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POLARITY COMPATIBLE WITH A CLOSURE SYSTEM

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A symmetric relation on a non empty set is called *a polarity*, in general. In [8] a *C-polarity* $\varrho_C(\Omega)$ *on a closure space* (S, Ω) is defined — S is a non empty set and Ω is a closure system on S — in the following way: $a\ \varrho_C(\Omega)\ b \Leftrightarrow \bar{a} \cap \bar{b} \subseteqq \bar{C}$, where $a, b \in S$, $C \subseteqq S$ and $\bar{M}$ denotes the closure of $M \subseteqq S$ in Ω. This C-polarity is a generalization of some polarities from [1], [3], [4], [5] and [7] defined in l-groups, po-groups, lattices and semigroups (see [8], § 3). We denote $p(A, C) = \{x \in S : \bar{x} \cap \cap\ \bar{a} \subseteqq \bar{C}$ for each $a \in A\}$, $p^{n+1}(A, C) = p[p^n(A, C)\ C]$ for every $A, C \subseteqq S$ and a positive integer n; $\Gamma_C(S, \Omega) = \{p(A, C) : A \subseteqq S\}$, $\Gamma(S, \Omega) = \bigcap\{\Gamma_C(S, \Omega) : C \subseteqq S\}$. A set $A \subseteqq S$, $A = p^2(A, C)$ is called a *C-polar*.

In § 1 of this paper we compare a C-polarity and a (general) polarity using the results of F. ŠIK (see [6]). It is shown that the set $\Gamma_C(S, \Omega)$ of all C-polars on a closure space (S, Ω) is a complete Boolean algebra for each $C \subseteqq S$, ordered by the set-inclusion (Corollary 1.4). Further, a polarity on (S, Ω) compatible with Ω is investigated. This polarity is characterized by the fact that all polars are closed. A C-polarity $\varrho_C(\Omega)$ is compatible with Ω for each $C \in \Omega$ if and only if Ω is an algebraic closure system and a distributive lattice (Theorem 1.11).

In § 2 we show that a C-polarity on a closure space is a polarity defined in [7] on a suitable semigroup and some connections of these polarities are given.

A C-polarity on special closure spaces (topological spaces of Bourbaki, spaces with closed points) is investigated in § 3.

1. POLARITY COMPATIBLE WITH A CLOSURE SYSTEM

1.1. Definition. A symmetric binary relation δ in a non empty set S is called *a polarity* in S. For each $A \subseteqq S$ we define a set $\delta(A) = \{x \in S : x\ \delta\ a$ for each $a \in A\}$ and $\delta^n(A) = \delta[\delta^{n-1}(A)]$ for each positive integer n. If $A = \delta^2(A)$, then A is called a *δ-polar*. The set of all δ-polars in S will be denoted by $\Gamma_\delta(S)$ (or briefly Γ).

1.2. ([6], Theorem 3.) A) *Let δ be a polarity in a set S. Then $\Gamma_\delta(S)$ is a complete lattice, infima in Γ are set meets, S and $A = \{s \in S : s\ \delta\ x$ for each $x \in S\}$ are the* *greatest and the least element of Γ, respectively, and the map $A \in \Gamma \mapsto \delta(A)$ is an involution, i.e. $\delta^2(A) = A$, $\delta(\bigvee A_\alpha) = \bigwedge \delta(A_\alpha)$, $\delta(\bigwedge A_\alpha) = \bigvee \delta(A_\alpha)$ for all $A, A_\alpha \in \Gamma$.*

B) *Let δ be an antireflexive polarity in a set S. Then $\Gamma_\delta(S)$ is complemented and $\delta(A)$ is a complement of $A \in \Gamma_\delta(S)$.*

C) *Let δ be an antireflexive polarity in a set S with a property $(D\beta)$:*

x non $\delta\ y \Rightarrow$ *there exists $z \in S$ such that z non $\delta\ z$, $z \prec x$, $z \prec y$, where $\prec$ is a quasiorder in S induced by $\delta(a \prec b \Leftrightarrow \{u\ \delta\ b \Rightarrow u\ \delta\ a\})$. Then $\Gamma_\delta(S)$ is a complete Boolean algebra.*

1.3. ([6], Theorem 4.) A) *Let $\mathfrak{B}$ be a complete lattice of subsets of a set S, let infima in $\mathfrak{B}$ be set meets and let $A \to A'$ be a map of $\mathfrak{B}$ into $\mathfrak{B}$, fulfilling $A'' = A$, $(\bigvee A_\alpha)' = \bigwedge A'_\alpha$ for all A, $A_\alpha \in \mathfrak{B}$. Denote by X the greatest element of $\mathfrak{B}$. Then there exists a unique polarity δ in X such that $\Gamma_\delta(X) = \mathfrak{B}$.*

