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F-QUASIGROUPS ISOTOPIC TO MOUFANG LOOPS

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One of the oldest tasks of the theory of quasigroups is to study generalizations of groups. One of the oldest examples of generalized groups is the class of F-quasigroups (the first article on this subject has appeared in 1939). The main aim of the present paper is to describe F-quasigroups isotopic to Moufang loops. It is shown that these quasigroups consist of three basic classes, namely groups, medial quasigroups and distributive Steiner quasigroups. Unfortunately, the author does not know whether there are any F-quasigroups not isotopic to a Moufang loop.

1. PRELIMINARIES

Some details concerning quasigroups and loops may be found in [1] and [2].

Let Q be a quasigroup and $a \in Q$. We denote by $e(a)$ and $f(a)$ the right and left local unit of a, respectively. Hence $f(a)\, a = a = a\, e(a)$ and e, f are mappings of Q into Q. Further we define two mappings L_a, R_a of Q into Q by $L_a(b) = ab$ and $R_a(b) = ba$ for every $b \in Q$. We denote by $\mathscr{M}(Q)$ the group generated by all these permutations $L_a, R_a, a \in Q$ and we put $\mathscr{I}(Q, a) = \{g \in \mathscr{M}(Q) \mid g(a) = a\}$. Further we put $S(a, b) = L_b^{-1} L_a^{-1} L_{ab}$, $T(a, b) = R_a^{-1} R_b^{-1} R_{ab}$ and $V(a) = R_a^{-1} L_a$ for all $a, b \in Q$.

Let Q be a left (right) loop. The left (right) unit of Q will be denoted by j. Further, if $a \in Q$ then $a^{-1} = b$ and ${}^{-1}a = c$, where $ab = j = ca$. The following result is well known (see [1], Theorem 4.4).

1.1. Lemma. *Let Q be a loop. Then the group $\mathscr{I}(Q, j)$ is generated by the permutations $S(a, b)$, $T(a, b)$, $V(a)$, $a, b \in Q$.*

A congruence r of a quasigroup Q is said to be normal if the factor Q/r is a quasigroup. A subquasigroup P of Q is said to be normal if P is a class of a normal congruence. The following result is classical (see [1], Theorem 4.5).

1.2. Lemma. *A subloop P of a loop Q is normal iff $g(a) \in P$ for all $a \in P$ and $g \in \mathscr{I}(Q, j)$.*

Let Q be a quasigroup. We put

$$\begin{aligned}
A(Q) &= \{a \in Q \mid ba \,.\, cd = bc \,.\, ad \quad \text{for all} \quad b, c, d \in Q\}\,,\\
B_1(Q) &= \{a \in Q \mid ab \,.\, cd = ac \,.\, bd \quad \text{for all} \quad b, c, d \in Q\}\,,\\
B_r(Q) &= \{a \in Q \mid bc \,.\, da = bd \,.\, ca \quad \text{for all} \quad b, c, d \in Q\}\,,\\
M(Q) &= \{a \in Q \mid ba \,.\, cb = bc \,.\, ab \quad \text{for all} \quad b, c \in Q\}\,,\\
D(Q) &= \{a \in Q \mid ab \,.\, ca = ac \,.\, ba \quad \text{for all} \quad b, c \in Q\}\,,\\
N_1(Q) &= \{a \in Q \mid a \,.\, bc = ab \,.\, c \quad \text{for all} \quad b, c \in Q\}\,,\\
N_r(Q) &= \{a \in Q \mid bc \,.\, a = b \,.\, ca \quad \text{for all} \quad b, c \in Q\}\,,\\
N_m(Q) &= \{a \in Q \mid ba \,.\, c = b \,.\, ac \quad \text{for all} \quad b, c \in Q\}\,,\\
N(Q) &= N_1(Q) \cap N_r(Q) \cap N_m(Q)\,,\\
K(Q) &= \{a \in Q \mid ab = ba \text{ for every } b \in Q\}\,,\\
C(Q) &= N(Q) \cap K(Q)\,,\\
\operatorname{Id} Q &= \{a \in Q \mid a = aa\}\,.
\end{aligned}$$

Moreover, we denote by Aut Q the automorphism group of Q.

1.3. Lemma. *Let Q be a quasigroup. Then:*

(i) *$N_1(Q)\,(N_r(Q))$ is non-empty iff Q is a left (right) loop. In this case, $N_1(Q)$ $(N_r(Q))$ is a subgroup of Q.*

(ii) *$N(Q)$ is non-empty iff Q is a loop. In this case, $N(Q)$ is a subgroup of Q.*

(iii) *$C(Q)$ is non-empty iff Q is a loop. In this case, $C(Q)$ is an abelian subgroup of Q.*

Proof. See [1], Theorem 1.1.

A quasigroup Q is called an S-quasigroup if $S(a, b)$ is an automorphism for all $a, b \in Q$. Similarly we define T-quasigroups, V-quasigroups, ST-quasigroups, STV-quasigroups. STV-loops are called also A-loops. The following lemma is clear from the definitions of $S(a, b)$, $T(a, b)$ and $V(a)$.

1.4. Lemma. *Let Q be a quasigroup. Then:*

(i) *$ab \,.\, c = a \,.\, bS(a, b)(c)$ and $c \,.\, ab = T(a, b)(c)\, a \,.\, b$ for all $a, b, c \in Q$.*

(ii) *$K(Q) = \{a \in Q \mid V(a) = 1\}$ and $K(Q) = \{a \in Q \mid V(b)(a) = a \text{ for every } b \in Q\}$.*

1.5. Lemma. *Let Q be an A-loop. Then:*

(i) *If P is a subloop of Q and $g(P) \subseteq P$ for every $g \in \mathrm{Aut}\, Q$ then P is normal.*

(ii) *$N(Q)$ and $C(Q)$ are normal subgroups of Q.*

(iii) *$K(Q)$ is a normal commutative subloop of Q.*

Proof. (i) follows from 1.1 and 1.2, (ii) follows from (i) and 1.3 and (iii) follows from (i) and 1.4(ii).

A quasigroup Q is called an LIP-quasigroup (RIP-quasigroup) if there is a mapping $p(q)$ of Q into Q such that $p(a) . ab = b$ $(ba . q(a) = b)$ for all $a, b \in Q$. Further, Q is called an IP-quasigroup if it is both LIP-quasigroup and RIP-quasigroup.

The following results are proved in [1], Chapter 5.

1.6. Lemma. *Let Q be an LIP-quasigroup. Then $p^2 = 1$, $pf = f$, $p(a)\, a = e(a)$, $a . p(a) = e\, p(a)$ and $a . p(a)\, b = b$ for all $a, b \in Q$. Moreover, if Q is a right loop then ${}^{-1}a = p(a) = a^{-1}$ for every $a \in Q$.*

1.7. Lemma. *Let Q be an IP-quasigroup. Then $p(ab) = q(b)\, q(a)$ and $q(ab) = p(b)\, p(a)$ for all $a, b \in Q$.*

1.8. Lemma. *Let Q be an IP-quasigroup. Then:*

(i) *If Q is commutative then $p = q \in \mathrm{Aut}\, Q$.*

(ii) *If Q is a loop then $p(a) = a^{-1} = q(a)$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in Q$.*

A quasigroup Q is called medial if $A(Q) = Q$. Further, Q is called monomedial (dimedial, trimedial) if each of its subquasigroups generated by one (two, three) element(s) is medial.

A quasigroup Q is called a WM-quasigroup if $aa . bc = ab . ac$ and $bc . aa = ba . ca$ for all $a, b, c \in Q$.

A quasigroup Q is called unipotent if $aa = bb$ for all $a, b \in Q$.

A quasigroup Q is called idempotent if $\mathrm{Id}\, Q = Q$.

A quasigroup Q is called distributive if $a . bc = ab . ac$ and $bc . a = ba . ca$ for all $a, b, c \in Q$.

A quasigroup Q is called a Steiner quasigroup if it is commutative idempotent and $a . ab = b$ for all $a, b \in Q$.

A mapping g of a quasigroup Q into itself is said to be (left) regular if there exists a mapping h such that $g(ab) = h(a)\, b$ for all $a, b \in Q$.

Remark. The notation $e(a), f(a), L_a, S(a, b)$, etc., will be used only if the basic operation is written multiplicatively. We shall write $e^*(a)$, $f^*(a)$, L_a^*, $S(a, b, *)$, when another symbol (say $*$) is used.

All results of this section are auxiliary and their proofs are only sketched. However, almost each of the following assertions is either implicity or explicitly contained in [1] and [2].

Let g be a permutation of a quasigroup Q and $x \in Q$ an element. We say that g is a left pseudoautomorphism with a left companion x of the quasigroup Q if $x\, g(ab) = x\, g(a) \,.\, g(b)$ for all $a, b \in Q$. Dually we define right pseudoautomorphisms.

