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p.p. RINGS AND REDUCED RINGS

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1. Introduction. G. BERGMAN [1] investigated commutative p.p. rings and centers of left p.p. rings (rings in which every left principal ideal is projective as a left module over the ring). W. VASCONCELOS [5] studied a class of p.p. rings called commutative almost hereditary rings, where a commutative almost hereditary ring is a commutative ring with identity 1 such that (1) it is reduced (a ring with no nonzero nilpotent elements), and (2) every ideal not contained in a minimal prime ideal is projective. Then the author [3] generalized a commutative almost hereditary ring to a non-commutative case. We note that any (left) almost hereditary ring is a (left) p.p. ring ([5] and [3], Theorem 1.1), and that not all p.p. rings are reduced rings. It is our purpose to find some conditions under which a p.p. ring is reduced. Thus the result gives an intrinsic relation between two conditions satisfied by an almost hereditary ring. We shall characterize the set of nilpotent elements of a p.p. ring R in terms of a chain of associated idempotents ([1], Section 3). Then the length of a chain of associated idempotents of an element r in R is defined and measures the nilpotency of r; and so some conditions are derived for a p.p. ring being reduced by using the concept of the length.

2. Preliminaries. We recall that a ring R is *a left p.p. ring* if every left principal ideal of R is projective as a left R-module ([1] and [2]). It is easy to see that R is a left p.p. ring if and only if the left annihilator $A(r)$ of an element r in R is equal to the left annihilator $A(e)$ of an idempotent e in R ([1], Section 3). Such an idempotent e is called *an associated idempotent of* r. Now, for a left p.p. ring R, we call the set of idempotents e_i of R *a chain of associated idempotents of the element* r *in* R if $A(r) = A(e_1)$ and $A(re_i) = A(e_{i+1})$ for each positive integer i. If there is a first integer n with $A(e_n) = R$ (hence $e_k = 0$ for all $k \geqq n$), we say that *the length of the chain of associated idempotents of* r is $n - 1$; the length of a chain is infinite if $e_i \neq 0$ for all i. We shall show that the length of different chains of associated idempotents of the same r is the same, so the length of chains for the element r is defined as this common integer.

Throughout, we assume that a p.p. ring means a left p.p. ring, that the annihilator of r means the left annihilator of r which is denoted by $A(r)$, and that R is a p.p. ring.

3. p.p. rings and reduced rings. Let R be a p.p. ring. We are going to define the length $L(r)$ of chains of associated idempotents of an element r in R. Then a nilpotent element r of R is characterized in terms of $L(r)$, and so R becomes a reduced ring if $L(r)$ is infinite for each nonzero r in R.

Proposition 3.1. *Let e_i and e'_i be two chains of associated idempotents of an element r in R. Then $A(e_i) = A(e'_i)$ for each $i = 1, 2, \ldots$.*

Proof. We prove this by induction. For $i = 1$, we have $A(r) = A(e_1) = A(e'_1)$ by the meaning of e_1 and e'_1. Assume that $A(e_k) = A(e'_k)$ for a positive integer k. To show that $A(e_{k+1}) = A(e'_{k+1})$ is the same as to show that $A(re_k) = A(re'_k)$ by the meaning of e_{k+1} and e'_{k+1}. Let t be in $A(re_k)$. We have $tre_k = 0$; and so (tr) is in $A(e_k)$. Since $A(e_k) = A(e'_k)$, $tre'_k = 0$. Hence t is in $A(re'_k)$. Thus $A(re_k) \subset A(re'_k)$. Similarly, $A(re'_k) \subset A(re_k)$. Thus the proof is complete.

The above proposition implies that $A(e_i) = R$ if and only if $A(e'_i) = R$, so the length of chains of associated idempotents of an element r is well defined, which is denoted by $L(r)$.

Next, we characterize a nilpotent element r in terms of $L(r)$. We begin with a lemma.

Lemma 3.2. *Let R be a p.p. ring with identity 1. If e is an associated idempotent of an element r in R, then $er = r$.*

Proof. Since $r = er + (1 - e)r$ and $(1 - e)e = 0$, $(1 - e)r = 0$ (for $A(e) = A(r)$), and so $r = er$.

