$ is nondecreasing in I and we get a contradiction with (18) and (19). Hence $\{t \in (t_0, t_1): (t, x(t)) \in \Omega_\varphi\} = \emptyset$.

Analogously we can observe $\{t \in (t_0, t_1) : (t, x(t)) \in \Omega_\psi\} = \emptyset$. In view of (13) we infer that $\psi(t) \leqq m(t) \leqq \varphi(t)$ for $t \in (t_0, t_1]$. Thus

$$V(t_0, x(t_0)) = m(t_0) = \lim_{t \to t_0+} \varphi(t) = \lim_{t \to t_0+} \psi(t) = 0$$

and, because of (4), $x(t_0) = x_0$. Therefore $x_1(t_0) = x_2(t_0) = x_0$ and the theorem is proved.

Note. The set of all points (t_0, x_0) having the properties from Theorem 1 is at most countable in the set $\{(t_0, x); x \in R^n\}$. This follows from the separability of R^{n+1} because we can define the map $(t_0, x_0) \mapsto M_{x_0} = \{(t, x) \in \Gamma(t_1, b) : \psi(t) < V(t, x) < \varphi(t)\}$, where each of M_x contains an open set P_x such that $P_{x_1} \cap P_{x_2} = \emptyset$ for $x_1 \neq x_2$.*)

Corollary 1. *Let a, b, $f(t, x)$, $\varkappa$, t_1 be as in Theorem 1 and assume that*

(i) *there exists a function $q(t, u) \in C[(t_0, t_1] \times R^+, R]$ nondecreasing in u and such that a certain solution $\varphi(t)$, $t \in (t_0, t_1]$ of*

$$u' = q(t, u)$$

satisfies the conditions

$$\varphi(t_1) > 0, \quad \lim_{t \to t_0+} \varphi(t) = 0;$$

(ii) *there exist a positive function $\varepsilon(t) \in C[(t_0, t_1), R^+]$ and a function $V(t, x) \in C[\Gamma(t_1, b), R^+]$ such that the conditions (4) and*

(20) $$V(t_1, y_0) < \varphi(t_1) \quad \textit{for some } y_0, \quad \|y_0 - x_0\| < (1 - \varkappa) b,$$

(21) $$V(t, x) \in \mathrm{Lip}_{\mathrm{loc}}(\Omega_\varphi),$$

(22) $$\dot{V}_f(t, x) \geqq q(t, V(t, x)) \quad \textit{on} \quad \Omega_\varphi$$

hold; Ω_φ being defined by (6).

Then the problem (1) has at least two different solutions on $[t_0, t_1]$.

*) This note was suggested by the referee.

Proof. Define

$$g(t, u) = \begin{cases} q(t, u) & \text{for} \quad (t, u) \in (t_0, t_1] \times R^+ , \\ q(t, 0) & \text{for} \quad (t, u) \in (t_0, t_1] \times R^- . \end{cases}$$

Setting $h(t, u) = -1$ for $(t, u) \in (t_0, t_1] \times R$, we can easily see that the assumptions of Theorem 1 are satisfied with $\psi(t) = t_0 - t$, $\Omega_\psi = \emptyset$.

Corollary 2. *Assume $a > 0$, $b > 0$, $f(t, x) \in C[\hat{\Gamma}(t_0 + a, b), R^n]$ and*

(i) *there exists a function $q(t, u) \in C[(t_0, t_0 + a] \times R^+, R]$ nondecreasing in u and such that a certain solution $\varphi(t)$, $t \in (t_0, t_0 + a]$ of*

$$u' = q(t, u)$$

satisfies the conditions

$$\lim_{t \to t_0 +} \varphi(t) = 0 , \quad \varphi(t) > 0 \quad \textit{for} \quad t \in (t_0, t_0 + a) ;$$

(ii) *there exist a differentiable function $z(t) \in C[[t_0, t_0 + a), R^n]$ and a positive function $\varepsilon(t) \in C[(t_0, t_0 + a), R^+]$ such that $z(t_0) = x_0$ and the inequality*

$$(23) \qquad (f(t, x) - z'(t)) \,.\, (x - z(t)) \geqq |x - z(t)| \, q(t, |x - z(t)|)$$

holds on $\hat{\Omega} = \{(t, x) : \varphi(t) < |x - z(t)| < \varphi(t) + \varepsilon(t), t_0 < t < t_0 + a, |x - x_0| < b\}$.
Then the problem (1) is nonunique.