2.1. Lemma. *Let a quasigroup Q possess a left pseudoautomorphism g with a left companion x. Then:*

(i) *Q is a left loop and $g(j) = e(x)$.*

(ii) *If Q is a loop then $g(j) = j$.*

(iii) *If Q is an RIP-quasigroup then $g(a^{-1}) = g(a)^{-1}$ for every $a \in Q$.*

(iv) *If Q is an IP-loop then g is a right pseudoautomorphism and x^{-1} is its right companion.*

Proof. (i) $x \,.\, g(g^{-1}\, e(x)\, a) = x\, g(a)$ for every $a \in Q$.

(ii) follows from (i).

(iii) $x = x\, g(aa^{-1}) = x\, g(a) \,.\, g(a^{-1})$ and $x\, g(a^{-1})^{-1} = x\, g(a)$ for every $a \in Q$.

(iv) $g(b^{-1}a^{-1})\, x^{-1} = g(ab)^{-1} \,.\, x^{-1} = (x\, g(a) \,.\, g(b))^{-1} = g(b^{-1}) \,.\, g(a^{-1})\, x^{-1}$ for all $a, b \in Q$.

2.2. Lemma. *Let g, h be two left pseudoautomorphisms of a left loop Q and let x, y be their respective left companions. Then gh is a left pseudoautomorphism and $x\, g(y)$ is its left companion.*

Proof. $x\, g(y) \,.\, g\, h(ab) = x\, g(y\, h(ab)) = x\, g(y\, h(a)) \,.\, g\, h(b) = (x\, g(y) \,.\, g\, h(a)) \,.\, g\, h(b)$ for all $a, b \in Q$.

2.3. Lemma. *Let g with a left companion x be a left pseudoautomorphism of a left loop Q. Then g^{-1} is a left pusedoautomorphism and $g^{-1}(x^{-1})$ is its left companion.*

Proof. $xg(g^{-1}(x^{-1})\, g^{-1}(a)) = a$ and $xg(g^{-1}(x^{-1})\, g^{-1}(ab)) = ab = (xg(g^{-1}(x^{-1})\, g^{-1}(a)))\, gg^{-1}(b) = xg(g^{-1}(x^{-1})\, g^{-1}(a) \,.\, g^{-1}(b))$ for all $a, b \in Q$.

2.4. Lemma. *Let Q be a left loop, g a permutation of Q and $x \in Q$. The following conditions are equivalent:*

(i) *$g \in \mathrm{Aut}\, Q$ and $x \in N_1(Q)$.*

(ii) *g is a left pseudoautomorphism, x is its left companion and $x \in N_1(Q)$.*

(iii) *$g \in \mathrm{Aut}\, Q$ and g is a left pseudoautomorphism with a left companion x.*

Proof. Easy.

2.5. Corollary. *Let Q be a left loop. Then every automorphism is a left pseudoautomorphism and left pseudoautomorphisms form a group.*

2.6. Lemma. *Let g with a companion x be a pseudoautomorphism of a commutative loop Q. Then g is an automorphism and x belongs to $N(Q)$.*

Proof. $x\,g(ab) = x\,g(a) \,.\, g(b) = x\,g(ba) = x\,g(b) \,.\, g(a)$ for all $a, b \in Q$. Hence $x \in N(Q)$ and the rest is clear.

2.7. Lemma. *Let Q be a left loop and $ab^{-1} \in N_1(Q)$ for some $a, b \in Q$. Then $a = ab^{-1} \,.\, b$.*

Proof. $e(ab^{-1}) = j$ since $N_1(Q)$ is a subgroup, and hence $ab^{-1} = ab^{-1} \,.\, bb^{-1} = (ab^{-1} \,.\, b)\, b^{-1}$.

2.8. Lemma. *Let Q be a left loop and g a left pseudoautomorphism with a left companion x. The following conditions are equivalent for any $y \in Q$:*

(i) *y is a left companion of g.*
(ii) *$yx^{-1} \in N_1(Q)$.*
(iii) *$y = ax$ for some $a \in N_1(Q)$.*

Proof. (i) implies (ii). By 2.2 and 2.3, $yx^{-1} = ygg^{-1}(x^{-1})$ is a left companion of $gg^{-1} = 1$.
(ii) implies (iii) by 2.7 and (iii) implies (i) trivially.

2.9. Lemma. *Let g, h be two left pseudoautomorphisms of a left loop Q. The following conditions are equivalent:*

(i) *g and h have a common left companion.*
(ii) *g and h have the same left companions.*
(iii) *$g^{-1}h$ is an automorphism.*

Proof. (i) implies (ii) by 2.8.
(ii) implies (iii). Let x be a left companion of both g and h. Then $g^{-1}(x^{-1})\, g^{-1}(x)$ is a left companion of $g^{-1}h$. But $x = xg(g^{-1}(x^{-1})\, g^{-1}(x))$, and so $g^{-1}(x^{-1}) \,.\, g^{-1}(x) = j$.
(iii) implies (i). Let x be a left companion of g. Then $x = x\,g(j) = x\,e(x)$ is a left companion of $gg^{-1}h = h$.

2.10. Lemma. *Let Q be a left loop and g a left pseudoautomorphism with a left companion x. Put $a * b = R_j^{-1}(a)\, b$ for all $a, b \in Q$ and suppose that $g(j) = j$.*

Then $Q(*)$ is a loop, g is a left pseudoautomorphism of $Q(*)$ and x is its left companion.

Proof. $x * g(aj * b) = x\,g(ab) = (x * g(a))\,j * g(b)$, and hence $x * g(aj) = (x * g(a))\,j$ and $x * g(aj * b) = (x * g(aj)) * g(b)$ for all $a, b \in Q$.

3. MOUFANG LOOPS

Details concerning Moufang loops may be found in [1] and [2].

3.1. Lemma. *Let Q be a Moufang loop. Then $K(Q) = M(Q)$ is a commutative subloop of Q.*

Proof. Let $x, y \in K(Q)$ and $a \in Q$. Then $x(a\,.\,xy) = (xa\,.\,x)\,y = y(xa\,.\,x) = y(x\,.\,ax) = (yx\,.\,a)\,x = x(yx\,.\,a) = x(xy\,.\,a)$. Hence $a\,.\,xy = xy\,.\,a$ and $xy \in K(Q)$. Further, $ax^{-1} = x^{-1}xax^{-1} = x^{-1}axx^{-1} = x^{-1}a$ and we have proved that $K(Q)$ is a subloop of Q. If $x \in K(Q)$ and $a, b \in Q$ then $ax\,.\,ba = (a\,.\,xb)\,a = (a\,.\,bx)\,a = ab\,.\,xa$ and x is contained in $M(Q)$. Similarly we can show that $M(Q)$ is a subset of $K(Q)$.

3.2. Lemma. *Let Q be a Moufang ST-loop. Then $K(Q)$ is a normal subloop.*

Proof. Clearly, $g(K(Q))$ is a subset of $K(Q)$ for every automorphism g of Q. Further, $V(a)\,(x) = x$ for all $a \in Q$ and $x \in K(Q)$. The rest is clear from 1.1 and 1.2.

3.3. Lemma. *Let Q be a Moufang loop, $K(Q)$ a normal subloop and $a \in Q$. Then the subloop generated by $\{a\} \cup K(Q)$ is commutative.*

Proof. Let P be the subloop generated by $K(Q) \cup \{a\}$. Then $K(Q)$ is contained in $K(P)$ and it suffices to show that $a \in K(P)$. Let $b \in P$ and let G be the subloop generated by $\{a, b\}$. Then G is a group and $G/G \cap K(Q)$ is a subgroup of $P/K(Q)$. However, $P/K(Q)$ is a cyclic Moufang loop, hence a group, and therefore $G/G \cap K(Q)$ is cyclic. On the other hand, $G \cap K(Q) \subseteq C(G)$ and consequently $G/C(G)$ is cyclic. Thus G is commutative and $ab = ba$.

3.4. Lemma. *Let Q be a Moufang loop. Then every mapping from $\mathscr{I}(Q, j)$ is a pseudoautomorphism of Q.*

Proof. See [1], Theorem 6.4.

3.5. Lemma. *Let Q be a Moufang loop. Then:*

(i) *If Q is commutative then Q is an A-loop.*

(ii) *If Q is a V-loop then $a^3 \in N(Q)$ for every $a \in Q$.*

Proof. (i) follows from 3.4 and 2.6, (ii) follows from 3.4, 2.4 and [1], Corollary 6.4.1.

3.6. Lemma. *Let g be a pseudoautomorphism of a Moufang loop Q. Then $g(ax) = g(a)\,g(x)$, $g(xa) = g(x)\,g(a)$ and $g(x) \in N(Q)$ for all $a \in Q$ and $x \in N(Q)$.*

Proof. See [1], Lemma 6.2 and its proof.

