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COHOMOLOGY AND SPECIAL EXTENSIONS OF GROUPS

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Cohomology of groups was first studied by EILENBERG and MACLANE [6]. Later, GRUENBERG [3] gave a different approach to the subject. Aside from Gruenberg's work, the treatment of the subject has seen little change. In this paper, we give new proofs of some basic results. For example, we give a rather simple proof of the theorem relating $H^2(G, A)$ and abelian extensions. The proof is free of cocycle calculations and is a direct analog of a proof of the corresponding theorem in module theory. Moreover, the proof can be copied exactly to prove the corresponding theorem in Lie algebras. Our proof is somewhat easier than that presented by BARR and REINHART [2].

The main new result of this paper concerns the interpretations of $H^3(G, A)$. Eilenberg and MacLane showed that $H^3(G, A)$ corresponds to obstructions to group extensions but unfortunately their correspondence is not bijective. We show that $H^3(G, A)$ corresponds to a set of equivalence classes of two term extensions of groups. The correspondence is bijective. Furthermore, this interpretation seems more natural because the theory of cohomology was motivated by the study of group extensions. A similar interpretation of $H^3(L, A)$ of Lie algebras was given by SHIMADA, UEHARA, BRENNEMAN and IWAI [7]. Their proof used triple cohomologies and is not suited for our treatment.

In the last section, we generalize the Schur-Zassenhaus theorem as follows. A group is called 2-finite if every subgroup generated by two elements is finite. We prove that extension of a finite group H by a 2-finite group G split if the orders of elements of G are relatively prime to the order of H. This result is probably known but we cannot find it in the literature.

1. Let G be a group and ZG the integral group ring. Let $IG = \operatorname{Ker}(ZG \to^{\varepsilon} Z)$. Then we can view $H^n(G, A)$ as $\operatorname{Ext}_G^{n-1}(IG, A)$ for $n \geqq 2$. A proof using cochains of the following proposition can be found in [6].

Proposition 1.1. *If* $0(G) = m$, *then* $m\, H^n(G, A) = 0$ *for* $n \geqq 2$.

Proof. Let

$$\xi\colon 0 \to A \to E_{n-1} \to \dots \to E_1 \to^{\eta} IG \to 0$$

represent an element of $\mathrm{Ext}_G^{n-1}(IG, A)$. Then, since IG is a free abelian group, there is a group homomorphism $s\colon IG \to E_1$ such that $\eta s = \mathrm{Id}$. Let $\psi\colon IG \to E_1$ be defined by $\psi(x) = \sum_{y \in G} y\, s(y^{-1}x)$. Then ψ is a G-map and $\eta \,.\, \psi = m\,\mathrm{Id}$. Now $m\xi$ is constructed as follows.

$$\begin{tikzcd}
\xi\colon 0 \arrow[r] & A \arrow[r] & E_{n-1} \arrow[r] & \dots \arrow[r] & E_1 \arrow[r, "\eta"] & IG \arrow[r] & 0 \\
m\xi\colon 0 \arrow[r] & A \arrow[r] \arrow[u, equal] & E_{n-1} \arrow[r] \arrow[u, equal] & \dots \arrow[r] & \bar{E}_1 \arrow[r, "\bar{\eta}"] \arrow[u] \arrow[ur, phantom, "\Sigma"] & IG \arrow[r] \arrow[u, "m"'] & 0
\end{tikzcd}$$

where m denotes the map "multiplication by m" and Σ is a pullback diagram. Define $\bar{\psi}\colon IG \to E_1 \times IG$ by $\bar{\psi}(x) = (\psi(x), x)$. Since $\eta\, \psi(x) = mx$, thus $\bar{\psi}(x) \in \bar{E}_1$. Obviously $\bar{\eta}\bar{\psi} = \mathrm{Id}$ and $[m\xi] = 0$.

Corollary 1.2. *If* $0(G) = m$, $\exp(A) = l$ *are relatively prime,* $H^n(G, A) = 0$, $n \geqq 2$.

