Ladislav Nebeský

On pancyclic line graphs

Czechoslovak Mathematical Journal, Vol. 28 (1978), No. 4, 650–655

Persistent URL: `http://dml.cz/dmlcz/101566`

Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

ON PANCYCLIC LINE GRAPHS

LADISLAV NEBESKÝ, Praha

(Received August 18, 1977)

By a graph we mean a graph in the sense of [1] or [4]. If G is a graph, then we denote by $V(G)$, $E(G)$, $\bar{G}$, and $L(G)$ its vertex set, edge set, complement, and line graph, respectively. The number of vertices of a graph is called its *order*. A graph containing a hamiltonian cycle is said to be *hamiltonian*.

In [6] the following theorem was proved:

Theorem 0. *Let G be a graph of order $p \geqq 5$. Then either* (a) *G is connected and $L(G)$ is hamiltonian, or* (b) *$\bar{G}$ is connected and $L(\bar{G})$ is hamiltonian.*

A graph isomorphic with its complement is said to be self-complementary. Note that if G is a self-complementary graph of order p, then $p \equiv 0$ or $1 \pmod 4$.

Corollary 0. *If G is a self-complementary graph of order $p \geqq 5$, then $L(G)$ is hamiltonian.*

In graph theory, various concepts stronger than that of a hamiltonian graph have been studied. Thus, it is natural to ask how Theorem 0 can be improved.

A graph G is called *strongly hamiltonian* if every edge of G belongs to a hamiltonian cycle of G (cf. [1], Chapter 11). A graph G is called *1-hamiltonian* if it is hamiltonian, and for each $u \in V(G)$, $G - u$ is also hamiltonian (cf. [3]). Unfortunately, if we substitute "strongly hamiltonian" or "1-hamiltonian" for "hamiltonian" in Theorem 0, the theorem does not hold for any $p \geqq 5$. Consider the infinite sequence of connected graphs in Fig. 1. The complements of these graphs are given in Fig. 2. It is not difficult to see that the line graph of any graph in Figs. 1 or 2 is neither strongly hamiltonian nor 1-hamiltonian.

A graph G of order $p \geqq 3$ is said to be pancyclic if G contains a cycle of length n for every integer n, $3 \leqq n \leqq p$ (cf. [2]). If G is a cycle of length five, then G, $\bar{G}$, and $L(G)$ are isomorphic, and so neither $L(G)$ nor $L(\bar{G})$ are pancyclic. Fortunately, the situation is different for $p \geqq 6$:

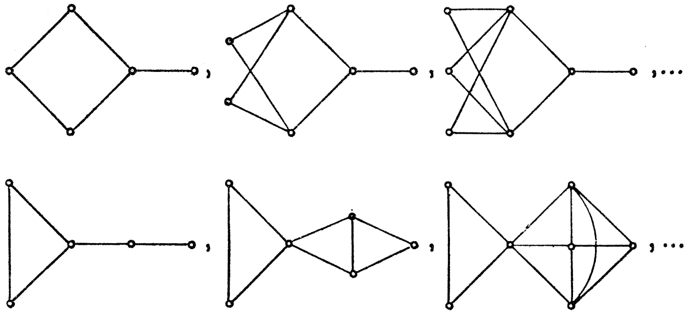

Fig. 1—2.

Theorem 1. *Let G be a graph of order $p \geqq 6$. Then either* (a) *G is connected and $L(G)$ is pancyclic, or* (b) *$\bar{G}$ is connected and $L(\bar{G})$ is pancyclic.*

A proof of this theorem is the main result of the present paper.

Following [6], we say that a graph F is an LH-subgraph of a graph G, if (i) F is either trivial or eulerian, (ii) F is a subgraph of G, and (iii) every edge of G is incident with a vertex of F. Harary and Nash-Williams [5] proved that if G is a connected graph with more than 2 edges, then $L(G)$ is hamiltonian if and only if G contains an LH-subgraph.