3.7. Lemma. *Let Q be a Moufang loop. Then $N(Q)$ is a normal subloop of Q.*

Proof. Apply 3.4, 3.6 and 1.2.

Let Q be a loop. We shall say that Q satisfies the condition NK if for every $a \in Q$ there exist $b \in N(Q)$ and $c \in K(Q)$ such that $a = bc$.

3.8. Lemma. *Let Q be a loop, $a, c \in N(Q)$ and $b, d \in K(Q)$. Then $ab\,.\,cd = ac\,.\,bd$.*

Proof. $ab\,.\,cd = a(b\,.\,cd) = a(cd\,.\,b) = a(c\,.\,db) = ac\,.\,db = ac\,.\,bd$.

3.9. Lemma. *Let Q be a Moufang loop satisfying NK. Then Q is a homomorphic image of the cartesian product $N(Q) \times K(Q)$.*

Proof. Define $g(a, b) = ab$ for all $a \in N(Q)$ and $b \in K(Q)$. According to 3.8, g is a homomorphism. However, Q satisfies NK, and hence g is onto Q.

3.10. Lemma. *Every Moufang loop satisfying NK is an A-loop.*

Proof. Every group is an A-loop and every commutative Moufang loop is an A-loop. On the other hand, as is easy to see, A-loops form a quasigroup variety. In particular, A-loops are closed under homomorphic images and we can apply 3.9.

3.11. Lemma. *Let Q be a Moufang loop satisfying NK. Then $N(K(Q))$ is contained in $C(Q)$.*

Proof. There is a homomorphism g of $N(Q) \times K(Q)$ onto Q. If $a \in N(K(Q))$ then $(j, a) \in C(N(Q) \times K(Q))$, and therefore $a = g(j, a) \in C(Q)$.

3.12. Lemma. *Let Q be a Moufang loop satisfying NK. Then every pseudoautomorphism of Q is an automorphism.*

Proof. Let g with a companion x be a pseudoautomorphism. Then $x = yz$ for some $y \in N(Q)$, $z \in K(Q)$ and z is a companion of g. Let $a, b \in K(Q)$ and $c \in N(Q)$. Then $g^{-1}(c) \in N(Q)$ and $g(a)\,c = g(a)\,g\,g^{-1}(c) = g(ag^{-1}(c)) = g(g^{-1}(c)\,a) = c\,g(a)$ by 3.6. Further, $g(a)\,.\,cb = g(a)\,c\,.\,b = c\,g(a)\,.\,b = c\,.\,b\,g(a) = cb\,.\,g(a)$. However, Q satisfies NK and it is evident that $g(a) \in K(Q)$. Similarly $g^{-1}(a) \in K(Q)$ and g with a companion z is a pseudoautomorphism of $K(Q)$. By 2.6 and 3.11, $z \in C(Q)$, and hence g is an automorphism of Q.

3.13. Lemma. *Let Q be a Moufang loop satisfying NK. Then:*

(i) *$K(Q)$ is a normal subloop of Q and $Q/K(Q)$ is a group.*

(ii) *If $a \in Q$ then the subloop generated by $\{a\} \cup K(Q)$ is commutative.*

Proof. (i) $K(Q)$ is normal by 3.10,3.2 and $Q/K(Q)$ is a group since Q is generated by $K(Q) \cup N(Q)$.

(ii) Use (i) and 3.3.

A Moufang loop Q is called primitive if Q is commutative and $a^3 = j$ for every $a \in Q$.

3.14. Lemma. *Let Q be a Moufang loop satisfying NK. Then:*

(i) *$N(Q)$ is a normal subloop and $Q/N(Q)$ is a primitive Moufang loop.*

(ii) *If $a, b \in Q$ then the subloop generated by $\{a, b\} \cup N(Q)$ is a group.*

Proof. (i) $N(Q)$ is normal by 3.7, $Q/N(Q)$ is commutative since Q is generated by $N(Q) \cup K(Q)$, and $Q/N(Q)$ is primitive by 3.5(ii).

(ii) With respect to 3.9, we can assume that Q is either a group or a commutative loop. The first case is obvious. If Q is commutative then $N(Q) = C(Q)$ and the assertion follows from the fact that Q is diassociative.

3.15. Lemma. *Let Q be a Moufang loop satisfying NK. Suppose that Q is isotopic to a commutative quasigroup. Then Q is commutative.*

Proof. There are two permutations g, h of Q such that $g(a)\,h(b) = g(b)\,h(a)$ for all $a, b \in Q$. Then $xa \,.\, b = xb \,.\, a$ for all $a, b \in Q$, where $x = g\,h^{-1}(j)$. However, $x = yz$, $y \in N(Q)$, $z \in K(Q)$ and $y(az \,.\, b) = y(za \,.\, b) = (y \,.\, za)\,b = (yz \,.\, a)\,b = (yz \,.\, b)\,a = y(zb \,.\, a) = y(a \,.\, zb)$ for all $a, b \in K(Q)$. Hence $z \in N(K(Q))$ and $z \in C(Q)$ by 3.11. It is easy to see that $x \in N(Q)$. Consequently $x \,.\, ab = xa \,.\, b = xb \,.\, a = x \,.\, ba$ for all $a, b \in Q$.

3.16. Lemma. *Let Q be a loop and g a mapping of Q into Q. The following conditions are equivalent:*

(i) *$ab \,.\, c\,g(a) = (a \,.\, bc)\,g(a)$ for all $a, b, c \in Q$.*

(ii) *Q is a Moufang loop and $a^{-1} \,.\, g(a) \in N(Q)$ for every $a \in Q$.*

Proof. See [11], Theorem 1.

3.17. Lemma. *Let Q be a Moufang loop and g a mapping of Q into Q. The following conditions are equivalent:*

(i) *$a\,g(a) \,.\, bc = ab \,.\, g(a)\,c$ for all $a, b, c \in Q$.*

(ii) *$g(a) \in K(Q)$ and $a^{-1} \,.\, g(a) \in N(Q)$ for every $a \in Q$.*

Proof. (i) implies (ii). $a \,.\, g(a)\, b = aj \,.\, g(a)\, b = a\, g(a) \,.\, b$, and hence $ab \,.\, g(a) = a \,.\, b\, g(a)$ by Moufang's theorem. Then $a \,.\, b\, g(a) = ab \,.\, g(a) = ab \,.\, g(a)\, j = a\, g(a) \,.\, b = a \,.\, g(a)\, b$ yields $g(a) \in K(Q)$. Now $ab \,.\, c\, g(a) = ab \,.\, g(a)\, c = a\, g(a) \,.\, bc = (a \,.\, bc)\,(g(a)\, j) = (a \,.\, bc)\, g(a)$ and we can apply 3.16.

(ii) implies (i.) $(ab \,.\, c\, g(a))\,(g(a)^{-1} \,.\, a) = ab \,.\, ca = (a \,.\, bc)\, a = ((a \,.\, bc)\, g(a)) \,.\, (g(a)^{-1} \,.\, a)$, and therefore $ab \,.\, c\, g(a) = (a \,.\, bc)\, g(a)$. In particular, $a\, g(a) \,.\, c\, g(a) = (a \,.\, g(a)\, c)\, g(a) = (a \,.\, c\, g(a))\, g(a)$ for all $a, c \in Q$. Thus $a\, g(a) \,.\, b = ab \,.\, g(a)$ and $a\, g(a) \,.\, bc = (a \,.\, bc)\, g(a) = ab \,.\, c\, g(a) = ab \,.\, g(a)\, c$ for all $a, b, c \in Q$.

3.18. Lemma. *Let Q be a Moufang loop such that $a\, g(a) \,.\, bc = ab \,.\, g(a)\, c$ for all $a, b, c \in Q$ and a mapping g. Then Q satisfies NK.*

Proof. We have $a = g(a)^{-1}\, a \,.\, g(a)$ and $g(a) \in K(Q)$, $g(a)^{-1}\, a \in N(Q)$ by 3.17.

3.19. Lemma. *Let Q be a Moufang loop and let $g \in$ Aut Q be such that $g^{-1}(a^{-1}) \,.\, a \in K(Q)$ for every $a \in Q$. Then $a^{-1} \,.\, g(a) \in K(Q)$.*

Proof. It is enough to take into account that $K(Q)$ is a subloop and $g(a^{-1}) = g(a)^{-1}$.

3.20. Lemma. *Let g be an automorphism of a Moufang loop Q. The following conditions are equivalent:*

(i) *$a^{-2} \,.\, g^{-1}(a) \in N(Q)$ for every $a \in Q$.*
(ii) *$a^{-1} \,.\, g(a^2) \in N(Q)$ for every $a \in Q$.*
(iii) *$g^{-1}(a) \,.\, a^{-2} \in N(Q)$ for every $a \in Q$.*
(iv) *$g(a^2) \,.\, a^{-1} \in N(Q)$ for every $a \in Q$.*

Proof. Similar to that of 3.19.