B) *Let $\mathfrak{B}$ be as in* A) *and in addition, let A' be a complement of A for any A in $\mathfrak{B}$. Then δ is antireflexive.*

C) *Let $\mathfrak{B}$ be a complete Boolean algebra of subsets of a set S, let infima in $\mathfrak{B}$ be set meets. Denote by X the greatest element of $\mathfrak{B}$. Then there exists a unique polarity δ in X such that $\Gamma_\delta(X) = \mathfrak{B}$. Furthermore, δ is antireflexive and δ has the property $(D\beta)$ from* 1.2.

Remark. The polarity δ from 1.3 is defined in the following way: $x\ \delta\ y \Leftrightarrow y \in \bar{x}'$, where $\bar{x} = \bigcap\{A \in \mathfrak{B} : x \in A\}$.

1.4. Corollary. *The set $\Gamma_C(S, \Omega)$ of all C-polars on a closure space (S, Ω) is a complete Boolean algebra for each $C \subseteqq S$, ordered by set-inclusion. Further, $\bigwedge_{i \in I} p(A_i, C) = \bigcap_{i \in I} p(A_i, C)$, $\bigvee_{i \in I} p(A_i, C) = p^2[\bigcup_{i \in I} p(A_i, C), C]$ for every $A_i \subseteqq S$, $i \in I \neq \emptyset$ and a complement of a C-polar $p(A, C)$ is $p^2(A, C)$ for each $A \subseteqq S$. The greatest element of $\Gamma_C(S, \Omega)$ is $S = p(\emptyset, C)$ and the smallest element of $\Gamma_C(S, \Omega)$ is $C = p(S, C)$.*

Proof. C-polarity $\varrho_C(\Omega)$ is a symmetric and antireflexive relation in S and we shall prove the property $(D\beta)$ from 1.2, C): If x non $\varrho_C(\Omega)\ y$, then $\bar{x} \cap \bar{y}$ non $\subseteqq \bar{C}$ and if we choose $z \in (\bar{x} \cap \bar{y}) \setminus \bar{C}$, then $\bar{z} \cap \bar{z} = \bar{z}$ non $\subseteqq \bar{C}$, i.e., z non $\varrho_C(\Omega)\ z$. Further, if $u\ \varrho_C(\Omega)\ x$, then $\bar{u} \cap \bar{x} \subseteqq \bar{C}$ and $\bar{u} \cap \bar{z} \subseteqq \bar{u} \cap (\bar{x} \cap \bar{y}) \subseteqq \bar{u} \cap \bar{x} \subseteqq \bar{C}$, i.e., $u\ \varrho_C(\Omega)\ z$ and $z \prec x$ in the quasiorder $\prec$ induced by $\varrho_C(\Omega)$ in S. Similarly, we can prove $z \prec y$. The rest follows from 1.2.

1.5. Proposition. *Let δ be an antireflexive polarity in S, $S \neq \emptyset$. Then $\varrho_\emptyset(\Gamma_\delta(S)) \supseteqq \delta$. Further, $\varrho_\emptyset(\Gamma_\delta(S)) = \delta$ if and only if $\delta^2(a) \cap \delta^2(b) = \delta^2(\emptyset)$ implies $a\ \delta\ b$ for $a, b \in S$.*

Proof. If $a, b \in S$, then $a \,\delta\, b$ implies $\delta^2(b) \subseteqq \delta(a)$ and $\delta^2(a) \cap \delta^2(b) \subseteqq \delta^2(a) \cap \cap\, \delta(a) = \delta^2(\emptyset)$. Further, $a \;\varrho_\emptyset(\Gamma_\delta(S))\; b \Leftrightarrow \bar{a} \cap \bar{b} \subseteqq \emptyset$ in $\Gamma_\delta(S) \Leftrightarrow \delta^2(a) \cap \delta^2(b) = = \delta^2(\emptyset)$.