Proofs of the following three lemmas may be left to the reader:

Lemma 1. *Let G be a graph belonging to the infinite sequence of graphs in* Fig. 3 *or to the infinite sequence of graphs in* Fig. 4. *Then $L(G)$ is pancyclic.*

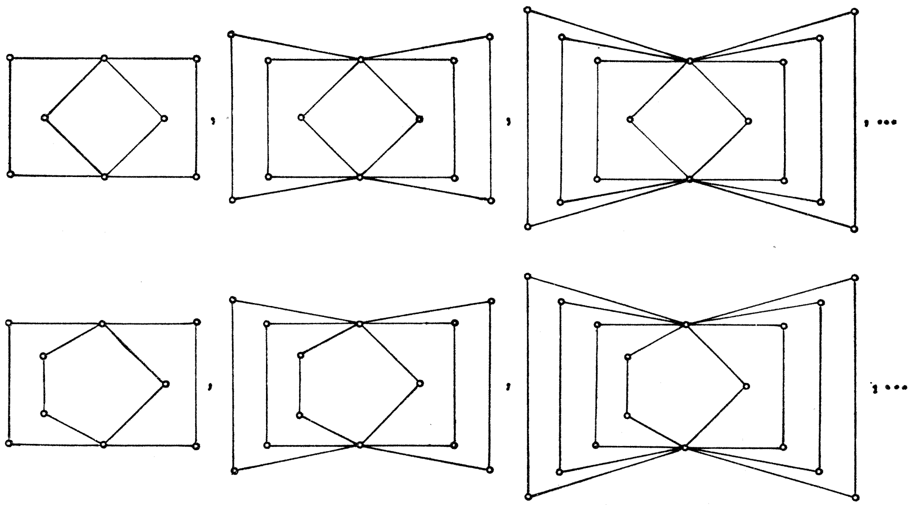

Fig. 3—4.

Lemma 2. *Let G_1 and G_2 be vertex-disjoint graphs, let G_1 be eulerian, and let G_2 be a triangle. Consider a graph G obtained from G_1 and G_2 by identifying one vertex of G_1 with one vertex of G_2. Then $L(G)$ contains a cycle of length n for $n = m$, $m + 1$, $m + 2$ and $m + 3$, where $m = |E(G_1)|$. If $L(G_1)$ is pancyclic, then $L(G)$ is also pancyclic.*

Lemma 3. *Let F be an LH-subgraph of a graph G. Then $L(G)$ contains a cycle of every length n for $|E(F)| \leqq n \leqq |E(G)|$. If $L(F)$ is pancyclic, then $L(G)$ is also pancyclic.*

Let G be a graph of order $p \geqq 5$. We shall say that G is an R-graph if there exist distinct $r, s, t, u \in V(G)$ such that $rs, st, tu, ur \in E(G)$ and, for every $v \in V(G - r - - s - t - u)$, either $rv \in E(G)$ or $tv \in E(G)$.

Lemma 4. *Let G be a graph of order $p \geqq 5$. If G is an R-graph, then $L(G)$ is pancyclic.*

Proof. Let G be an R-graph, and let r, s, t and u be the same as in the definition of an R-graph. For every edge of $G - r - t$ we shall define a certain subgraph of G.

Let $e \in E(G - r - t)$. Then there exist distinct $v, w \in V(G)$ such that $e = vw$. If $\{rv, rw\} \subseteq E(G)$, then we denote by $A(e)$ the triangle induced by $\{r, v, w\}$. If $\{rv, rw\} - - E(G) \neq \emptyset$ and $\{tv, tw\} \subseteq E(G)$, then we denote by $A(e)$ the triangle induced by $\{t, v, w\}$. Let $\{rv, rw\} - E(G) \neq \emptyset \neq \{tv, tw\} - E(G)$. Since G is an R-graph, there exists exactly one $r - t$ path P in G with the property that P contains exactly three edges, and one of these edges is vw This means that either $E(P) = \{rv, vw, wt\}$ or $E(P) = \{rw, wv, vt\}$. We denote P by $A(e)$.