Let Q be a Moufang loop and g, h two automorphisms of Q. We shall say that g, h satisfy the condition (F) if $gh = hg$, $a^{-1} \,.\, g(a)$, $a^{-1} \,.\, h(a) \in K(Q)$ and $a^{-1} \,.\, g(a^2)$, $a^{-1} \,.\, h(a^2) \in N(Q)$ for every $a \in Q$.

3.21. Lemma. *Let Q be a Moufang loop and let $g, h \in$ Aut Q satisfy* (F). *Then:*

(i) *$a \,.\, bc = (g(a)\, b)\,(g(a^{-1})\, a \,.\, c)$ for all $a, b, c \in Q$.*
(ii) *$bc \,.\, a = (b \,.\, a\, h(a^{-1}))\,(c\, h(a))$ for all $a, b, c \in Q$.*
(iii) *$K(Q) = \{a \in Q \mid g^2(b)\, a \,.\, c\, h^2(b) = g^2(b)\, c \,.\, a\, h^2(b)$ for all $b, c \in Q\}$.*
(iv) *Q satisfies NK.*

Proof. (i) Put $k(a) = a^{-1} \,.\, g^{-1}(a)$ for every $a \in Q$. It is easy to see that $k(a) \in K(Q)$ and $a^{-1} \,.\, k(a) \in N(Q)$ for every $a \in Q$. According to 3.17, $a\, k(a) \,.\, bc = ab \,.\, k(a)\, c$ for all $a, b, c \in Q$. The rest is clear.

(ii) is similar to (i).

(iii) Since $a^{-1} \,.\, h(a^2)$, $a^{-2} \,.\, g^{-1}(a) \in N(Q)$ for every $a \in Q$, $h(a^{-2}) \,.\, h\, g^{-1}(a)$ and $a^{-1} \,.\, h\, g^{-1}(a)$ belong to $N(Q)$. Hence $h\, g^{-1}(a^{-1}) \,.\, h^2\, g^{-2}(a)$ and $a^{-1} \,.\, h^2\, g^{-2}(a)$ are contained in $N(Q)$. By 3.16, $ax \,.\, bh^2\, g^{-2}(a) = (a \,.\, xb)\, h^2\, g^{-2}(a) = (a \,.\, bx)\, h^2\, g^{-2}(a) = ab \,.\, xh^2\, g^{-2}(a)$ for all $x \in K(Q)$ and $a, b \in Q$. Conversely, if $g^2(a)\, b \,.\, x\, h^2(a) = g^2(a)\, x \,.\, b\, h^2(a)$ for all $a, b \in Q$ then the substitution $a = j$ yields $x \in K(Q)$.

(iv) $a = a^{-1}\, g(a^2) \,.\, g(a^{-2})\, a^2$ for every $a \in Q$.

3.22. Lemma. *Let Q be a Moufang loop and let $g \in \mathrm{Aut}\, Q$ be such that $a^{-2} \,.\, g(a) \in N(Q)$ for every $a \in Q$. Then $a\, g(a) \in N(Q)$, provided Q is a V-loop.*

Proof. Apply 3.5(ii).

4. BASIC PROPERTIES OF F-QUASIGROUPS

A quasigroup Q is called an LF-quasigroup (RF-quasigroup) if $a \,.\, bc = ab \,.\, e(a)\, c$ ($cb \,.\, a = c\, f(a) \,.\, ba$) for all $a, b, c \in Q$. If Q is both an LF and an RF-quasigroup then we shall say that Q is an F-quasigroup.

4.1. Lemma. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an LF-quasigroup.*

(ii) *$L_a(b) \,.\, L_{e(a)}(c) = L_a(bc)$ for all $a, b, c \in Q$.*

(iii) *$L_a^{-1}(b) \,.\, L_{e(a)}^{-1}(c) = L_a^{-1}(bc)$ for all $a, b, c \in Q$.*

(iv) *$S(a, b) = L_{e(a)}^{-1}$ for all $a, b \in Q$.*

(v) *$S(a, b) = S(a, c)$ for all $a, b, c \in Q$.*

Proof. The equivalence of (i), (ii), (iii) and the implication (iv) implies (v) are obvious.

(i) implies (iv). We have $L_{ab}(c) = ab \,.\, c = ab \,.\, L_{e(a)} L_{e(a)}^{-1}(c) = a \,.\, b L_{e(a)}^{-1}(c) = L_a L_b L_{e(a)}^{-1}(c)$. Hence $S(a, b) = L_{e(a)}^{-1}$.

(v) implies (iv). $S(a, b) = S(a, e(a)) = L_{e(a)}^{-1} L_a^{-1} L_{ae(a)} = L_{e(a)}^{-1}$.

(iv) implies (i). $a \,.\, bc = L_a L_b(c) = L_{ab} S(a, b)^{-1}\,(c) = L_{ab} L_{e(a)}(c) = ab \,.\, e(a)\, c$ for all $a, b, c \in Q$.

4.2. Lemma. *Let Q be an LF-quasigroup. Then $ef = fe$ and e is an endomorphism of Q.*

Proof. $a \,.\, ef(a)\, e(a) = f(a)\, a \,.\, ef(a)\, e(a) = f(a) \,.\, a\, e(a) = a = a\, e(a)$ for every $a \in Q$. Further, $ab \,.\, e(a)\, e(b) = a \,.\, b\, e(b) = ab$ for all $a, b \in Q$.

4.3. Lemma. *Let Q be an LF-quasigroup and $a, b \in Q$. Then $L_b R_a = R_a L_b$ iff $e(b) = f(a)$.*

Proof. If L_b and R_a commute then $ba = R_a L_b\, e(b) = L_b R_a\, e(b)$ and $a = e(b)\, a$. Conversely, if $e(b) = f(a)$ then $b \,.\, ca = bc \,.\, e(b)\, a = bc \,.\, a$ for every $c \in Q$.

4.4. Lemma. *Let Q be an LF-quasigroup and $a \in Q$. Then $L_{f(a)} R_{e(a)} = R_{e(a)} L_{f(a)}$ and $eL_a = L_{e(a)} e$.*

Proof. The former equality follows from 4.3 and 4.2, the latter from the fact that e is an endomorphism of Q.

4.5. Lemma. *Let Q be an LF-quasigroup and $j \in \mathrm{Id}\, Q$. Then L_j is an automorphism, $L_j R_j = R_j L_j$, $L_j e = eL_j$ and $L_j f = fL_j$.*

Proof. Use 4.1(ii), 4.3, 4.4 and the fact that $ja = j\, f(a) \,.\, ja$.

4.6. Proposition. *A quasigroup is a group iff it is an LF-quasigroup (or an RF-quasigroup) and a loop.*

Proof. Easy.

4.7. Proposition. *Let Q be an LF-quasigroup, $j \in \mathrm{Id}\, Q$, $K = \{a \in Q \mid e(a) = j\}$, $H = \{a \in Q \mid f(a) = j\}$ and $G = H \cap K$. Then K is a normal subquasigroup and a right loop, H is a subquasigroup and a left lopp and G is a subgroup of Q. Moreover. if Q is an F-quasigroup then G is a normal subgroup.*

Proof. By 4.2, e is an endomorphism of Q and K is one of classes of the corresponding normal congruence. However, j is an idempotent and consequently K is a normal subquasigroup. Further, L_j is an automorphism and $H = \{a \mid L_j(a) = a\}$. Hence H is a subquasigroup. Now it is evident that G is a subloop, and therefore a subgroup by 4.6. If Q is an F-quasigroup then H is normal, and so G is normal as well.

4.8. Lemma. *Let Q be an LF-quasigroup, $x \in Q$, $j = e(x)$ and $a * b = a \,.\, L_j^{-1}(b)$ for all $a, b \in Q$. Then $Q(*)$ is a left loop, j is its left unit and $Q(*)$ is an LF-quasigroup.*

Proof. Clearly, $j * a = j \,.\, L_j^{-1}(a) = a$ for every $a \in Q$. Further, $a * j\, e(a) = a$, $j\, e(a) = e^*(a)$ and $a * (b * c) = a \,.\, L_j^{-1}(bL_j^{-1}(c)) = a(L_j^{-1}(b) \,.\, L_{e(j)}^{-1} L_j^{-1}(c)) = (a\, L_j^{-1}(b))\,(e(a) \,.\, L_{e(j)}^{-1} L_j^{-1}(c)) = aL_j^{-1}(b) \,.\, L_j^{-1}(j\, e(a) \,.\, L_j^{-1}(c)) = (a * b) * (e^*(a) * * c)$ for all $a, b, c \in Q$.

In the following four lemmas we shall assume that Q is an LF-quasigroup with a left unit j. These lemmas are implicitly contained in [1].

5.1. Lemma. *Let $a \in Q$. Then $L_{e(a)}$ is a left pseudoautomorphism of Q and aj is its left companion.*

Proof. $aj(e(a) . bc) = a(j . bc) = a . bc = ab . e(a)\, c = (aj . e(a)\, b) . e(a)\, c$ for all $b, c \in Q$.