1.6. Corollary. *If δ is an antireflexive polarity in S, which fulfils $\delta \neq \emptyset$ and has the property $(D\beta)$ from 1.2, then $\delta = \Gamma_\emptyset(\Gamma_\delta(S))$.*

Proof. If x non $\delta\, y$, $x, y \in S$, then $(D\beta)$ implies the existence of an element $z \in S$ such that z non $\delta\, z$, $z \prec x$, $z \prec y$. The relation $z \prec x$ means: $s \,\delta\, x \Rightarrow s \,\delta\, z$, $(s \in S)$, i.e., $z \in \delta^2(x)$. Similarly $z \in \delta^2(y)$. Further, $\delta^2(\emptyset) = \delta(S) = \{s \in S : s \,\delta\, x$ for every $x \in S\}$. If $z \in \delta^2(\emptyset)$, then $z \,\delta\, z$, a contradiction. Then $\delta^2(\emptyset) \neq \delta^2(x) \cap \delta^2(y)$ and the rest follows from 1.5.

Remark. C-polarity $\varrho_C(\Omega)$ is antireflexive and has the property $(D\beta)$ (see the proof of 1.4) and thus $\varrho_C(\Omega) = \varrho_\emptyset(\Gamma_C(S, \Omega))$.

1.7. Definition. Let δ be a relation on a closure space (S, Ω). We say that *δ is compatible with Ω*, when $s \,\delta\, A \Rightarrow s \,\delta\, \bar{A}$ for every $s \in S$ and $A \subseteqq S$.

Remark. $s \,\delta\, A$ means $s \,\delta\, a$ for each $a \in A$.

1.8. Proposition. *Let δ be a symmetric relation on a closure space (S, Ω). Then it holds:*

1) *δ is compatible with Ω if and only if $\delta(\bar{A}) = \delta(A)$ for each $A \subseteqq S$.*

2) *If δ is compatible with Ω, then $\Gamma(\delta) \subseteqq \Omega$.*

Proof. 1) is clear. 2) If $x \in \overline{\delta(A)}$, then $x \in \delta^2(\{x\}) \subseteqq \delta^2(\overline{\delta(A)}) = \delta^2(\delta(A)) = \delta(A)$ and $\overline{\delta(A)} \subseteqq \delta(A)$.

1.9. Proposition. 1) *C-polarity $\varrho_C(\Omega)$ is a symmetric relation and $\varrho_C(\Omega)(A) = = p(A, C)$ for every $A, C \subseteqq S$. $\Gamma_C(S, \Omega) = \Gamma(\varrho_C(\Omega))$.*

2) *C-polarity $\varrho_C(\Omega)$ is compatible with Ω if and only if $\Gamma_C(S, \Omega) \subseteqq \Omega$.*

3) *C-polarity $\varrho_C(\Omega)$ is compatible with Ω for each $C \subseteqq S$ if and only if $\Gamma(S, \Omega) = = \Omega$.*

Proof. 1) $\varrho_C(\Omega)(A) = \{s \in S : s \;\varrho_C(\Omega)\; a$ for each $a \in A\} = p(A, C)$.

2) $\Leftarrow$: $\Gamma_C(S, \Omega) \subseteqq \Omega$ implies $p(A, C) \cap \bar{A} \subseteqq p(A, C) \cap p^2(A, C) = p(A, C) \cap \cap\, p^2(A, C) = \bar{C}$ and similarly $p(\bar{A}, C) \cap \bar{A} \subseteqq \bar{C}$. From [8], 1.7 we have $p(A, C) \subseteqq \subseteqq p(\bar{A}, C) \subseteqq p(A, C)$. $\Rightarrow$: see 1.8,2.

3) $\Leftarrow$: $\Gamma(S, \Omega) = \Omega$ implies $\Gamma_C(S, \Omega) \subseteqq \Omega$ for each $C \subseteqq S$ and the rest follows from 2. $\Rightarrow$: see 1.8,2.

1.10. Lemma. *Let (S, Ω) be a closure system and Ω a distributive lattice with operations $\bar{A} \wedge \bar{B} = \bar{A} \cap \bar{B}$, $\bar{A} \vee \bar{B} = \overline{A \cup B}$ for every $A, B \subseteqq S$. Then $\bar{x} \cap \cap \bigcup\{\bar{N} : N \subseteqq A$ finite$\} \subseteqq \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}}$ for every $x \in S$, $A \subseteqq S$.*

Proof. $\bar{x} \cap \bigcup\{\bar{N} : N \subseteq A \text{ finite}\} = \bigcup\{\bar{x} \cap \bar{N} : N \subseteq A \text{ finite}\} = \bigcup\{\bar{x} \cap \overline{\{a_{1N}, \ldots, a_{kN}\}} : N = \{a_{1N}, \ldots, a_{kN}\} \subseteq A \text{ finite}\} = \bigcup\{\bar{x} \cap \overline{(\bar{a}_{1N} \cup \ldots \cup \bar{a}_{kN})} : N = \{a_{1N}, \ldots, a_{kN}\} \subseteq A \text{ finite}\} = \bigcup\{\overline{(\bar{x} \cap \bar{a}_{1N}) \cup \ldots \cup (\bar{x} \cap \bar{a}_{kN})} : N = \{a_{1N}, \ldots, a_{kN}\} \subseteq A \text{ finite}\} \subseteq \overline{\bigcup\{(\bar{x} \cap \bar{a}_{1N}) \cup \ldots \cup (\bar{x} \cap \bar{a}_{kN}) : N = \{a_{1N}, \ldots, a_{kN}\} \subseteq A \text{ finite}\}} = \overline{\bigcup\{\bar{x} \cap \bar{a} : a \in A\}} = \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}}$.

