Alois Švec

An integral formula for non-Codazzi tensors

Czechoslovak Mathematical Journal, Vol. 28 (1978), No. 3, 434–438

Persistent URL: `http://dml.cz/dmlcz/101548`

Czechoslovak Mathematical Journal, 28 (103) 1978, Praha

AN INTEGRAL FORMULA FOR NON-CODAZZI TENSORS

ALOIS ŠVEC, Olomouc

(Received September 22, 1976)

The purpose of the following paper is to prove an integral formula for quadratic differential forms on an orientable Riemannian manifold M. Let $Q = a_{ij}\,du^i\,du^j$, $a_{ij} = a_{ji}$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a_{ij} (at $m \in M$), $w_1, \ldots, w_n$ the corresponding orthonormal eigenvectors and K_{ij} the sectional curvature corresponding to $\{w_i, w_j\}$. Our integral formula reads

$$(*) \qquad \int_{\partial M} * \left(a^{\alpha}_{.\beta}\,\nabla_\gamma a^{\gamma}_{.\alpha} - a^{\alpha\gamma}\,\nabla_\gamma a_{\alpha\beta}\right) du^\beta =$$

$$= \int_M \{\nabla_\beta a^{\alpha\beta}\,\nabla_\gamma a^{\gamma}_{.\alpha} - \nabla^\gamma a^{\alpha\beta}\,\nabla_\beta a_{\alpha\gamma} - \sum_{\alpha<\beta} (\lambda_\alpha - \lambda_\beta)^2 K_{\alpha\beta}\}\, do\,.$$

For a Codazzi tensor $a_{ij} = a_{ji}$ satisfying $\nabla_k a_{ij} = \nabla_i a_{kj}$ we get the known formula; see [1].

1. Let M be a connected orientable n-dimensional Riemannian manifold, its metric being ds^2. In a suitable domain $U \subset M$, let us consider a field of coframes $(\omega^1, \ldots, \omega^n)$ such that

$$(1) \qquad ds^2 = \delta_{\alpha\beta}\omega^\alpha\omega^\beta\,;$$

throughout the paper, we are going to use the summation convention and $i, j, \ldots$ $\ldots, \alpha, \beta, \ldots = 1, \ldots, n$. Then there is, in U, a unique field of matrices of 1-forms $\|\omega_i^j\|$ such that

$$(2) \qquad d\omega^i = \omega^\alpha \wedge \omega^i_\alpha\,, \quad \delta_{i\alpha}\omega^\alpha_j + \delta_{j\alpha}\omega^\alpha_i = 0\,.$$

Indeed, let us write

$$(3) \qquad d\omega^i = A^i_{\alpha\beta}\omega^\alpha \wedge \omega^\beta\,, \quad A^i_{jk} + A^i_{kj} = 0\,.$$

Suppose the existence of matrices $\|\omega_i^j\|$ satisfying (2). From (2_1) and (3).

$$\omega^\alpha \wedge (\omega_\alpha^i - A^i_{\alpha\beta}\omega^\beta) = 0\,, \tag{4}$$

and we get the existence of functions B^i_{jk} such that

$$\omega_i^j = A^j_{i\alpha}\omega^\alpha + B^j_{i\alpha}\omega^\alpha\,, \quad B^i_{jk} - B^i_{kj} = 0\,. \tag{5}$$

Substituting into (2_2), we get

$$\delta_{i\alpha}A^\alpha_{jk} + \delta_{j\alpha}A^\alpha_{ik} + \delta_{i\alpha}B^\alpha_{jk} + \delta_{j\alpha}B^\alpha_{ik} = 0\,, \tag{6}$$

and permutations of indices lead to

$$\begin{gathered}\delta_{i\alpha}A^\alpha_{kj} + \delta_{k\alpha}A^\alpha_{ij} + \delta_{i\alpha}B^\alpha_{kj} + \delta_{k\alpha}B^\alpha_{ij} = 0\,,\\ \delta_{k\alpha}A^\alpha_{ji} + \delta_{j\alpha}A^\alpha_{ki} + \delta_{k\alpha}B^\alpha_{ji} + \delta_{j\alpha}B^\alpha_{ki} = 0\,.\end{gathered} \tag{7}$$