5.2. Lemma. *Let $a, b \in Q$. The left pseudoautomorphisms $L_{e(a)}L_{e(b)}$ and $L_{e(R_j^{-1}(a.bj))}$ have the same left companions.*

Proof. By 5.1 and 2.2, $(aj)(e(a) . bj) = a . bj$ is a left companion of $L_{e(a)}L_{e(b)}$. On the other hand, $R_j(R_j^{-1}(a . bj)) = a . bj$ is a left companion of the other pseudoautomorphism.

5.3. Lemma. *Let $a, b \in Q$. Then $L_{e(b)}^{-1}L_{e(a)}^{-1}L_{e(R_j^{-1}(a.bj))}$ is an automorphism of Q.*

Proof. Apply 5.2 and 2.9.

5.4. Lemma. *Let $a \circ b = R_j^{-1}(a) . b$ for all $a, b \in Q$. Then $Q(\circ)$ is an S-loop.*

Proof. We can write $R_j(R_j(a) \circ b) \circ c = ab . c = a . b\, L_{e(a)}^{-1}(c) = R_j(a) \circ (R_j(b) \circ L_{e(a)}^{-1}(c))$ for all $a, b, c \in Q$. Hence $R_j(a \circ b) \circ c = a \circ (R_j(b) \circ L_{g(a)}^{-1}(c))$, where $g(a) = eR_j^{-1}(a)$. In particular, $R_j(a) \circ c = a \circ L_{g(a)}^{-1}(c)$ for all $a, c \in Q$, and hence $R_j(a \circ b) \circ c = (a \circ b) \circ L_{g(a \circ b)}^{-1}(c)$ and $R_j(b) \circ L_{g(a)}^{-1}(c) = b \circ L_{g(b)}^{-1} L_{g(a)}^{-1}(c)$ for all $a, b, c \in Q$. This implies the equalities $a \circ (R_j(b) \circ L_{g(a)}^{-1}(c)) = (a \circ b) \circ L_{g(a \circ b)}^{-1}(c)$ and $(a \circ b) \circ c = a \circ (b \circ L_{g(b)}^{-1}L_{g(a)}^{-1}L_{g(a \circ b)}(c))$ for all $a, b, c \in Q$. It is evident that $S(a, b, \circ) = L_{g(b)}^{-1}L_{g(a)}^{-1}L_{g(a \circ b)}$ for all $a, b \in Q$. Let $a, b \in Q$, $c = R_j^{-1}(a)$, $d = R_j^{-1}(b)$ and $h = S(a, b, \circ)$. Then $h = L_{e(c)}^{-1}L_{e(d)}^{-1}L_{e(R_j^{-1}(c.dj))}$ is an automorphism of Q by 5.3. Hence $h(j) = j$, $hR_j = R_j h$ and $hR_j^{-1} = R_j^{-1}h$. Finally, $h(x \circ y) = h\, R_j^{-1}(x) . h(y) = R_j^{-1}\, h(x) . h(y) = h(x) \circ h(y)$ for all $x, y \in Q$.

5.5. Proposition. *Let Q be an F-quasigroup, $x \in Q$ and $y = e\, f(x)$. Put $a \circ b = R_y^{-1}(a) . L_y^{-1}(b)$ for all $a, b \in Q$. Then $Q(\circ)$ is an ST-loop, provided $y \in \mathrm{Id}\, Q$.*

Proof. Let $a * b = a . L_y^{-1}(b)$ for all $a, b \in Q$. Then $Q(*)$ is an LF-quasigroup and a left loop with left unit y (see 4.8). Further, $a \circ b = R_y^{-1}(a) * b = (R_y^*)^{-1}(a) * b$ for all $a, b \in Q$. By 5.4, $Q(\circ)$ is an S-loop. On the other hand, $y = f\, e(x)$ and we can use the right hand forms of 4.8 and 5.4 to prove that $Q(\circ)$ is a T-loop.

5.6. Corollary. *Every F-quasigroup with non-empty* Id Q *is isotopic to na ST-loop.*

6.1. Proposition. *Let Q be a WM-quasigroup. Then $D(Q) = Q$.*

Proof. See [9], Corollary 2.

6.2. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is a WM-quasigroup and an LF-quasigroup.*

(ii) *Q is a WM-quasigroup and an RF-quasigroup.*

(iii) *Q is an F-quasigroup and there exists $x \in Q$ such that $f(x)\, a \,.\, b\, e^2(x) = f(x)\, b \,.\, a\, e^2(x)$ for all $a, b \in Q$.*

(iv) *Q is an F-quasigroup and there exists $x \in Q$ such that $f^2(x)\, a \,.\, b\, e(x) = f^2(x)\, b \,.\, a\, e(x)$ for all $a, b \in Q$.*

(v) *Q is a trimedial quasigroup.*

(vi) *There are a commutative Moufang loop $Q(\circ)$, $g, h \in \operatorname{Aut} Q(\circ)$ and $x \in N(Q(\circ))$ such that $gh = hg$, $a \circ g(a)$, $a \circ h(a) \in N(Q(\circ))$ and $ab = g(a) \circ h(b) \circ x$ for all $a, b \in Q$.*

Proof. Apply 6.1 and [7], Theorem 2.

6.3. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an idempotent WM-quasigroup.*

(ii) *Q is a distributive quasigroup.*

(iii) *Q is an idempotent F-quasigroup.*

Proof. Obvious.

6.4. Proposition. *Every distributive quasigroup is trimedial.*

Proof. See [1], Corollary 8.6.1.

6.5. Proposition. *Every commutative F-quasigroup is trimedial.*

Proof. See [7], Corollary 6.

6.6. Proposition. *Let Q be a trimedial quasigroup. Then there is a normal congruence r of Q such that every class of r is a medial subquasigroup of Q and Q/r is a distributive Steiner quasigroup.*

Proof. See [7], Theorem 3.

6.7. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an F-quasigroup and $A(Q)$ is non-empty.*

(ii) *Q is an F-quasigroup and Q is isotopic to a group.*

(iii) *There are a group $Q(\circ)$, $g, h \in \mathrm{Aut}\, Q(\circ)$ and $x \in C(Q(\circ))$ such that $gh = hg$, $a^{-1} \circ g(a)$, $a^{-1} \circ h(a) \in C(Q(\circ))$ and $ab = g(a) \circ h(b) \circ x$ for all $a, b \in Q$.*

Proof. See [6], Theorem 3.2.

6.8. Proposition. *Let Q be an F-quasigroup isotopic to a group. Then $A(Q)$ is a normal subquasigroup of Q and $Q/A(Q)$ is a group.*

Proof See [6], Theorem 3.4.

6.9. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an LF-quasigroup and a right loop.*

(ii) *There is an element $j \in Q$ such that $a \,.\, bc = ab \,.\, jc$ for all $a, b, c \in Q$.*

(iii) *Q is a right loop and $A(Q)$ is non-empty.*

(iv) *There are a group $Q(\circ)$ and $h \in \mathrm{Aut}\, Q(\circ)$ such that $ab = a \circ h(b)$ for all $a, b \in Q$.*

Proof. The implications (i) implies (ii) and (iv) implies (i) are easy.

(ii) implies (iii). We have $a = a \,.\, j\, e^2(a)$, $e(a) = j \,.\, e^2(a)$ and $j = e(a)$ for every $a \in Q$. Hence j is a right unit and $aj \,.\, bc = a \,.\, bc = ab \,.\, jc$ for all $a, b, c \in Q$.

(iii) implies (iv). By Theorem 1.1 from [6] there are a group $Q(\circ)$ and $g, h \in \mathrm{Aut}\, Q(\circ)$ such that $ab = g(a) \circ h(b)$ for all $a, b \in Q$ and j is the unit of $Q(\circ)$. Now $a = aj = g(a) \circ h(j) = g(a)$ and $g = 1$.

6.10. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an F-quasigroup and a right loop.*

(ii) *There are a group $Q(\circ)$ and $h \in \mathrm{Aut}\, Q(\circ)$ such that $a^{-1} \circ h(a) \in C(Q(\circ))$ and $ab = a \circ h(b)$ for all $a, b \in Q$.*

Proof. Apply 6.7 and 6.9.

6.11. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an F-quasigroup and $B_1(Q)$ is non-empty.*

(ii) *Q is an F-quasigroup and $B_r(Q)$ is non-empty.*

(iii) *Q is a medial quasigroup.*

(iv) *Q is an F-quasigroup and Q is isotopic to an abelian group.*

(v) *There are an abelian group $Q(+)$, $g, h \in \mathrm{Aut}\, Q(+)$ and $x \in Q$ such that $gh = hg$ and $ab = g(a) + h(b) + x$ for all $a, b \in Q$.*

Proof. Apply 6.7, [1], Theorem 2.10 and [6], Theorem 3.6.