1.11. Theorem. *The following assertions are equivalent:*

1) *C-polarity $\varrho_C(\Omega)$ is compatible with Ω for each $C \in \Omega$.*
2) $\bar{x} \cap \bar{A} = \bar{x} \cap \overline{\bigcup\{\bar{a} : a \in A\}}$ *for every $x \in S$, $A \subseteq S$.*
3) *Ω is an algebraic closure system and a distributive lattice with operations $\bar{A} \wedge \bar{B} = \bar{A} \cap \bar{B}$, $\bar{A} \vee \bar{B} = \overline{A \cup B}$ for every $A, B \subseteq S$.*

Remark. 1. An algebraic closure system Ω on S is a closure system with the property: $\bar{A} = \bigcup\{\bar{N} : N \subseteq A \text{ finite}\}$ for each $A \subseteq S$ (see [2]).

2. The assertion 2 is the same kind of distributivity in $\Omega : \bar{x} \cap \overline{\bigcup\{\bar{a} : a \in A\}} = \bar{x} \cap \bar{A} = \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}} = \overline{\bigcup\{\bar{x} \cap a : \bar{a} \in A\}}$.

Proof. 1 ⇒ 2: The fact $\bigcup\{\bar{x} \cap \bar{a} : a \in A\} = \bar{x} \cap \bigcup\{\bar{a} : a \in A\} \subseteq \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}}$ implies $x \in p(A, \bar{x} \cap \bigcup\{\bar{a} : a \in A\}) = p(\bar{A}, \bar{x} \cap \bigcup\{\bar{a} : a \in A\})$. Thus $\bar{x} \cap \bar{A} = \bar{x} \cap \bigcup\{\bar{b} : b \in \bar{A}\} = \bigcup\{\bar{x} \cap \bar{b} : b \in \bar{A}\} \subseteq \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}}$. The inclusion $\bar{x} \cap \bar{A} \supseteq \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}}$ is clear.

2 ⇒ 1: If $x \in p(A, C)$, $A \subseteq S$, $C \in \Omega$, then $\bar{x} \cap \bigcup\{\bar{a} : a \in A\} = \bigcup\{\bar{x} \cap \bar{a} : a \in A\} \subseteq C$ and thus $\bar{x} \cap \bar{A} = \overline{\bar{x} \cap \bigcup\{\bar{a} : a \in A\}} \subseteq C$. It means that $x \in p(\bar{A}, C)$ and $p(A, C) \subseteq p(\bar{A}, C)$. The inclusion $p(A, C) \supseteq p(\bar{A}, C)$ follows from [7], 1.2,d).

1 ⇒ 3: Let $X, Y, Z \subseteq S$. Then $(\bar{X} \cup \bar{Z}) \cap (\bar{Y} \cup \bar{Z}) = (\bar{X} \cap \bar{Y}) \cup \bar{Z} \subseteq \overline{(\bar{X} \cap \bar{Y}) \cup \bar{Z}}$. If we denote $\overline{(\bar{X} \cap \bar{Y}) \cup \bar{Z}} = K$, then $\bar{X} \cup \bar{Z} \subseteq p(\bar{Y} \cup \bar{Z}, K) = p(\overline{\bar{Y} \cup \bar{Z}}, K)$. It means $(\bar{X} \cup \bar{Z}) \cap \overline{(\bar{Y} \cup \bar{Z})} \subseteq K$ and $\overline{\bar{Y} \cup \bar{Z}} \subseteq p(\bar{X} \cup \bar{Z}, K) = p(\overline{\bar{X} \cup \bar{Z}}, K)$. Finally, $\overline{\bar{X} \cup \bar{Z}} \cap \overline{\bar{Y} \cup \bar{Z}} \subseteq K = \overline{(\bar{X} \cap \bar{Y}) \cup \bar{Z}}$ and Ω is a distributive lattice. If $x \in \bar{A}$, then $\bar{x} \subseteq \bar{A}$ and $\bar{x} \cap \bigcup\{\bar{N} : N \subseteq A \text{ finite}\} \subseteq \bar{x} \cap \overline{\bigcup\{\bar{a} : a \in A\}} = \bar{x} \cap \bar{A} = \bar{x}$ (see Lemma 1.10 and 1 ⇔ 2). This fact implies $x \in \bar{x} \subseteq \bigcup\{\bar{N} : N \subseteq A \text{ finite}\}$ and $\bar{A} \subseteq \bigcup\{\bar{N} : N \subseteq A \text{ finite}\}$. The inclusion $\bar{A} \supseteq \bigcup\{\bar{N} : N \subseteq A \text{ finite}\}$ is clear and Ω is an algebraic closure system.