Proof. Since $H^n(G, A)$ is a subquotient of $\mathrm{Hom}_G(P_n, A)$ for some G-projective module P_n, $lH^n(G, A) = 0$. This together with Proposition 1.1 proves the assertion.

2. Let G be a group. Let A, B be two G-modules. All commutative squares of the form

(2.1)
$$\begin{tikzcd}
P \arrow[r, "\bar{f}"] \arrow[d, "\bar{d}"'] & G \arrow[d, "d"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

with d a fixed G-derivation, f a fixed G-homomorphism, $\bar{f}$ a group homomorphism and $\bar{d}$ a P-derivation, (A acquires a P-module structure via $\bar{f}$), form a category $\mathbf{C}(d, f)$ with obvious maps. The terminal object in this category is called a *mixed pullback diagram.*

Proposition 2.1. *There exist mixed pullbacks.*

Proof. Let $A \times_s G$ be the semidirect product of A and G. Let $P = \{(a, x) \in A \times_s G\colon f(a) = d(x)\}$. Let $\bar{d}$ and $\bar{f}$ be the projections. Obviously, $\bar{f}$ is a group homomorphism and calculations show that $\bar{d}$ is a P-derivation. It is routine to check that, with these maps, we do have a mixed pullback. (Indeed P is even the set-theoretical pullback.)

Proposition 2.2. *In the pullback diagram* (2.1), $\mathrm{Ker}\, f \simeq \mathrm{Ker}\, \bar{f}$.

Proof. Let $K = \operatorname{Ker} f$. Define $i : K \to P$ by $i(a) = (a, 1)$. Then $i(a + b) = (a + b, 1) = i(a)\, i(b)$ and $\bar{f} i = 0$. On the other hand, if $(a, x) \in \operatorname{Ker} \bar{f}$, then $x = 1$ and $f(a) = d(1) = 0$ so that $a \in K$. Thus $K = \operatorname{Ker} \bar{f}$.

Proposition 2.3. *In the pullback diagram* (2.1), *if f is epic, $\bar{f}$ is epic.*

Proof. Obvious from construction.

Proposition 2.4. *Consider the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, "g"'] & P \arrow[r, "\bar{f}"] \arrow[d, "\bar{d}"'] \arrow[dr, phantom, "\Sigma"] & G \arrow[r] \arrow[d, "d"] & 1 \\
0 \arrow[r] & K \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & 1
\end{tikzcd}$$

where the top row is an exact sequence of groups, the bottom row an exact sequence of G-modules and $\Sigma \in \mathbf{C}(d, f)$. Then g is a group homomorphism and is an isomorphism iff Σ is a pullback.

Proof. K is a trivial K'-group since its P-action is induced from G. Thus g, being a derivation, is a group homomorphism. By Proposition 2.2 it is an isomorphism when Σ is a mixed pullback.

Suppose g is an isomorphism. Any mixed pullback

$$\begin{tikzcd}
P'' \arrow[r] \arrow[d] & G \arrow[d, "d"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

induces an exact commutative group diagram

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r] \arrow[d, "g'"'] & P'' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & K' \arrow[r] & P \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

Since g and (by Proposition 2.2) gg' are isomorphisms, so is g'. Thus $P'' \to P$ is an isomorphism by the "Five Lemma".

3. Let E and G be groups. We call G an *E-group* if there is a group homomorphism $E \to \operatorname{Aut} G$. Thus E acts on G and we denote this action by $e \,.\, x$ for $e \in E$, $x \in G$. Clearly, E is an E-group by conjugation and any group homomorphism $\pi : E \to G$ induces an E-action on G rendering π an E-map.

A *one-fold special extension* of G by a G-module A is an exact sequence of E-groups

$$0 \to A \to E \to G \to 1$$

where the E-action on A via G coincides with the E-action on A by conjugation. The one fold extensions are known as *abelian extensions.*

A *two-fold special extension* of G by A is an exact sequence of E_1-groups

$$0 \to A \to E_2 \to^{\phi} E_1 \to G \to 1$$

where the E_1-action on A coincides with that induced by G, the E_2-action on A via E_1 coincides with that by conjugation.