Subtracting (7) from (6), we have

$$B^i_{jk} = \delta^{\alpha i}\delta_{\beta j}A^\beta_{k\alpha} + \delta^{\alpha i}\delta_{\beta k}A^\beta_{j\alpha}\,, \tag{8}$$

i.e.,

$$\omega_i^j = (A^j_{i\gamma} + \delta^{\alpha j}\delta_{\beta i}A^\beta_{\gamma\alpha} + \delta^{\alpha j}\delta_{\beta\gamma}A^\beta_{j\alpha})\,\omega^\gamma\,. \tag{9}$$

Now, it is easy to see that $\|\omega_i^j\|$ (see (9)) satisfy (2).

The curvature tensor R^j_{ikl} of M is defined by the relations

$$\mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j - \tfrac{1}{2}R^j_{ikl}\omega^k \wedge \omega^l, \quad R^j_{ikl} + R^j_{ilk} = 0\,; \tag{10}$$

it satisfies the well known symmetry relations

$$\begin{gathered}\delta_{i\alpha}R^\alpha_{jkl} + \delta_{j\alpha}R^\alpha_{ikl} = 0\,, \quad \delta_{j\alpha}R^\alpha_{ikl} = \delta_{l\alpha}R^\alpha_{kij}\,,\\ \delta_{j\alpha}R^\alpha_{ikl} + \delta_{k\alpha}R^\alpha_{ilj} + \delta_{l\alpha}R^\alpha_{ijk} = 0\,.\end{gathered} \tag{11}$$

Let us change our coframes $(\omega^1, \ldots, \omega^n)$; let the new ones be $(\tau^1, \ldots, \tau^n)$, and let

$$\mathrm{d}s^2 = \delta_{\alpha\beta}\tau^\alpha\tau^\beta\,, \tag{12}$$

$$\omega^i = R^i_\alpha\tau^\alpha\,. \tag{13}$$

The matrix $\|R_i^j\|$ is orthogonal, i.e.,

$$\delta_{\alpha\beta}R^\alpha_i R^\beta_j = \delta_{ij}\,. \tag{14}$$

Denote by $\|\tilde{R}_i^j\|$ the inverse matrix to $\|R_i^j\|$, i.e., let

$$R^\alpha_i\tilde{R}^j_\alpha = \tilde{R}^\alpha_i R^j_\alpha = \delta^j_i\,. \tag{15}$$

Let the matrix $\|\tau_i^j\|$ be associated with the coframes $(\tau^1, \ldots, \tau^n)$, i.e., let τ_i^j satisfy

(16) $$d\tau^i = \tau^\alpha \wedge \tau_\alpha^i, \quad \delta_{i\alpha}\tau_j^\alpha + \delta_{j\alpha}\tau_i^\alpha = 0.$$

Then we have the following assertion: *It is*

(17) $$\tau_i^j = \tilde{R}_\alpha^j \, dR_i^\alpha + \tilde{R}_\alpha^j R_i^\beta \omega_\beta^\alpha.$$

The proof is obvious.