3 ⇒ 1: Let $x \in p(A, C)$, $A \subseteq S$. Then $\bar{x} \cap \bar{a} \subseteq C$ for each $a \in A$, i.e., $\bar{x} \cap \bigcup\{\bar{a} : a \in A\} = \bigcup\{\bar{x} \cap \bar{a} : a \in A\} \subseteq C$. Now, if N is a finite subset in A, $N = \{a_1, \ldots, a_k\}$, then $\bar{x} \cap \bar{N} = \bar{x} \cap \overline{\bar{a}_1 \cup \ldots \cup \bar{a}_k} = \overline{(\bar{x} \cap \bar{a}_1) \cup \ldots \cup (\bar{x} \cap \bar{a}_k)} \subseteq C$, because $\bar{x} \cap \bar{a}_i \subseteq C$ $(i = 1, \ldots, k)$. Further, $\bar{x} \cap \bar{A} = \bar{x} \cap \bigcup\{\bar{N} : N \subseteq A \text{ finite}\} = \bigcup\{\bar{x} \cap \bar{N} : N \subseteq A \text{ finite}\} \subseteq C$, i.e., $x \in p(\bar{A}, C)$. Finally, $p(A, C) \subseteq p(\bar{A}, C)$, [8], 1.2,d) implies $p(A, C) \supseteq p(\bar{A}, C)$ and $\varrho_C(\Omega)$ is compatible with Ω.

1.12. Proposition. *Let (S, Ω) be a closure system. Then it holds*: 1. *Let $A, C, D \subseteq \subseteq S$, $A \supseteq C$. Then $p(A, C) \cap p(C, D) = p(A, D)$ if and only if $\varrho_D(\Omega)$ is compatible with Ω and $\bar{D} \subseteq \bar{C}$.*

2. *If $\varrho_D(\Omega)$ is compatible with Ω and $\bar{D} \subseteq \bar{C}$, then $\bar{D} = \bar{C} \cap p(C, D)$.*

Proof. 1. $\Rightarrow$: $p(\bar{A}, D) = p(\bar{A}, A) \cap p(A, D) = S \cap p(A, D) = p(A, D)$ and 1.8.1 implies the compatibility of $\varrho_D(\Omega)$ with Ω. $\bar{D} = p(S, D) = p(S, C) \cap p(C, D) \subseteq \subseteq p(S, C) = \bar{C}$.

$\Leftarrow$: If $x \in p(A, D)$, then $\bar{x} \cap \bar{a} \subseteq \bar{D} \subseteq \bar{C}$ for each $a \in A$ and $x \in p(A, C)$. Further, $\bar{x} \cap \bar{c} \subseteq \bar{x} \cap \bigcup\{\bar{a} : a \in A\} \subseteq \bar{D}$ for each $c \in C$. It means that $x \in p(C, D)$, i.e., $p(A, D) \subseteq p(A, C) \cap p(C, D)$. If $x \in p(A, C) \cap p(C, D)$, then $\bar{x} \cap \bar{a} \subseteq \bar{C}$, $\bar{x} \cap \bar{c} \subseteq \bar{D}$ for every $a \in A$, $c \in C$. Compatibility of $\varrho_D(\Omega)$ with Ω implies $x \in p(C, D) = p(\bar{C}, D)$, i.e., $\bar{x} \cap \bar{C} = \bar{x} \cap \bigcup\{\bar{y} : y \in \bar{C}\} = \bigcup\{\bar{x} \cap \bar{y} : y \in \bar{C}\} \subseteq \bar{D}$. Finally, $\bar{x} \cap \bar{a} \subseteq \bar{x} \cap \cap \bar{C} \subseteq \bar{D}$ and $x \in p(A, D)$, $p(A, C) \cap p(C, D) \subseteq p(A, D)$.