We introduce an equivalence relation between special extensions similar to that between extension of modules. The sets of equivalence classes are denoted by $\mathrm{Sext}^i(G, A)$, $i = 1, 2$, respectively.

Note that if $0 \to A \to E_2 \to^{\phi} E_1 \to G \to 1$ is a 2-fold special extension, then A is central in E_2. For if $e_2 \in E_2$, $a \in A$, then $e_2 a e_2^{-1} = e_2 . a = \phi(e_2) . a = a$.

4. Let $1 \to H \to E \to G \to 1$ be an exact sequence of groups. Let $H_{ab} = H/[H, H]$. Then there is an exact sequence of G-modules [4]

$$0 \to H_{ab} \to^{i} ZG \otimes_E IE \to^{\pi} IG \to 1 \tag{4.1}$$

where $i(h + [H, H]) = 1 \otimes (h - 1)$.

Proposition 4.1. *If*

$$0 \to A \to E \to G \to 1 \tag{4.2}$$

is an abelian extension, then

(a) *the diagram*

$$\begin{array}{ccc} E & \longrightarrow & G \\ {\scriptstyle \bar{d}}\downarrow & & \downarrow{\scriptstyle d} \\ ZG \otimes_E IE & \longrightarrow & IG \end{array}$$

where $\bar{d}(e) = 1 \otimes (e - 1)$, $dx = x - 1$, *is a mixed pullback.*

(b) *If* $E = A \times_s G$, *then*

$$\begin{array}{ccc} A \times_s G & \xrightarrow{\bar{\pi}} & G \\ \downarrow & & \downarrow \\ A \times IG & \xrightarrow{\pi} & IG \end{array}$$

wherw $\bar{d}(a, x) = (a, x - 1)$, $dx = x - 1$, *is a mixed pullback.*

Proof. To demonstrate (a) we first claim that it is easy to verify $\bar{d}$ is a derivation. Then from sequence 4.1 it is clear that $\operatorname{Ker} \bar{\pi} \simeq \operatorname{Ker} \pi$ and hence by Proposition 2.4, the diagram is a mixed pullback. (b) follows from (a) since when $E = A \times_s G$ the unique homomorphism $\theta : ZG \otimes_E IE \to A \times IG$ such that $\theta(\sigma \otimes (a, x) - 1) = (^{\sigma}a, \sigma x - 1)$ is an E-isomorphism with $\theta^{-1}(a, x - 1) = 1 \otimes [(a, x) - 1]$.

Theorem 4.2. $H^2(G, A) \simeq^{\text{nat}} \operatorname{Sext}^1(G, A)$.

Proof. We show that $\operatorname{Ext}^1_G(IG, A) \simeq^{\text{nat}} \operatorname{Sext}^1(G, A)$. Let $\xi : 0 \to A \to E \to IG \to 0$ represent an element in $\operatorname{Ext}^1_G(IG, A)$. Construct the commutative diagram

$$\begin{tikzcd} \psi(\xi): & 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, "\bar{d}"'] \arrow[dr, phantom, "\Sigma"] & G \arrow[r] \arrow[d] & 1 \\ \xi: & 0 \arrow[r] & A \arrow[r] & E \arrow[r] & IG \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

where Σ is a mixed pullback. Define $\psi : \operatorname{Ext}^1_G(IG, A) \to \operatorname{Sext}^1(G, A)$ by $\psi([\xi]) = [\psi(\xi)]$. It is obvious that ψ is well-defined.

Conversely, given $\chi : 0 \to A \to P \to G \to 1$ representing an element in $\operatorname{Sext}^1(G, A)$, define $\psi^{-1} : \operatorname{Sext}^1(G, A) \to \operatorname{Ext}^1_G(IG, A)$ by $\psi^{-1}([\chi]) = [0 \to A \to ZG \otimes_P IP \to IG \to 0]$. Proposition 4.1a says that $\psi\psi^{-1} = \mathrm{Id}$. On the other hand, if P is obtained as in (4.3), define $g : ZG \otimes_P IP \to E$ by linearity and $g(y \otimes ((e, x) - 1)) = ye$. Then the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & ZG \otimes_P IP \arrow[r] \arrow[d, "g"] & IG \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r] & IG \arrow[r] & 0 \end{tikzcd}$$

is commutative. Thus $\psi^{-1}\psi = \mathrm{Id}$.