6.12. Proposition. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an ST-quasigroup and an LF-quasigroup.*

(ii) *Q is an ST-quasigroup and an RF-quasigroup.*

(iii) *Q is an F-quasigroup and $e(Q), f(Q) \subseteq \mathrm{Id}\ Q$.*

(iv) *$\mathrm{Id}\ Q$ is a normal distributive subquasigroup of Q, $Q/\mathrm{Id}\ Q$ is a group and Q is isomorphic to the cartesian product $\mathrm{Id}\ Q \times Q/\mathrm{Id}\ Q$.*

Proof. See [5], Theorem 12.

6.13. Proposition. *Let Q be an IP-quasigroup and an LF-quasigroup. Then Q is isotopic to a Moufang loop.*

Proof. See [3], Theorem 1.

6.14. Lemma. *Let an F-quasigroup Q be isotopic to a Moufang loop. Then every loop isotopic to Q is a Moufang loop.*

Proof. It is well known that Moufang loops are closed under loop isotopies.

7. SEVERAL LEMMAS ON F-QUASIGROUPS

Throughout this section, let Q be an F-quasigroup and let $x, y \in Q$ be such that $R_x L_y = L_y R_x$. Further put $g = R_x$, $h = L_y$, $j = yx$, $u = e(y)\,x$, $v = y f(x)$, $k = g R_{f(x)} g^{-1}$, $t = h L_{e(y)} h^{-1}$, $z = g(j) = yx \,.\, x$, $w = h(j) = y \,.\, yx$, $m = gheg^{-1}$, $n = hgfh^{-1}$ and $a \circ b = g^{-1}(a) \,.\, h^{-1}(b)$ for all $a, b \in Q$. As is easy to see, $Q(\circ)$ is a loop and j is its unit.

7.1. Lemma. *$g(a \circ b) = k(a) \circ g(b)$ and $h(a \circ b) = h(a) \circ t(b)$ for all $a, b \in Q$.*

Proof. $g(g(a) \circ h(b)) \circ h(c) = ab \,.\, c = a f(c) \,.\, bc = g(g(a) \circ h f(c)) \circ h(g(b) \circ h(c))$ for all $a, b, c \in Q$. Hence $g(a \circ b) \circ c = g(a \circ hfh^{-1}(c)) \circ h(gh^{-1}(b) \circ c)$ for all $a, b, c \in Q$. If $c = j$ then $g(a \circ b) = g(a \circ hfh^{-1}(j)) \circ g(b)$. However, $hfh^{-1}(j) = h f(x) = v$ and $g(a \circ v) = g(g^{-1}(a) \,.\, h^{-1}(v)) = g R_{f(x)} g^{-1}(a) = k(a)$. The latter identity can be proved in a similar way.

7.2. Lemma. *$a \circ (b \circ c) = (k(a) \circ b) \circ (m(a) \circ c)$, $(a \circ mk^{-1}(a)) \circ (b \circ c) = (a \circ b) \circ (mk^{-1}(a) \circ c)$, $(b \circ c) \circ a = (b \circ n(a)) \circ (c \circ t(a))$ and $(b \circ c) \circ (nt^{-1}(a) \circ a) = (b \circ nt^{-1}(a)) \circ (c \circ a)$ for all $a, b, c \in Q$.*

Proof. We have $g(a) \circ (h\, g(b) \circ t\, h(c)) = a \,.\, bc = ab \,.\, e(a)\, c = (k\, g(a) \circ g\, h(b)) \circ (hg\, e(a) \circ t\, h(c))$ for all $a, b, c \in Q$. Hence $a \circ (b \circ c) = (k(a) \circ b) \circ (m(a) \circ c)$ for all $a, b, c \in Q$. In particular, $a = k(a) \circ m(a)$ and the rest is clear.

7.3. Lemma. *$g(a) = k(a) \circ z$ for every $a \in Q$ and k is a right pseudoautomorphism with a right companion z of the loop $Q(\circ)$.*

Proof. The equality $g(a) = k(a) \circ z$ is an immediate consequence of 7.1. Further, $k(a \circ b) \circ z = g(a \circ b) = k(a) \circ g(b) = k(a) \circ (k(b) \circ z)$ for all $a, b \in Q$.

7.4. Lemma. *$h(a) = w \circ t(a)$ for every $a \in Q$ and t is a left pseudoautomorphism with a left companion w of the loop $Q(\circ)$.*

Proof. Similar to that of 7.3.

In the remaining part of this section we shall assume that $Q(\circ)$ is a Moufang loop.

7.5. Lemma. *$m\, k^{-1}(a)$, $n\, t^{-1}(a) \in K(Q(\circ))$, $a^{-1} \circ m\, k^{-1}(a)$, $n\, t^{-1}(a) \circ a^{-1}$ belong to $N(Q(\circ))$ for every $a \in Q$. Moreover, $Q(\circ)$ satisfies NK.*

Proof. Apply 7.2,3.17 (and its right hand form) and 3.18.

7.6. Lemma. *$a^{-1} \circ k(a)$, $a^{-1} \circ t(a)$, $gh\, e(a)$, $gh\, f(a) \in K(Q(\circ))$ for every $a \in Q$.*

Proof. $m\, k^{-1}(a) = gheR_{f(x)}\, g^{-1}(a) \in K(Q(\circ))$ for every $a \in Q$. However, k^{-1} and $R_{f(x)}k^{-1}$ are permutations. Hence $m(a)$ and $gh\, e(a)$ belong to $K(Q(\circ))$. Further, $a = k(a) \circ m(a)$ (see 7.2), and hence $k(a^{-1}) \circ a \in K(Q(\circ))$. Now we can use 7.5, 3.12, 7.3 and 3.19.

7.7. Lemma. *$k, t \in$ Aut $Q(\circ)$ and $z, w \in N(Q(\circ))$.*

Proof. Apply 7.3, 7.4, 7.5 and 3.12, 2.4.

7.8. Lemma. *$a^{-1} \circ k(a^2)$, $a^{-1} \circ t(a^2) \in N(Q(\circ))$ for every $a \in Q$.*

Proof. $a^{-1} \circ m\, k^{-1}(a) = a^{-2} \circ k^{-1}(a)$ and $n\, t^{-1}(a) \circ a^{-1} = t^{-1}(a) \circ a^{-2}$ belong to $N(Q(\circ))$ and we can use 3.20.

7.9. Lemma. *Let $z \in K(Q(\circ))$. Then $z \in C(Q(\circ))$ and $kt = tk$.*

Proof. We can write $k(w) \circ k\, t(a) \circ z = g\, h(a) = h\, g(a) = w \circ t\, k(a) \circ t(z)$. Consequently $k(w) \circ z = w \circ t(z)$. Moreover, $z \in N(Q(\circ)) \cap K(Q(\circ)) = C(Q(\circ))$, and therefore $k(w) \circ z \circ k\, t(a) = w \circ t(z) \circ t\, k(a)$. Thus $k\, t(a) = t\, k(a)$.

7.10. Lemma. *Let $x = e(p)$ and $y = e(q)$ for some $p, q \in Q$. Then $e(Q) \subseteq K(Q(\circ))$ and $z \in C(Q(\circ))$.*

Proof. $gh\, e(a) = e(q \,.\, a)\, e(p) = e(qa \,.\, p) \in K(Q(\circ))$ for every $a \in Q$. However, R_pL_q is a permutation. Finally, $z = g(j) = yx \,.\, x = e(qp \,.\, p)$ and we can use 7.9.

7.11. Lemma. *Let $x = f(p)$ and $y = f(q)$ for some $p, q \in Q$. Then $f(Q) \subseteq K(Q(\circ))$ and $w \in C(Q(\circ))$.*

Proof. Similar to that of 7.10.

7.12. Lemma. *Let $z \in K(Q(\circ))$ (or $w \in K(Q(\circ))$). Then k, t satisfy* (F).

Proof. Apply 7.6, 7.7 and 7.9, 7.8.

7.13. Lemma. *Let $x = e\,f(p)$ and $y = e\,f(q)$ for some $p, q \in Q$. Then $z, w, z \circ w \in C(Q(\circ))$ and k, t satisfy* (F). *Moreover, $ab = k(a) \circ t(b) \circ z \circ w$ for all $a, b \in Q$.*

Proof. Apply 7.12, 7.10 and 7.11.

8. QUASIGROUPS LINEAR OVER MOUFANG LOOPS

In this section, let $Q(\circ)$ be a Moufang loop, let $k, t \in \operatorname{Aut} Q(\circ)$ satisfy (F) and $q \in C(Q(\circ))$. By 3.21, the loop $Q(\circ)$ satisfies the condition NK. We put $ab = k(a) \circ t(b) \circ q$ for all $a, b \in Q$. As is easy to see, Q is a quasigroup.