2. From 1 it follows that $D = p(S, D) = p(S, C) \cap p(C, D) = \bar{C} \cap p(C, D)$.

2. POLARITY ON SEMIGROUPS

2.1. Definition. (See [7].) Let $(S, \cdot)$ be a semigroup. A mapping $x : \exp S \to \exp S$ fulfilling the following conditions:

I. $A \subseteq S \Rightarrow A \subseteq A_x$,

II. $A, B \subseteq S$, $A \subseteq B_x \Rightarrow A_x \subseteq B_x$,

III. $A \subseteq S \Rightarrow S . A_x \subseteq A_x$,

IV. $A, B \subseteq S \Rightarrow A . B_x \subseteq (A . B)_x$

is called *an ideal mapping* and a set $A \subseteq S$ with the property $A_x = A$ is called *an x-ideal in S*. A system of all x-ideals in S for a given ideal mapping is called *an x-system.*

2.2. Proposition. *If Ω is a closure system on a set S and $x : \exp(\exp S) \to \to \exp(\exp S)$ such that $\mathscr{A}_x = \{X \in \exp S : X \subseteq \bar{A},\ X \neq \emptyset$ for a suitable $A \in \mathscr{A}\}$ for each $\mathscr{A} \subseteq \exp S$, then x is an ideal mapping in the commutative semigroup $(\exp S, \cdot)$, where $A . B = \bar{A} \cap \bar{B}$ for every $A, B \subseteq S$.*

Proof. I. $\mathscr{A} \subseteq \mathscr{A}_x$ is clear. II. If $\mathscr{A} \subseteq \mathscr{B}_x$, $Y \in \mathscr{A}_x$, then there exists $A \in \mathscr{A}$ such that $Y \subseteq \bar{A}$. It means that $A \in \mathscr{B}_x$, i.e., there exists $B \in \mathscr{B}$ such that $A \subseteq \bar{B}$ and $Y \subseteq \bar{A} \subseteq \bar{\bar{B}} = \bar{B}$, $Y \in \mathscr{B}_x$. Finally, $\mathscr{A}_x \subseteq \mathscr{B}_x$. III. If $\mathscr{A} \subseteq \exp S$, $X \in \mathscr{A}_x$, $Y \in \exp S$, then there exists $A \in \mathscr{A}$ such that $X \subseteq \bar{A}$, i.e., $X . Y = \bar{X} \cap \bar{Y} \subseteq \bar{\bar{A}} \cap \bar{Y} \subseteq \bar{A}$. It implies $X . Y \in \mathscr{A}_x$ and $\mathscr{A}_x . \exp S \subseteq \mathscr{A}_x$. IV. If $\mathscr{A}, \mathscr{B} \subseteq \exp S$, $X \in \mathscr{A}$, $Y \in \mathscr{B}_x$, then there exists $B \in \mathscr{B}$ such that $Y \subseteq \bar{B}$ and thus $X . Y = \bar{X} \cap \bar{Y} \subseteq \overline{\bar{X} \cap \bar{Y}} \in \in (\mathscr{A} . \mathscr{B})_x$.

2.3. Definition. (See [7].) Let $(S, \cdot, e)$ be a commutative semigroup with a zero e (i.e., $s \,.\, e = e \,.\, s = e$ for each $s \in S$). We define a symmetric relation δ' in S, called *δ'-polarity*, in the following way:

$$x \, \delta' \, y \Leftrightarrow x \,.\, y = e \quad \text{for} \quad x, y \in S\,, \quad x \neq y\,,$$

$$x \, \delta' \, x \Leftrightarrow x = e \qquad \text{for} \quad x \in S\,.$$

2.4. Proposition. *If $(S, \cdot, e)$ is a commutative semigroup with a zero e, Ω is an x-system on S and $s \,.\, s = e$ implies $s = e$, for $s \in S$, then it holds: $\delta' = \varrho_{\{e\}}(\Omega) \Leftrightarrow \{e\}_x = \{e\}$.*

Proof. $\Rightarrow$: $p \in \{e\}_x \Rightarrow \{p\}_x \cap \{p\}_x = \{p\}_x \subseteqq \{e\}_x \Rightarrow p\varrho_{\{e\}}(\Omega)\, p \Rightarrow p \,\delta'\, p \Rightarrow p = e \Rightarrow \{e\}_x = \{e\}$.