The naturality of ψ is clear. Note that by Proposition 4.1 the neutral element of $\operatorname{Ext}^1_G(IG, A)$ corresponds to the *split* abelian extension in $\operatorname{Sext}^1(G, A)$.

Corollary 4.3. *If* $0(G)$ *and* $0(A)$ *are relatively prime, the special extension* $0 \to A \to E \to G \to 0$ *splits.*

Proof. Corollary 1.2.

5. Suppose $0 \to H \to E \to^{\pi} G \to 1$ is an exact sequence of groups and $\sigma : G \to E$ is a section of π with $\sigma(1) = 1$. Then we can view E as $H \times G$ with

$$(h_1, x) . (h_2, y) = (h_1 + {}^{\sigma(x)}h_2 + f(x, y), xy) \tag{5.1}$$

where $f(x, y) = \sigma(x)\,\sigma(y)\,\sigma(xy)^{-1}$ is a normalized non-abelian 2 cocycle:

$$f(x, 1) = 0 = f(1, y) \tag{5.2}$$

$$^{\sigma(x)}f(y, z) + f(x, yz) = f(x, y) + f(xy, z) \tag{5.3}$$

Now suppose $\xi : 0 \to A \to H \to E \to^{\pi} G \to 1$ is a special extension and $\sigma : G \to E$ a section of π with $\sigma(1) = 1$. Since $A \to H$ is an E map

$$^{\sigma(x)}a = {}^{x}a \ \forall a \in A\,. \tag{5.4}$$

For $x, y \in G$ choose $f(x, y) \in H$ normalized to satisfy (5.2) such that its class in $H/A = \bar{H} \subseteq E$ represents $\sigma(x)\,\sigma(y)\,\sigma(xy)^{-1}$. Since $H \to E$ is an H-map we have

$$^{\sigma(x)\sigma(y)}h + f(x, y) = f(x, y) + {}^{\sigma(xy)}h\,. \tag{5.5}$$

As above we view E as $\bar{H} \times G$ with multiplication as in (5.1) (with appropriate bars) and condition (5.3) translates to

$$^{\sigma(x)}f(y, z) + f(x, yz) = f(x, y) + f(xy, z) + k(x, y, z) \tag{5.6}$$

for a unique $k(x, y, z) \in A$.

The centrality of A in H ensures that k is a 3 cocycle in A. Notice that if $k = 0$, then formula (5.1) renders $\tilde{E} = H \times G$ into a group such that the natural exact sequence of sets

$$0 \to A \to \tilde{E} \to E \to 1 \tag{5.7}$$

is a group extension. Thus we may call k the 3 cocyle obstruction to group extension. In fact even if $\tilde{E}$ is not a group the maps in (5.7) preserve the identity and multiplication and so (5.7) can be viewed as a "multiplicative extension" of A by E. We can then impose the usual equivalence relation on such extensions.

Theorem 5.1. *For each k the set of (equivalence classes of) multiplicative extensions (5.7) is in 1-1 correspondence with $Z^2(G, A)\,(H^2(G, A))$. Moreover $k = 0$ iff each $\tilde{E}$ is associative iff $\tilde{E}$ is a group.*

Proof. The second statement is clear. The multiplication in $\tilde{E}$ must be given by (5.1) where $f(x, y) \in H$ satisfies (5.2), (5.5), (5.6). If $f'(x, y)$ is another solution, then by (5.1) $f'(x, y) = f(x, y) + a(x, y)$ and by (5.6) $a \in Z^2(G, A)$.

f and $f' = f + a$ give equivalent solutions iff there is a multiplicative map $A : \tilde{E}_f \to \tilde{E}_{f'}$, rendering