Consider a matching M in $G - r - s - t - u$ which is maximal with respect to $\subseteq$. We denote by F_0 the subgraph of G induced by the set of edges

$$\{rs, st, tu, ur\} \cup \bigcup_{e \in M} E(A(e)) .$$

Clearly, every edge of G is incident with a vertex of F_0. It is obvious that there exists $j \geqq 2$ such that (i) exactly one block of F_0, say B, is homeomorphic to the complete bipartite graph $K(2, j)$, and (ii) if $F_0 \neq B$, then every block of F_0 different from B is a triangle.

We distinguish the following cases:

(I) Assume that j is even. Then F_0 is an LH-subgraph of G.

(IA) Assume that $j = 2$. Then B is a cycle of length four. Since $p \geqq 5$, $L(G)$ contains a triangle. If $F_0 = B$, then it is clear that $L(G)$ is pancyclic. If $F_0 \neq B$, then $L(F_0)$ is pancyclic, and therefore $L(G)$ is also pancyclic.

(IB) Assume that $j \neq 2$. Then $j \geqq 4$ and B is isomorphic to one of the graphs in the infinite sequence given in Fig. 3. Lemmas 1 and 2 imply that $L(F_0)$ is pancyclic. Hence, $L(G)$ is pancyclic.

(II) Assume that j is odd. Then $p \geqq 6$, and $L(G)$ contains a cycle of length 3 and a cycle of length 4. Obviously, M is a matching in $G - r - s - t$. If M is a matching in $G - r - s - t$ which is maximal with respect to $\subseteq$, then we denote the graph $F_0 - u$ by F. Otherwise, there exists $u' \in V(G - r - s - t - u)$ incident with no edge in M such that $uu' \in E(G)$, and we denote by F the graph obtained from $F_0 - u$ by adding the triangle $A(uu')$. It is easy to see that F is an LH-subgraph of G. Obviously, F contains the block $B - u$.

(IIA) Assume that $B - u$ is a cycle. Then it is a cycle of length 5. It is clear that $L(G)$ is pancyclic.

(IIB) Assume that $B - u$ is not a cycle. Then $B - u$ is isomorphic to one of the graphs in the infinite sequence given in Fig. 4. Hence, $L(B - u)$ is pancyclic. This implies that $L(F)$ is pancyclic, and therefore $L(G)$ is pancyclic, which completes the proof.

Proof of Theorem 1. (I) Assume that $p = 6$. (Note that the list of graphs of order six is to be found in [4], Appendix 1.) Since the complete graph of order six has exactly 15 edges, without loss of generality we shall assume that $|E(G)| \geqq 8$.

(IA) Assume that G is disconnected. Then G consists of exactly two components, and one of them is trivial. Hence, $\bar{G}$ contains a spanning star and $5 \leqq |E(\bar{G})| \leqq 7$. Clearly, $\bar{G}$ is connected and $L(\bar{G})$ is pancyclic.

(IB) Assume that G is connected.

(IB1) Assume that G consists of at least three blocks. Then G consists of two acyclic blocks and one complete block of order four. This means that either G or $\bar{G}$ is an R-graph. According to Lemma 4, either $L(G)$ or $L(\bar{G})$ is pancyclic.

(IB2) Assume that G consists of two blocks. If one of the blocks is a triangle, then G contains a spanning eulerian subgraph; otherwise, there is a cycle of length four which is an LH-subgraph of G. Clearly, $L(G)$ is pancyclic.

(IB3) Assume that G is 2-connected. Since $p = 6$, it is not difficult to see that there exists a cycle which is an LH-subgraph of G. Therefore, $L(G)$ contains a cycle of length n for $n = 6, \dots$ and $|E(G)|$.