$\Leftarrow$: If $a \,\delta'\, b$, $a \neq b$, $a, b \in S$, then $a \,.\, b = e$, $b \in \delta'(a)$, $\{b\}_x \subseteqq \delta'(a)$, $\{a\}_x \subseteqq \delta''(a)$ – see [7], 1.7. Now, if $p \in \{a\}_x \cap \{b\}_x$, then $p \in \delta''(a) \cap \delta'(a) = \{e\}_x = \{e\}$ and $\{a\}_x \cap \{b\}_x \subseteqq \{e\}_x$, i.e., $a\, \varrho_{\{e\}}(\Omega)\, b$. If $a \,\delta'\, b$, $a = b$, then $a = b = e$ and $\{a\}_x \cap \{b\}_x \subseteqq \{e\}_x$, i.e., $a\, \varrho_{\{e\}}(\Omega)\, b$.

Conversely, if $a\, \varrho_{\{e\}}(\Omega)\, b$, $a \neq b$, then $\{a\}_x \cap \{b\}_x \subseteqq \{e\}_x = \{e\}$ and $a \,.\, b \in \{a\}_x \cap \{b\}_x = \{e\}$, i.e., $a \,.\, b = e$, $a \,\delta'\, b$. If $a\, \varrho_{\{e\}}(\Omega)\, b$, $a = b$, then $\{a\}_x = \{e\}$ and $a = e$, i.e., $a \,\delta'\, b$.

Notation. $(\exp S)_C = \{X \in \exp S : X \supseteqq C\}$, $\Omega(\exp S) = \{\mathscr{A}_x : \mathscr{A} \subseteqq \exp S\}$, $\Omega_C(\exp S) = \{\mathscr{A}_x : \mathscr{A} \subseteqq (\exp S)_C\}$.

2.5. Corollary. *If (S, Ω) is a non empty set with a closure system Ω, $C \in \Omega$, then $((\exp S)_C, \cdot, C)$ is a commutative semigroup with a zero C, where $A \,.\, B = \bar{A} \cap \bar{B}$ for every $A, B \in (\exp S)_C$, and the restriction x_C of the mapping x from Proposition 2.2 on $(\exp S)_C$ is an ideal mapping in $(\exp S, \cdot)$ and also in $((\exp S)_C, \cdot)$. Further, a δ'-polarity in $((\exp S)_C, \cdot)$ is a $\{C\}$-*polarity *$\varrho_{\{C\}}(\Omega_C(\exp S))$ in $((\exp S)_C, \Omega_C(\exp S))$.*

Proof. The first part of Corollary can be proved similarly as Proposition 2.2. The second part follows from Proposition 2.4 and the fact $\{C\}_{x_C} = \{X \in (\exp S)_C : X \subseteqq \bar{C}\} = \{C\}$.

2.6. Proposition. *For a C-polarity $\varrho_C(\Omega)$ on a closure system (S, Ω) and a δ'-polarity on a commutative semigroup $((\exp S)_C, \cdot)$, where $C \in \Omega$, it holds:*

1. *$X \in \delta'(A) \Leftrightarrow \bar{X} \subseteqq p(\bar{A}, C)$ for $A, X \in (\exp S)_C$.*

2. *Moreover, if $\varrho_C(\Omega)$ is compatible with Ω, then $\delta'(A) = (p(A, C))_{x_C}$, $\delta''(A) = (p^2(A, C))_{x_C}$, $\delta'[(p(A, C))_{x_C}] = \delta'(p(A, C))$, where $A \in (\exp S)_C$ and x_C is the ideal mapping on $(\exp S)_C$ from 2.5.*

Proof. 1. $X \in \delta'(A) \Leftrightarrow X \,.\, A = C \Leftrightarrow \bar{X} \cap \bar{A} = C \Leftrightarrow \bar{X} \subseteqq p(\bar{A}, C)$.

2. The fact $\delta'(A) = (p(A, C))_{xc}$ follows from 1 and 1.9, 2): $X \in \delta'(A) \Leftrightarrow \bar{X} \subseteq p(\bar{A}, C) \Leftrightarrow \bar{X} \in \{p(\bar{A}, C)\}_{xc} \Leftrightarrow X \in \{p(\bar{A}, C)\}_{xc}$. Further, for each $X \in \delta'(p(\bar{A}, C))$ and each $Z \in [p(A, C)]_{xc}$ we have $\bar{Z} \subseteq p(\bar{A}, C)$ and $X . Z = \overline{\bar{X} \cap \bar{Z}} \subseteq \overline{\bar{X} \cap p(\bar{A}, C)} = \overline{\bar{X} \cap \overline{p(\bar{A}, C)}} = X . p(\bar{A}, C) = C$, i.e., $X \in \delta'[(p(A, C))_{xc}]$. It means $\delta'(p(\bar{A}, C)) \subseteq \delta'[(p(A, C)_{xc}]$ and from $p(\bar{A}, C) \in (p(A, C))_{xc}$ we have $\delta'(p(\bar{A}, C)) \supseteq \delta'[(p(\bar{A}, C)_{xc}]$. Finally, $\delta''(A) = \delta'(\delta'(A)) = \delta'[(p(\bar{A}, C)_{xc}] = \delta'(p(\bar{A}, C)) = (p^2(\bar{A}, C))_{xc}$.