2. On M, let a quadratic differential form Q be given; let us restrict our considerations to the domain U. By means of the coframes $(\omega^1, \ldots, \omega^n)$ or $(\tau^1, \ldots, \tau^n)$, respectively, Q may be written as

(18) $$Q = a_{\alpha\beta}\omega^\alpha\omega^\beta = \tilde{a}_{\alpha\beta}\tau^\alpha\tau^\beta; \quad a_{ij} = a_{ji}, \quad \tilde{a}_{ij} = \tilde{a}_{ji}.$$

Hence

(19) $$\tilde{a}_{ij} = a_{\alpha\beta}R_i^\alpha R_j^\beta.$$

The covariant derivatives $b_{ijk} = b_{jik}$ of the tensor a_{ij} with respect to the coframes $(\omega^1, \ldots, \omega^n)$ let be defined by

(20) $$da_{ij} - a_{i\alpha}\omega_j^\alpha - a_{\alpha j}\omega_i^\alpha = b_{ij\alpha}\omega^\alpha.$$

Substituting into the analogous equations

(21) $$d\tilde{a}_{ij} - \tilde{a}_{i\alpha}\tau_j^\alpha - \tilde{a}_{\alpha j}\tau_i^\alpha = \tilde{b}_{ij\alpha}\tau^\alpha,$$

we get

(22) $$\tilde{b}_{ijk} = b_{\alpha\beta\gamma}R_i^\alpha R_j^\beta R_k^\gamma,$$

and b_{ijk} are components of a tensor.

The exterior differentiation of (20) yields

(23) $$(db_{ij\beta} - b_{\alpha j\beta}\omega_i^\alpha - b_{i\alpha\beta}\omega_j^\alpha - b_{ij\alpha}\omega_\beta^\alpha) \wedge \omega^\beta = \\ = \tfrac{1}{2}(a_{i\alpha}R_{j\beta\gamma}^\alpha + a_{\alpha j}R_{i\beta\gamma}^\alpha)\,\omega^\beta \wedge \omega^\gamma$$

as well as the existence of functions $c_{ijkl} = c_{jikl}$ such that

(24) $$db_{ijk} - b_{\alpha jk}\omega_i^\alpha - b_{i\alpha k}\omega_j^\alpha - b_{ij\alpha}\omega_k^\alpha = c_{ijk\alpha}\omega^\alpha,$$

(25) $$c_{ijkl} - c_{ijlk} = -a_{i\alpha}R_{jkl}^\alpha - a_{\alpha j}R_{ikl}^\alpha.$$

On U, define the 1-forms

(26) $$\tau_1 = \delta^{\alpha\beta}\delta^{\gamma\delta}a_{\alpha\varepsilon}b_{\beta\gamma\delta}\omega^\varepsilon, \quad \tau_2 = \delta^{\alpha\beta}\delta^{\gamma\delta}a_{\alpha\gamma}b_{\beta\varepsilon\delta}\omega^\varepsilon.$$

Because of (19) and (22), *the forms* τ_1 *and* τ_2 *are globally defined over all* M.

Let the well known $*$-operator be given by

(27) $$*\omega^i = (-1)^{i+1}\,\omega^1 \wedge \ldots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \ldots \wedge \omega^n ,$$

$$\text{i.e.,}\quad \omega^i \wedge *\omega^i = \omega^1 \wedge \ldots \wedge \omega^n =: \mathrm{d}o .$$

We easily obtain

(28) $$\mathrm{d} * \tau_1 = \delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\varepsilon\varphi}\big(b_{\alpha\varepsilon\varphi}b_{\beta\gamma\delta} + a_{\alpha\varepsilon}c_{\beta\gamma\delta\varphi}\big)\,\mathrm{d}o ,$$

$$\mathrm{d} * \tau_2 = \delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\varepsilon\varphi}\big(b_{\alpha\gamma\varepsilon}b_{\beta\varphi\delta} + a_{\alpha\gamma}c_{\beta\varepsilon\delta\varphi}\big)\,\mathrm{d}o .$$

Using (25) we have

(29) $$\mathrm{d} * (\tau_1 - \tau_2) = \{\delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\varepsilon\varphi}\big(b_{\alpha\varepsilon\varphi}b_{\beta\gamma\delta} - b_{\alpha\gamma\varepsilon}b_{\beta\varphi\delta}\big) -$$

$$- \delta^{\alpha\beta}a_{\gamma\alpha}a_{\delta\varepsilon}\big(\delta^{\gamma\delta}\delta^{\varphi\psi} + \delta^{\gamma\varphi}\delta^{\delta\psi}\big)\,R^{\varepsilon}_{\varphi\psi\beta}\}\,\mathrm{d}o .$$