(IB3a) Assume that G contains a triangle, say T. Since G is 2-connected, T contains at least two vertices of degree $\geqq 3$ in G. Hence, $L(G)$ contains a cycle of length n for $n = 3, 4$ and 5. It is obvious that $L(G)$ is pancyclic.

(IB3b) Assume that G does not contain any triangle. It is easy to see that G contains a cycle C of length four. Since C contains a vertex of degree $\geqq 3$ in G, we have that $L(G)$ is pancyclic.

(II) Assume that $p = p_0 \geqq 7$, and that the theorem is proved for $p = p_0 - 1$. The case when either G or $\bar{G}$ is complete is obvious. Assume that neither G nor $\bar{G}$ are complete. Then there exists $r \in V(G)$ such that $1 \leqq d \leqq p - 2$ and $1 \leqq \bar{d} \leqq p - 2$, where d or $\bar{d}$ denotes the degree of r in G or in $\bar{G}$, respectively. By the induction assumption, at least one of the graphs $G - r$ and $\overline{G - r}$, say G_1, is connected and $L(G_1)$ is pancyclic. Since $\overline{G - r}$ is identical with $\bar{G} - r$, we shall assume without loss of generality that G_1 is identical with $G - r$. Thus $G - r$ is connected and $L(G - r)$ is pancyclic. This means that $L(G)$ contains a cycle of length n for $n = 3, \ldots$ and $|E(G - r)|$. Since $L(G - r)$ is hamiltonian, there exists an LH-subgraph of $G - r$. Consider such an LH-subgraph F of $G - r$ which has the maximum order among the LH-subgraphs of $G - r$.

Since $d \geqq 1$, G is connected. If $L(G)$ is pancyclic, then the theorem is proved. We shall assume that $L(G)$ is not pancyclic.

Assume that F is an LH-subgraph of G. According to Lemma 3, $L(G)$ contains a cycle of every length n for $|E(F)| \leqq n \leqq |E(G)|$. Since $|E(F)| \leqq |E(G - r)|$, we have that $L(G)$ is pancyclic, which is a contradiction. Therefore, F is not an LH-subgraph of G. This implies that there exists $t \in V(G - r)$ such that $rt \in E(G)$ and $t \notin E(F)$.

(IIA) Assume that F is trivial. Then $G - r$ is a star. It is clear that if $d \geqq 2$, then $L(G)$ is pancyclic, which is a contradiction. If $d = 1$, then $\bar{G}$ is an R-graph, and thus $\bar{G}$ is connected and – according to Lemma 4 – $L(\bar{G})$ is pancyclic.

(IIB) Assume that F is nontrivial. Then F is eulerian. We denote by F_r or F_t or F_{rt} the graph obtained from F by adding the vertex r or the vertex t or the vertices r and t, respectively.

Consider an arbitrary vertex v of F. Assume that $rv, tv \in E(G)$. Then $F_{rt} + rv + tv + rt$ is an LH-subgraph of G. Lemmas 2 and 3 imply that $L(G)$ is pancyclic, which is a contradiction. Thus either $rv \in E(\bar{G})$ or $tv \in E(\bar{G})$.

Consider an arbitrary edge ww' of F. If $tw, tw' \in E(G)$, then $F_t - ww' + tw + tw'$ is an LH-subgraph of $G - r$ with more than $|V(F)|$ vertices; a contradiction. Assume that either $rw, rw' \in E(G)$, or $rw, tw' \in E(G)$, or $rw', tw \in E(G)$. Then either $F_r - ww' + rw + rw'$, or $F_{rt} - ww' + rw + tw'$, or $F_{rt} - ww' + rw' + tw$, respectively, is an LH-subgraph of G. Hence, $L(G)$ contains a cycle of every length n, for $|E(F)| \leqq n \leqq |E(G)|$. This implies that $L(G)$ is pancyclic, which is a contradiction. Thus we have that either $rw, tw \in E(\bar{G})$ or $rw', tw' \in E(\bar{G})$.