3. POLARS ON SPECIAL CLOSURE SPACES

3.1. Proposition. *Let $\varrho_C(\Omega)$ be compatible with Ω for each $C \subseteq S$. Then Ω defines a topological space of Bourbaki on S if and only if $\Gamma(S, \Omega)$ is a sublattice of the lattice* $(\exp S, \cup, \cap)$.

Proof. $\Rightarrow$: For every $P, Q \in \Gamma(S, \Omega)$ it holds $P \cup Q = \bar{P} \cup \bar{Q} = \overline{P \cup Q} \in \Omega = \Gamma(S, \Omega)$, see 1.9,3.

$\Leftarrow$: If $A, B \subseteq S$, then $\bar{A} \cup \bar{B} = p(S \setminus A, A) \cup p(S \setminus B, B) \in \Gamma(S, \Omega) = \Omega$ (see 1.9,3 and [8], 1.5,a)), i.e., $\bar{A} \cup \bar{B} = \overline{\bar{A} \cup \bar{B}} = \overline{A \cup B}$.

3.2. Proposition. *If $\bar{\emptyset} \in \Omega$, then $p(\bar{A}, \emptyset) = \{s \in S : \bar{s} \subseteq S \setminus \bar{A}\}$ for each $A \subseteq S$.*

Proof. If $s \in p(\bar{A}, \emptyset)$, then $\bar{s} \cap \bar{a} \subseteq \bar{\emptyset} = \emptyset$ for each $a \in \bar{A}$, i.e., $\bar{s} \cap \bar{A} = \emptyset$ and $\bar{s} \subseteq S \setminus \bar{A}$. Further, if $\bar{s} \subseteq S \setminus \bar{A}$, then $\bar{s} \cap \bar{a} \subseteq (S \setminus \bar{A}) \cap \bar{A} = \emptyset$ for each $a \in \bar{A}$ and $s \in p(\bar{A}, \emptyset)$.

3.3. Theorem. *If $\{s\} \cup \bar{\emptyset} \in \Omega$ for each $s \in S$, then $p(A, C) = (S \setminus A) \cup \bar{C}$ for every $A, C \subseteq S$. If $\varrho_C(\Omega)$ is compatible with Ω and $p(A, C) = (S \setminus A) \cup \bar{C}$ for every $A, C \subseteq S$, then $\{s\} \cup \bar{\emptyset} \in \Omega$ for each $s \in S$.*

Proof. If $x \in p(A, C) \setminus \bar{C}$, then $\bar{x} \cap \bar{a} \subseteq C$ for every $a \in A$ and $x \neq a$, i.e., $x \in S \setminus A$, $p(A, C) \subseteq \bar{C} \cup (S \setminus A)$. Further, $\bar{C} \subseteq p(A, C)$ (see [8], 1.1,a)) and $\bar{s} \cap \bar{a} = (\{s\} \cup \bar{\emptyset}) \cap (\{a\} \cup \bar{\emptyset}) = (\{s\} \cap \{a\}) \cup \bar{\emptyset} = \bar{\emptyset} \subseteq C$ for every $s \in S \setminus A$ and $a \in A$, i.e., $\bar{C} \cup (S \setminus A) \subseteq p(A, C)$.

If $\varrho_C(\Omega)$ is compatible with Ω, then we have $p(S \setminus \{s\}, \emptyset) = (S \setminus (S \setminus \{s\})) \cup \bar{\emptyset} = \{s\} \cup \bar{\emptyset}$ for each $s \in S$ and thus $\{s\} \cup \bar{\emptyset} \in \Omega$, see 1.8,2.

3.4. Corollary. *Let $\{s\} \in \Omega$ for each $s \in S$. Then $\varrho_\emptyset(\Omega)$ is compatible with Ω if and only if $\Omega = \exp S$.*

Proof. $\Rightarrow$: It is $\emptyset \in \Omega$ and $S \setminus A = p(A, \emptyset) = p(\bar{A}, \emptyset) = S \setminus \bar{A}$ (see 3.3), i.e., $A \in \Omega$ for each $A \subseteq S$. The second implication is clear.

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