Proof. Let $\varkappa = \frac{1}{2}$ and $\alpha = \min(a, b/2K)$, where $K = \max\{|f(t, x)| : (t, x) \in \in \hat{\Gamma}(t_0 + a, b)\}$. There exists a number $t_2 \in (t_0, t_0 + \alpha)$ such that $|z(t) - x_0| < b/2$ for $t \in (t_0, t_2]$. Choose $t_1 \in (t_0, t_2]$ and define

$$V(t, x) = |x - z(t)| \quad \text{for} \quad (t, x) \in \hat{\Gamma}(t_1, b) .$$

Since

$$\dot{V}_f(t, x) = \frac{1}{|x - z(t)|} (f(t, x) - z'(t)) \,.\, (x - z(t))$$

is true for $t_0 \leqq t < t_1$, $|x - x_0| < b$, $x \neq z(t)$, we get, in view of (23),

$$\dot{V}_f(t, x) \geqq q(t, |x - z(t)|) , \quad (t, x) \in \hat{\Omega} .$$

Corollary 1 implies that (1) has at least two different solutions on $[t_0, t_1]$, which completes the proof.

Example 1. Let $a > 0$, $\alpha > 0$, $\beta > 0$, $m > 2$ and $\gamma > \alpha m/2$. Define

$$F(t, v) = \begin{cases} \dfrac{\gamma t^{m-1} v}{\alpha t^m + \beta v^2} & \text{for} \quad 0 \leqq t \leqq a , \quad v \in R , \quad t^2 + v^2 \neq 0 , \\ 0 & \text{for} \quad t = v = 0 . \end{cases}$$

It is easy to verify that

$$|F(t, v)| \leqq \frac{\gamma t^{m/2-1}}{2\sqrt{(\alpha\beta)}} \quad \text{for} \quad 0 \leqq t \leqq a\,, \quad v \in R\,,$$

which implies that $F(t, v)$ is continuous. We want to show that the problem

(24) $$v' = F(t, v)\,, \quad v(0) = 0$$

is nonunique.

Let b be any positive number. Set

$c = (2\gamma - m\alpha)^{1/4}(4\beta m)^{-1/4}$, $q(t, u) = \frac{1}{2}cmt^{m/4-1}\sqrt{u}$ for $(t, u) \in (0, a] \times R^+$, $\varphi(t) = c^2 t^{m/2}$ for $t \in (0, a]$, $\varepsilon(t) = \varphi(t)$ for $t \in (0, a)$, $z(t) = 0$ for $t \in [0, a)$. For $(t, v) \in \hat{\Omega} = \{(t, v) : \varphi(t) < |v| < \varphi(t) + \varepsilon(t),\ 0 < t < a,\ |v| < b\}$ we have

$$v\,F(t, v) = \frac{\gamma t^{m-1} v^2}{\alpha t^m + \beta v^2} \geqq \frac{\gamma c t^{m+m/4-1}|v|\sqrt{|v|}}{\alpha t^m + 4\beta c^4 t^m} = \frac{\gamma c t^{m/4-1}|v|\sqrt{|v|}}{\alpha + 4\beta c^4} = |v|\,q(t, |v|)\,.$$

The assumptions of Corollary 2 are satisfied with $x = v$, $t_0 = 0$, $x_0 = 0$ and $f(t, x) = F(t, x)$ for $0 \leqq t \leqq a$, $|x| \leqq b$. Hence the problem (24) is nonunique.