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "1"] & \tilde{E}_f \arrow[r] \arrow[d, "A"] \arrow[dr, phantom, "2"] & E \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r] & \tilde{E}_{f'} \arrow[r] & E \arrow[r] & 1
\end{tikzcd}$$

commutative. By the commutativity of 2 we see $\Lambda(h, x) = (\Lambda'(h, x), x)$ for some $\Lambda' : \tilde{E} \to H$. Since Λ is multiplicative, $\Lambda'' = \Lambda'$ on H is a group homomorphism. The commutativity of 1 and 2 renders

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d, "\Lambda''"] & \bar{H} \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r] & H \arrow[r] & \bar{H} \arrow[r] & 1
\end{tikzcd}$$

commutative. Thus Λ'' is the identity, hence $\Lambda(h, x) = (h + \Lambda'(1, x), x)$ with $\Lambda'(1, x) \in A$. It is now easily checked that $\Lambda : \tilde{E}_f \to \tilde{E}_{f+a}$ is multiplicative iff $a(x, y) = -d(\Lambda'(1, x))$. Thus the equivalence classes of extensions corresponds to $H^2(G, A)$.

Note that in the commutative exact diagram (5.8) of sets, we have shown that a multiplication on $\tilde{E}$ rendering the middle row a group extension also renders the middle column a group extension. The analysis of MacLane [6] page 128 is on this middle column.

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & \bar{H} \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & A \arrow[r] & \tilde{E} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 1 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd} \tag{5.8}$$

(In a sense k is the obstruction to (5.8) being a group diagram.) Similar to MacLane's analysis, we can use the Baer sum concept to show directly that for a particular k, $H^2(G, A) \simeq \operatorname{Ext}^1(G, A)$ operates transitively and faithfully on solutions of (5.7).

Theorem 5.2. $\theta : \operatorname{Sext}^2(G, A) \to H^3(G, A)$, *given by* $\theta([\xi]) = [k]$ *as above, is a natural equivalence.*

Proof. We omit the verification (similar to that on page 128 of [6]) that θ is a well-defined natural transformation.

Note that to reconstruct the special extension ξ we need only

1) the central embedding $A \subset H$
2) for $x \in G$ an action $\sigma(x)$ on H
3) for $x, y \in G$, elements $f(x, y) \in H$

with (5.2), (5.4), (5.5) and (5.6) satisfied. For then we can use (5.1) to define the multiplication on $E = \bar{H} \times G$, define an operation of E on H by

$$(\bar{h}, x)(h_2) = h + {}^{\sigma(x)}h_2 - h$$

and let $H \to E \to G$ be the natural maps. (5.6) renders E a group, (5.4) renders $A \to H$ an E-map. $H \to E \to G$ are automatically E-maps. (5.5) renders the E-action on H associative.

Thus to define an inverse ψ to θ we need for each $k \in Z^3(G, A)$ to define a group H_k containing A in its center, an action of $\sigma(x)$ on H_k and elements $f_k(x, y) \in H$ satisfying (5.2), (5.4), (5.5) and (5.6). We do this as follows. Let F_k be the group on the symbols $f_k(x, y)$ $(x, y \in G)$ subject to the relations (5.2). Let $H_k = A \times F_k$. Define $\sigma(x)$ on H_k by using (5.4) on A and (5.6) on generators of F_k. We need only verify (5.5). However the set of h satisfying (5.5) is clearly a subgroup of H containing A and (by a simple calculation) containing each generator $f_k(z, w)$ of F_k. Thus (5.5) holds.

We let $\psi(k) = \xi_k$ be the resulting special extension. It is easy to see that for $a(x, y) \in \in A$ the commutative diagram

$$\begin{array}{lccccccccccc}
\psi(k): & 0 & \longrightarrow & A & \longrightarrow & H_k & \longrightarrow & E_k & \longrightarrow & G & \longrightarrow & 1 \\
 & & & \| & & \downarrow{\scriptstyle A} & & \| & & \| & & \\
\psi(k'): & 0 & \longrightarrow & A & \longrightarrow & H_{k'} & \longrightarrow & E_{k'} & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