Further,

(30) $$\delta^{\alpha\beta}a_{\gamma\alpha}a_{\delta\varepsilon}\big(\delta^{\gamma\delta}\delta^{\varphi\psi} + \delta^{\gamma\varphi}\delta^{\delta\psi}\big)\,R^{\varepsilon}_{\varphi\psi\beta} =$$

$$= \sum_{\alpha}(a_{\alpha\alpha})^2 \cdot \sum_{\beta\neq\alpha} R^{\alpha}_{\beta\beta\alpha} - 2\sum_{\alpha\neq\beta} a_{\alpha\alpha}a_{\beta\beta}R^{\beta}_{\alpha\alpha\beta} + \sum_{\gamma\neq\alpha}\ \sum_{\delta\neq\varepsilon}\ \sum_{\beta,\varphi,\psi} a_{\gamma\alpha}a_{\delta\varepsilon}\delta^{\alpha\beta}\big(\delta^{\gamma\delta}\delta^{\varphi\psi} + \delta^{\gamma\varphi}\delta^{\delta\psi}\big)\,R^{\varepsilon}_{\varphi\psi\beta} ;$$

(31) $$\tau_1 = \sum_{\alpha,\beta} a_{\alpha\alpha}b_{\alpha\beta\beta}\omega^{\alpha} + \sum_{\alpha\neq\varepsilon}\ \sum_{\beta,\gamma,\delta} a_{\alpha\varepsilon}\delta^{\alpha\beta}\delta^{\gamma\delta}b_{\beta\gamma\delta}\omega^{\varepsilon} ,$$

$$\tau_2 = \sum_{\alpha,\beta} a_{\beta\beta}b_{\alpha\beta\beta}\omega^{\alpha} + \sum_{\alpha\neq\gamma}\ \sum_{\beta,\delta,\varepsilon} a_{\alpha\gamma}\delta^{\alpha\beta}\delta^{\gamma\delta}b_{\beta\varepsilon\delta}\omega^{\varepsilon} .$$

Let $m \in M$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of Q at m. Then the point m is called an *umbilical point of* Q if $\lambda_1 = \ldots = \lambda_n$.

Theorem. *Let M be an oriented Riemannian manifold and Q a quadratic differential form on M. Let us suppose:* (i) *all points of the boundary ∂M of M are umbilical points of Q;* (ii) *all sectional curvatures of M are positive;* (iii) *for the invariant*

(32) $$B := \delta^{\alpha\beta}\delta^{\gamma\delta}\delta^{\varepsilon\varphi}\big(b_{\alpha\varepsilon\varphi}b_{\beta\gamma\delta} - b_{\alpha\gamma\varepsilon}b_{\beta\varphi\delta}\big)$$

of Q, we have

(33) $$B \leqq 0$$

on M. Then all points of M are umbilical for Q.

Proof. Because of (29)–(31), we have the integral formula

(34) $$0 = \int_{\partial M} * (\tau_1 - \tau_2) = \int_M \{B - \sum_{\alpha<\beta}(\lambda_\alpha - \lambda_\beta)^2\,K_{\alpha\beta}\}\,\mathrm{d}o .$$

Here $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Q at m, $w_1, \ldots, w_n$ the corresponding orthonormal eigenvectors and $K_{\alpha\beta}$ the sectional curvature corresponding to $\{w_\alpha, w_\beta\}$.

Q.E.D.

Bibliography

[1] *U. Simon:* A further method in global differential geometry. Abh. Math. Sem. Hamburg, *44*, 1975, 52—69.

Author's address: 771 46 Olomouc, Leninova 26, ČSSR (Přírodovědecká fakulta UP).