Example 2. Let $a > 0$, $\alpha \in (0, 1)$, $\gamma \in C[[0, a], R]$. Define

$$G(t, v) = \begin{cases} v^\alpha + \gamma(t) & \text{for } 0 \leqq t \leqq a\,, \quad v > 0\,, \\ \gamma(t) & \text{for } 0 \leqq t \leqq a\,, \quad v \leqq 0 \end{cases}$$

and assume

(25) $$-(1 - \alpha)^{1/(1-\alpha)}\,\alpha^{\alpha/(1-\alpha)}\,t^{\alpha/(1-\alpha)} \leqq \gamma(t) \leqq 0\,.$$

We shall show that the problem

(26) $$v' = G(t, v)\,, \quad v(0) = 0$$

is nonunique.

Let b be any positive number. Set

$$c = \tfrac{1}{2}\alpha^{1/(1-\alpha)}(1 - \alpha)^{1/(1-\alpha)},\ q(t, u) = \frac{c}{1 - \alpha}\,t^{\alpha/(1-\alpha)} \quad \text{for} \quad (t, u) \in (0, a] \times R^+\,,$$

$$\varphi(t) = ct^{1/(1-\alpha)} \quad \text{for} \quad t \in (0, a]\,, \quad \varepsilon(t) = 1 \quad \text{for} \quad t \in (0, a)\,,$$

$$z(t) = ct^{1/(1-\alpha)} \quad \text{for} \quad t \in [0, a)\,.$$

Since

$$(1 - \alpha)^{1/(1-\alpha)}\,\alpha^{\alpha/(1-\alpha)} = 2^\alpha c^\alpha - \frac{2c}{1 - \alpha}\,,$$

we get, in view of (25), the inequality

$$\gamma(t) + 2^\alpha c^\alpha t^{\alpha/(1-\alpha)} \geqq \frac{2c}{1-\alpha} t^{\alpha/(1-\alpha)} .$$

Hence

$$v^\alpha + \gamma(t) - \frac{c}{1-\alpha} t^{\alpha/(1-\alpha)} \geqq \frac{c}{1-\alpha} t^{\alpha/(1-\alpha)}$$

holds for

$$0 < t < a , \quad 2ct^{1/(1-\alpha)} < v < b .$$

This implies that $(G(t, v) - z'(t)) (v - z(t)) \geqq |v - z(t)| q(t, |v - z(t)|)$ for $0 < t < < a$, $\varphi(t) + z(t) < v < b$.

By (25) we have

$$\frac{c}{1-\alpha} t^{\alpha/(1-\alpha)} - \gamma(t) \geqq \frac{c}{1-\alpha} t^{\alpha/(1-\alpha)} ,$$

which leads to the inequality $(G(t, v) - z'(t)) (v - z(t)) \geqq |v - z(t)| q(t, |v - z(t)|)$ for $0 < t < a$, $-b < v < z(t) - \varphi(t)$. Thus

$$(G(t, v) - z'(t)) (v - z(t)) \geqq |v - z(t)| q(t, |v - z(t)|)$$

for $(t, v) \in \hat{\Omega} = \{(t, v) : \varphi(t) < |v - z(t)| < \varphi(t) + \varepsilon(t), 0 < t < a, |v| < b\}$.

The assumptions of Corollary 2 are fulfilled with $x = v$, $t_0 = 0$, $x_0 = 0$ and $f(t, x) = G(t, x)$ for $0 \leqq t \leqq a$, $|x| \leqq b$. Therefore the problem (26) is nonunique.

Note. No nonuniqueness condition for (1) having a form

$$W(t, x_1 - x_2, f(t, x_1) - f(t, x_2)) \geqq q(t, V(t, x_1 - x_2)) ,$$

where q, V, W are suitable scalar functions, can be applied to the problem

$$v' = \sqrt[3]{v} , \quad v(0) = 0 . \tag{27}$$

Indeed, if this were not the case this condition would also be satisfied for the problem

$$v' = 1 + \sqrt[3]{v} , \quad v(0) = 0 ,$$

which has a unique solution.

On the other hand, the assumptions of Corollary 2 are fulfilled for the problem (27), e.g. with

$$q(t, u) = \tfrac{2}{3} \sqrt{t} , \quad \varphi(t) = \tfrac{4}{9} t \sqrt{t} , \quad \varepsilon(t) = 1 \quad \text{and} \quad z(t) = 0 .$$

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