8.1. Lemma. *Q is an F-quasigroup.*

Proof. We can write $a \,.\, bc = (k(a) \circ (k\, t(b) \circ t^2(c)) \circ q \circ t(q) = ((k^2(a) \circ k\, t(b)) \circ ((k^2(a^{-1}) \circ k(a)) \circ t^2(c)) \circ q \circ t(q)$ for all $a, b, c \in Q$ (use 3.21). Hence $a \,.\, bc = k(k(a) \circ t(b) \circ q) \circ t(kt^{-1}(k(a^{-1}) \circ a \circ q^{-1}) \circ t(c) \circ q) \circ q = ab \,.\, e(a)\, c$ since $k(a) \circ tt^{-1}(k(a^{-1}) \circ a \circ q^{-1}) \circ q = a$. Similarly we can show that Q is an RF-quasigroup.

8.2. Lemma. *If $Q(\circ)$ is commutative then Q is trimedial.*

Proof. By 3.22, $a \circ k(a)$, $a \circ t(a) \in N(Q(\circ))$ for every $a \in Q$ and we can use 6.2.

8.3. Lemma. *Let $P(\circ) \subseteq Q(\circ)$ be a subloop such that $K(Q(\circ)) \subseteq P(\circ)$. Then P is a subquasigroup of Q. Furthermore, if $P(\circ)$ is a normal subloop then P is a normal subquasigroup and the corresponding normal congruences coincide.*

Proof. Since $a^{-1} \circ k(a), a^{-1} \circ k^{-1}(a), a^{-1} \circ t(a), a^{-1} \circ t^{-1}(a)$ belong to $K(Q(\circ)) \subseteq P$ for every $a \in Q$, we see that $k(P) = P = t(P)$. Further, $q \in P$ and it is evident that P is a subquasigroup of Q. Let r be a normal congruence of $Q(\circ)$ such that $P(\circ)$ is one of its classes. If arb then $a \circ b^{-1} \in P$, $k(a \circ b^{-1}) = k(a) \circ k(b^{-1}) \in P$ and $k(a)\, r\, k(b)$. Hence $ac = k(a) \circ t(c) \circ q$ and bc are in r. The rest is similar.

8.4. Lemma. *$M(Q) = K(Q(\circ)) = M(Q(\circ))$.*

Proof. Let $x \in M(Q)$. Then $(k^2(a) \circ k\, t(x) \circ q) \circ (k\, t(b) \circ t^2(a) \circ q) \circ q = ax \,.\, ba = ab \,.\, xa = (k^2(a) \circ k\, t(b) \circ q) \circ (k\, t(x) \circ t^2(a) \circ q) \circ q$ for all $a, b \in Q$. Hence

$(k^2(a) \circ k\,t(x)) \circ (b \circ t^2(a)) = (k^2(a) \circ b) \circ (k\,t(x) \circ t^2(a))$ for all $a, b \in Q$ and $k\,t(x) \in K(Q(\circ))$ by 3.21(iii). But $K(Q(\circ))$ is invariant under automorphisms, and therefore $x \in K(Q(\circ))$. If $x \in K(Q(\circ))$ then $x \in M(Q)$, as one can show proceeding conversely. Finally $M(Q(\circ)) = K(Q(\circ))$ by 3.1.

8.5. Lemma. $e(Q), f(Q) \subseteq M(Q)$.

Proof. We have $e(a) = t^{-1}(k(a^{-1}) \circ a \circ q^{-1})$ and $k(a^{-1}) \circ a \in K(Q(\circ))$ for every $a \in Q$. Hence $e(a) \in K(Q(\circ)) = M(Q)$. The rest is similar.

8.6. Lemma. *$M(Q)$ is a normal subquasigroup of Q and $Q/M(Q)$ is a group.*

Proof. $M(Q)$ is a normal subquasigroup by 8.3, 8.4, 3.13, and $Q/M(Q)$ is a group by 8.5 and 4.6.

8.7. Lemma. *Let $a \in Q$. Then the subquasigroup generated by $\{a\} \cup M(Q)$ is trimedial.*

Proof. Apply 8.3, 3.13(ii), 8.4 and 8.2.

8.8. Lemma. *Let $P(\circ) \subseteq Q(\circ)$ be a subloop such that $N(Q(\circ)) \subseteq P$. Then P is a subquasigroup of Q. Furthermore, if $P(\circ)$ is a normal subloop then P is a normal subquasigroup and the corresponding normal congruences coincide.*

Proof. Similar to that of 8.3 (use 3.22 and the fact that q is contained in $N(Q(\circ))$).

In the sequel, let H denote the nucleus $N(Q(\circ))$.

8.9. Lemma. *H is a normal subquasigroup of Q and Q/H is a distributive Steiner quasigroup.*

Proof. H is normal by 8.8 and 3.7. Let r be the corresponding normal congruence (r is a congruence of both $Q(\circ)$ and Q). If $a, b \in Q$ then $c = a\,.\,ab = k(a) \circ (k\,t(a) \circ t^2(b)) \circ q \circ t(q)$. Since $q \circ t(q) \in C(Q(\circ)) \subseteq H$, we have $c\; r\; k(a) \circ (k\,t(a) \circ t^2(b))$. Further, $a^{-1} \circ k(a^{-1}) \in H$ by 3.22, and consequently $c\; r\; a^{-1} \circ (k\,t(a) \circ t^2(b))$. Finally, $k\,t(a^{-1}) \circ k(a^{-1})$, $k(a) \circ a$, $t(b^{-1}) \circ t^2(b^{-1})$, $b \circ t(b)$ are contained in H, and hence $k\,t(a^{-1}) \circ a \in H$, $c\; r\; t^2(b)$, $c\; r\; t(b^{-1})$ and $c\; r\; b$. From this we see that $x\,.\,xy = y$ for all $x, y \in Q/H$. Similarly we can show that $ab\; r\; a^{-1} \circ b^{-1}$ and $ba\; r\; b^{-1} \circ a^{-1}$ for all $a, b \in Q$. But $Q(\circ)/H(\circ)$ is commutative by 3.14, and therefore $b^{-1} \circ a^{-1}\; r\; a^{-1} \circ b^{-1}$ and $ab\; r\; ba$. Thus Q/H is commutative. Finally, $aa = k(a) \circ t(a) \circ q\; r\; a^{-2}\; r\; a$, since a^3 is contained in H by 3.5, and it is seen that Q/H is idempotent. An application of 6.3 completes the proof.

8.10. Lemma. *Let $a, b \in Q$. Then the subquasigroup generated by $\{a, b\} \cup H$ is isotopic to a group.*

Proof. Apply 8.8 and 3.14(ii).

8.11. Lemma. *Let $a, c \in H$, $b \in M(Q)$ and $d \in Q$. Then $ab \,.\, cd = ac \,.\, bd$.*

Proof. $a, c \in N(Q(\circ))$, $b \in K(Q(\circ))$, and therefore $ab \,.\, cd = (k^2(a) \circ k\,t(b) \circ k(q)) \circ (k\,t(c) \circ t^2(d) \circ t(q)) \circ q = (k^2(a) \circ ((k\,t(b) \circ k\,t(c)) \circ t^2(d)) \circ k(q) \circ t(q) \circ q = (k^2(a) \circ ((k\,t(c) \circ k\,t(b)) \circ t^2(d)) \circ k(q) \circ t(q) \circ q = (k^2(a) \circ k\,t(c) \circ k(q)) \circ (k\,t(b) \circ t^2(d) \circ t(q)) \circ q = ac \,.\, bd$.

8.12. Lemma. *For every $a \in Q$ there are $b \in H$ and $c \in M(Q)$ such that $a = bc$.*

Proof. There are $x \in N(Q(\circ))$ and $y \in K(Q(\circ))$ such that $a = x \circ y$. Put $b = k^{-1}(x)$ and $c = t^{-1}(y \circ q^{-1})$. Then $b \in H$, $c \in M(Q)$ and $a = bc$.

8.13. Lemma. *The quasigroup Q is a homomorphic image of $H \times M(Q)$.*

Proof. With respect to 8.11 and 8.12, we can define a homomorphism g of $H \times M(Q)$ onto Q by $g(a, b) = ab$ for all $a \in H$ and $b \in M(Q)$.

8.14. Lemma. *If Q is unipotent then $Q(\circ)$ is an abelian group.*

Proof. We have $aa = k(a) \circ t(a) \circ q = bb = k(b) \circ t(b) \circ q$ for all $a, b \in Q$. Hence $k(a) \circ t(a) = j$, $k(a) = t(a^{-1})$ and the mapping $a \to a^{-1}$ is an automorphism of $Q(\circ)$. It is easy to see that $Q(\circ)$ is commutative. Further, $a \circ k(a)$, $a \circ t(a) = a \circ k(a^{-1})$ and a^{-3} are contained in $N(Q(\circ))$ (by 3.22 and 3.5). Consequently $a^{-1} \in N(Q(\circ))$ and $N(Q(\circ)) = Q$.