renders $\psi(k) \sim \psi(k')$ where $k' = k + da$ and $A(f_k(x, y)) = f_{k'}(x, y) - a(x, y)$. Thus $\psi([k]) = [\psi(k)]$ is well defined. Clearly $\theta\, \psi([k]) = [k]$ and $\psi\theta$ is the identity since

$$\begin{array}{lccccccccccc}
\psi(k): & 0 & \longrightarrow & A & \longrightarrow & H_k & \longrightarrow & E_k & \longrightarrow & G & \longrightarrow & 1 \\
 & & & \| & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \Gamma} & & \| & & \\
\xi: & 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

commutes when $A(f_k(x, y)) = f(x, y)$ and $\Gamma(w, x) = (\overline{A(w)}, x)$. One need only verify that A is an E_k-map.

6. A group is 2-finite if every subgroup generated by two elements is finite.

Lemma 6.1. *If G is 2-finite and the orders of its elements are relatively prime to the order of the finite abelian group A, then every extension of A by G splits.*

Proof. Since A is abelian, every extension $0 \to A \to E \to G \to 1$ is special when A has the induced G-action. Let K be the kernel of $G \to \bar{G} \subseteq \operatorname{Aut}(A)$. Thus K acts on A trivially. Therefore, $H^1(K, A) = \operatorname{Der}(K, A) = \operatorname{Hom}(K, A) = 0$. We have the following exact sequence [5],

$$0 \to H^2(\bar{G}, A^K) \to H^2(G, A) \to H^2(K, A)\,.$$

Since $\mathrm{Sext}^1(G, A) \simeq H^2(G, A)$ by Theorem 4.2 and $H^2(\bar{G}, A^K) = 0$ by Corollary 1.2, it suffices to show that $H^2(K, A) = 0$. Since K acts on A trivially, every element of $H^2(K, A)$ is represented by a central extension of A by K.

$$0 \to A \to E' \to^{\pi} K \to 1 .$$

Because of the centrality of A in E', any splitting of π on a subgroup of K is unique. For each subgroup K' generated by 2 elements in K, $H^2(K', A) = 0$ by Corollary 1.2. Putting all the splittings on the subgroups K' together, we have a splitting of π.

Theorem 6.2. *If G is 2-finite and the orders of the elements of G are relatively prime to the order of the finite group H, then each extension $\xi : 1 \to H \to E \to G \to$ $\to 1$ splits.*

Proof. We induct on the order of H. View H as an E-group via conjugation. Suppose L is any proper E-subgroup of H. Then by induction the induced exact sequence

$$\bar{\xi} : 1 \to H/L \to \bar{E} \to G \to 1$$

splits. Hence we can find a normalized section σ of π with

$$f(x, y) = \sigma(x)\,\sigma(y)\,\sigma(xy)^{-1} \in L .$$

Viewing E as $H \times G$ with multiplication as in (5.1) we see $\tilde{E} = L \times G$ is a subgroup and by induction

$$\tilde{\xi} : 1 \to L \to \tilde{E} \to G \to 1$$

splits. Thus ξ also splits.

Thus we may assume H is E-simple and, by Lemma 6.1, non-abelian. Hence H cannot be a p-group and thus must contain a proper sylow group P. Let N be the normalizer of P in E. For $e \in E$, there is $h \in H$ with $ePe^{-1} = hPh^{-1}$ and thus $e \in$ $\in hN \subseteq HN$. Thus $HN = E$ and so

$$\xi' : 1 \to N \cap H \to N \to^{\pi'} G \to 1$$

is exact. Since P is not an E-subgroup of H it is non-normal in H. Thus $H \cap N \neq H$ and so by induction ξ' splits and so ξ splits.

Remark. Note the 2-finiteness condition on G is strictly weaker than the usual local finiteness. It is not clear that any less stringent finiteness condition on G will suffice. In particular Lemma 6.1 is false even when G is a periodic group with finite exponent prime to the finite order of A. For $m > 1$ Adyan [1] displays a non split exact sequence

$$0 \to Z/(2) \to \bar{A}(m, n) \to B(m, n) \to 1$$

where $B(m, n)$ is the universal m-generated group of odd exponent $n \geqq 4381$.

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