Theorem 3.3. *Let R be a p.p. ring with identity 1. Then the element r in R is nilpotent if and only if $L(r)$ is finite.*

Proof. For the necessity, let $r^n = 0$ for some positive integer n. If $r = 0$, the associated idempotent is 0. Hence $L(r) = 0$, and we are done. Let $r \neq 0$, and $\{e_1, e_2, \ldots\}$ be a chain of associated idempotents of r. We first note that $A(t) = R$ if and only if $t = 0$ since R has identity 1. Now, in case $re_1 = 0$, we have $A(re_1) = A(e_2) = R$ with $e_1 \neq 0$ (for $r \neq 0$). Hence $L(r) = 1$. In case $re_1 \neq 0$, we have $r^n e_1 = 0$. Since $e_1 r = r$ by Lemma 3.2, $r^n e_1 = (re_1)^n = 0$. But $A(r) = A(e_1) \subset A(e_2) = A(re_1)$, so $R(1 - e_1) = A(e_1) \subset A(e_2)$. Hence $e_2 = e_1 e_2 + (1 - e_1) e_2 = e_1 e_2$. By Lemma 3.2 again, $e_2(re_1) = re_1$, so $(re_1)^n = (re_1)^{n-1}(re_1) = 0$ implies that $(re_1)^{n-1} e_2 = 0$ which is $(re_2)^{n-1}$ (for $A(re_1) = A(e_2)$). Thus $(re_2)^{n-1} = 0$. Using the above argument on (re_2) and the associated idempotent e_3 or (re_2), we conclude that either $L(r) = 2$ or $re_2 \neq 0$ with $(re_3)^{n-2} = 0$. Since n is finite, the process stops at some k such that e_k is the first zero idempotent; that is, $e_{k-1} \neq 0$ with $re_{k-1} = 0$. Thus $L(r) = k - 1$.

Conversely, let $L(r) = k$ for a non-negative integer k, and $\{e_1, \ldots\}$ a chain of associated idempotents of r. Then e_{k+1} is the first zero idempotent, equivalently, $A(re_k) = R$ with the minimum k. This implies that $re_k = 0$. Since $A(e_k) = A(re_{k-1})$, $rre_{k-1} = 0$. Using the fact that $A(e_i) = A(re_{i-1})$ for each i, we have $rrre_{k-2} = 0, \ldots$, and $r^k e_1 = 0$, and so $(re_1)^k = 0$ (for $e_1 r = r$). But then $r^k = r^k e_1 + r^k(1 - e_1) = r^k(1 - e_1)$. Thus $r^{2k} = r^k r^k = r^k(1 - e_1)\, r^k(1 - e_1) = 0$ since $(1 - e_1)\, e_1 = 0$ and $A(e_1) = A(r)$. This proves that r is nilpotent.

We call the positive integer n of the element r in R *the exponent of* r, $\operatorname{Exp}(r) = n$, if $r^n = 0$ and $r^{n-1} \neq 0$, $\operatorname{Exp}(0) = 0$, and $\operatorname{Exp}(r)$ is infinite if r is not nilpotent. Call *the ring R of exponent n* if $\operatorname{Exp}(r) \leqq n$ for each nilpotent r in R. From the proof of Theorem 3.3, we have a relation between $L(r)$ and $\operatorname{Exp}(r)$ for each r in R.

Theorem 3.4. *Let R be a p.p. ring with* 1 *and r a nilpotent element in R. Then*

$$\operatorname{Exp}(r)/2 \leqq L(r) \leqq \operatorname{Exp}(r)\,, \quad \textit{or equivalently}\,, \quad L(r) \leqq \operatorname{Exp}(r) \leqq 2\,L(r)\,.$$

Proof. From the proof of the necessity of Theorem 3.3, we have $L(r) \leqq \operatorname{Exp}(r)$, and the proof of the sufficiency gives $\operatorname{Exp}(r) \leqq 2\,L(r)$. Combining these two inequalities, we have the theorem.

Now we derive a characterization of a reduced ring. The proof is immediate from Theorems 3.3 and 3.4.

Corollary 3.5. *Let R be a p.p. ring with* 1. *If $L(r) \leqq n$ for each nilpotent element r in R, then the exponent of $R \leqq 2n$.*

Corollary 3.6. *Let R be a p.p. ring with* 1. *Then the following statements are equivalent*:

(1) *R is reduced.*

(2) *The length $L(r)$ is infinite for each $r \neq 0$ in R.*

(3) *$re_i \neq 0$ for each e_i in a chain of associated idempotents of $r \neq 0$ for each r in R.*

Remarks: 1. W. Vasconcelos [5] and the author ([3], Theorem 1.1) have shown that any almost hereditary ring (commutative or not) is a p.p. ring. Here, using Corollary 3.6, we are able to redefine an almost hereditary ring in terms of associated idempotents: A ring R with identity 1 is called *an almost hereditary ring* (left) if every (left) principal ideal and (left) ideal not contained in any minimal prime ideal are projective such that for each $r \neq 0$, $re_i \neq 0$ for each e_i in a chain of associated idempotents of r.

2. There exist p.p. rings which are not reduced. For example, a zero ring R ($R^2 = 0$) is p.p. and it is not reduced.

3. There are reduced rings which are not p.p., since any reduced p.p. ring with exactly two idempotents 0 and 1 must be a domain; but there are reduced rings with exactly two idempotents 0 and 1 which are not domains, so they are not p.p.

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