8.15. Lemma. *Let $aa \,.\, ba = ab \,.\, aa$ for all $a, b \in Q$. Then $Q(\circ)$ is commutative.*

Proof. We have $(k^2(a) \circ k\,t(a)) \circ (b \circ t^2(a)) = (k^2(a) \circ b) \circ (k\,t(a) \circ t^2(a))$ for all $a, b \in Q$. However, $a^{-1} \circ t^2\,k^{-2}(a) \in N(Q(\circ))$ and hence $(a \circ (t\,k^{-1}(a) \circ b)) \circ t^2\,k^{-2}(a) = (a \circ t\,k^{-1}(a)) \circ (b \circ t^2\,k^{-2}(a)) = (a \circ b) \circ (t\,k^{-1}(a) \circ t^2\,k^{-2}(a)) = (a \circ (b \circ t\,k^{-1}(a)) \circ t^2\,k^{-2}(a)$ by 3.16. Thus $t\,k^{-1}(a) \circ b = b \circ t\,k^{-1}(a)$.

8.16. Lemma. *Q is an IP-quasigroup iff $k^2 = 1 = t^2$.*

Proof. The proof is purely of technical character and hence omitted.

8.17. Lemma. *Let p be a mapping of Q into Q. Then p is a regular mapping of Q iff there is $x \in N(Q(\circ))$ such that $p(a) = x \circ a$ for every $a \in Q$.*

Proof. Easy.

8.18. Lemma. *Define a relation r on Q by $a\,r\,b$ iff $a = p(b)$ for a regular mapping p of Q. Then r is a normal congruence of Q and H is one of its classes.*

Proof. Apply 8.8, 8.9 and 8.17.

By an FM-quasigroup or an FG-quasigroup we mean an F-quasigroup which is isotopic to a Moufang loop or a group, respectively.

9.1. Theorem. The following statements are equivalent for every quasigroup Q:

(i) *Q is an FM-quasigroup.*
(ii) *There are a Moufang loop $Q(\circ)$, $k, t \in \operatorname{Aut} Q(\circ)$ and $q \in C(Q(\circ))$ such that $kt = tk$, $a^{-1} \circ k(a)$, $a^{-1} \circ t(a) \in K(Q(\circ))$, $a^{-1} \circ k(a^2)$, $a^{-1} \circ t(a^2) \in N(Q(\circ))$ and $ab = k(a) \circ t(b) \circ q$ for all $a, b \in Q$.*

Proof. (i) implies (ii). Let $p \in Q$, $x = fe^2(p)$, $y = ef^2(p)$, $g = R_x$, $h = L_y$, $z = yx \,.\, x$, $w = y \,.\, yx$, $k = gR_{f(x)}g^{-1}$, $t = hL_{e(y)}h^{-1}$ and $q = z \circ w$, where $a \circ b = g^{-1}(a) \,.\, h^{-1}(b)$ for all $a, b \in Q$. Then $Q(\circ)$ is a Moufang loop. Further, $f(x) = e^2f^2(p) = e(y)$ and $gh = hg$ by 4.3. On the other hand, $x = efe(p)$, $y = eff(p)$ and we can apply 7.13.
(ii) implies (i). Use 8.1.

9.2. Theorem. *Let Q be an FM-quasigroup and $M = M(Q)$. Then:*

(i) *$e(Q), f(Q) \subseteq M$ and M is a normal subquasigroup of Q.*
(ii) *M is a trimedial quasigroup and Q/M is a group.*
(iii) *If $a \in Q$ then the subquasigroup generated by $\{a\} \cup M$ is trimedial.*

Proof. Apply 9.1, 8.5, 8.6 and 8.7.

9.3. Corollary. *Let Q be an FM-quasigroup. Then $e(Q)$ and $f(Q)$ are trimedial subquasigroups of Q.*

9.4. Corollary. *Every FM-quasigroup is monomedial.*

9.5. Corollary. *Every simple FM-quasigroup is either medial or it is a group.*

Proof. Let Q be a simple FM-quasigroup. By 9.2, Q is either a group or a trimedial quasigroup. However, every simple trimedial quasigroup is medial (see [7], Corollary 6).

Remark. Simple medial quasigroups are completely described in [4].

9.6. Theorem. *Let Q be an FM-quasigroup and $a \mathrel{r} b$ iff $a = p(b)$ for a regular mapping p of Q. Then:*

(i) *r is a normal congruence of Q.*
(ii) *Every class of r is a subquasigroup of Q and, moreover, an FG-quasigroup.*
(iii) *Q/r is a distributive Steiner quasigroup.*

(iv) *If H is a class of r and $a, b \in Q$ then the subquasigroup generated by $\{a, b\} \cup H$ is an FG-quasigroup.*
(v) *If H is a class of r then Q is a homomorphic image of the cartesian product $H \times M(Q)$.*

Proof. (i) and (iii) follow from 9.1 and 8.18, 8.9. Since Q/r is idempotent, every class of r is a subquasigroup. Now let H be a class of r. With respect to the proof of 9.1, we can assume that the unit of the corresponding Moufang loop $Q(\circ)$ is contained in H. Then $H = N(Q(\circ))$ and we can apply 8.9, 8.10 and 8.13.

9.7. Corollary. *Let Q be an FM-quasigroup. Then every subquasigroup generated by at most two elements is an FG-quasigroup.*

9.8. Theorem. *Every unipotent FM-quasigroup is medial.*

Proof. Apply 9.1, 8.14 and 6.11.

9.9. Theorem. *Let Q be an FM-quasigroup. Then Q is trimedial, provided at least one of the following conditions holds:*

(i) *e is a mapping onto Q.*
(ii) *e is a one-to-one mapping.*
(iii) *f is a mapping onto Q.*
(iv) *f is a one-to-one mapping.*
(v) *Q is dimedial.*
(vi) *Q is isotopic to a commutative quasigroup.*
(vii) *$D(Q)$ is non-empty.*
(viii) *$ab \,.\, aa = aa \,.\, ba$ for all $a, b \in Q$.*

Proof. (i) and (iii) follow from 9.3, (v) and (viii) follow from 8.15 and 8.2, (vi) follows from 3.15 and 8.2.
(ii) and (iv). If e is one-to-one then e is an isomrophism of Q onto $e(Q)$ and we can use 9.3. Similarly the other case. (vii) If $D(Q)$ is non-empty and $x \in D(Q)$ then the quasigroup $Q(*)$ where $a * b = L_x(a) \,.\, R_x(b)$, is commutative and (vi) may be used.

9.10. Corollary. *Every F-quasigroup isotopic to a commutative Moufang loop is trimedial.*

9.11. Theorem. *The following conditions are equivalent for a quasigroup Q:*

(i) *Q is an IP-quasigroup and an F-quasigroup.*
(ii) *There are a Moufang loop $Q(\circ)$, $k, t \in \mathrm{Aut}\, Q(\circ)$ and $q \in C(Q(\circ))$ such that $kt = tk$, $k^2 = 1 = t^2$, $a^{-1} \circ k(a)$, $a^{-1} \circ t(a) \in K(Q(\circ))$, $a^{-1} \circ k(a^2)$, $a^{-1} \circ t(a^2) \in$ $\in N(Q(\circ))$ and $ab = k(a) \circ t(b) \circ q$ for all $a, b \in Q$.*

Proof. Apply 9.1, 8.16 and 6.13.

References

[1] *V. D. Belousov:* Foundations of the theory of quasigroups and loops. (Russian), Moskva 1967.

[2] *R. H. Bruck:* A survey of binary systems, Springer Verlag, 1966.

[3] *I. A. Florja, M. I. Ursul:* F-quasigroups with the inverse property. (Russian), Questions of the theory of quasigroups and loops, Kišiněv 1971.

[4] *J. Ježek, T. Kepka:* Varieties of abelian quasigroups, Czech. Math. J. *27* (1977), 473—503.

[5] *T. Kepka:* On one class of quasigroups, Čas. Pěst. Mat. *97* (1972), 347—356.

[6] *T. Kepka:* Quasigroups which satisfy certain generalized form s of the abelian identity, Čas. Pěst. Mat. *100* (1975), 46—60.

[7] *T. Kepka:* Structure of triabelian quasigroups, Comment. Math. Univ. Carolinae *17* (1976), 229—240.

[8] *T. Kepka:* Structure of weakly abelian quasigroups, (to appear).

[9] *T. Kepka:* A note on WA-quasigroups, Acta Univ. Carolinae Math. Phys. *19*/2 (1978), 61—62.

[10] *D. G. Murdoch:* Quasigroups which satisfy certain generalized associative laws, Amer. J. Math. *61* (1939), 509—522.

[11] *H. Orlik-Pflugfelder:* A special class of Moufang loops, Proc. Amer. Math. Soc. *26* (1970), 583—586.

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