Since F contains a cycle, there exist distinct $s, u \in V(F)$ such that $rs, ru, ts, tu \in E(\bar{G})$.

Let $v \in V(G - r - s - t - u)$. If $v \in V(F)$, then – as we have proved – either $rv \in E(\bar{G})$ or $tv \in E(\bar{G})$. If $v \notin V(F)$, then from the fact that F is an LH-subgraph of $G - r$ it follows that $tv \in E(\bar{G})$. Thus we have that $\bar{G}$ is an R-graph. Hence, $\bar{G}$ is connected. According to Lemma 4, $L(\bar{G})$ is pancyclic, which completes the proof.

In graph theory concepts stronger than that of a pancyclic graph have been studied, first of all the concept of a vertex-pancyclic graph. A graph G of order $p \geqq 3$ is called vertex-pancyclic if for every $u \in V(G)$ and every n, $3 \leqq n \leqq p$, there exists a cycle of length n in G which contains u (cf. [2]). However, if we substitute "vertex-pancyclic" for "pancyclic" in Theorem 1, the theorem does not hold. Consider the infinite sequence of graphs in Fig. 5. The complements of these graphs are given in Fig. 6. It is easy to see that the line graph of none of the graphs in Figs. 5 or 6 is vertex-pancyclic.

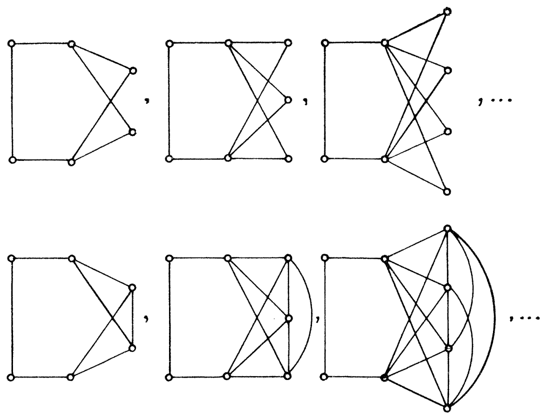

Fig. 5—6.

For $p \geqq 8$, Corollary 0 can be improved as follows:

Corollary 1. *If G is a self-complementary graph of order $p \geqq 8$, then $L(G)$ is pancyclic.*

In [2] BONDY stated the metaconjecture that almost every nontrivial sufficient condition for a graph to be hamiltonian is also a sufficient condition for a graph to be pancyclic (a simple family of exceptional graphs is allowed). Corollaries 0 and 1 represent one of partial confirmations of Bondy's metaconjecture.

References

[1] *M. Behzad* and *G. Chartrand:* Introduction to the Theory of Graphs. Allyn and Bacon, Boston 1971.

[2] *J. A. Bondy:* Pancyclic Graphs. Congressus Numeratium III (Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing, eds. R. C. Mullin, K. B. Reid, D. P. Roselle and R. S. D. Thomas), Utilitas Mathematica Publishing Inc., Winnipeg 1972, pp. 167—172.

[3] *G. Chartrand, S. F. Kapoor* and *D. R. Lick:* n-Hamiltonian graphs. J. Combinatorial Theory *9* (1970), 308—312.

[4] *F. Harary:* Graph Theory. Addison-Wesley, Reading (Mass.) 1969.

[5] *F. Harary* and *C. St. J. A. Nash-Williams:* On eulerian and hamiltonian graphs and line graphs. Canadian Math. Bull. *8* (1965), 701—709.

[6] *L. Nebeský:* A theorem on hamiltonian line graphs. Comment. Math. Univ. Carolinae *14* (1973), 107—112.

Author's address: 116 38 Praha 1, nám. Krasnoarmějců 2, ČSSR (Filozofická fakulta Karlovy univerzity).