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KINEMATIC GEOMETRY OF REGULAR MOTIONS IN HOMOGENEOUS SPACE

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1. INTRODUCTION

The present paper is devoted to the study of one-parametric motions in a homogeneous space G/H. The class of all Lie groups G in consideration is given by explicit conditions and includes all real semisimple Lie groups. It contains also some non-semisimple Lie groups, which are of interest in kinematic geometry (for example the Lie groups of all euclidean or affine transformations of n-dimensional affine space).

In part two we show that the problem of finding all invariants of a motion is equivalent to the problem of finding all invariants of a curve in a certain homogeneous space — we get a generalisation of centroids and axoids known in classical kinematics. Third part gives explicit formulas for compact and complex semisimple Lie groups. The case of euclidean motion in E_n was treated by the author in [5].

The last part is a realisation of the described method for unitary motions. The unitary group has been chosen because it has not been treated yet and so it can demonstrate, that our method can be used without knowing the classical results. The paper was partially written during my stay at the University of Kuwait.

2. MOTION IN A HOMOGENEOUS SPACE

Let G/H be a homogeneous space with elements represented as left cosets γH for $\gamma \in G$. A one-parametric motion in G/H is an immersion i of an open interval I of the real line in the group G. The curve $i(I)\,\gamma H$ in G/H is called *the trajectory of the point* $\gamma H \in G/H$. Kinematic geometry studies the system of trajectories of all points of G/H regardless of the parametrization of the curve $i(I)$. We shall write $g(t)$ for $i(I)$ without paying special attention to parameter changes of the curve $i(I)$ because we usually can choose a canonical parameter for it. Let further $\bar{\Omega}$ resp. Ω be the left resp. right canonical form on G. This by definition means that $\bar{\Omega}(X) = L^*_{g^{-1}}X$ resp. $\Omega(X) = R^*_{g^{-1}}X$ for $X \in T_g(G)$. L resp. R is the left resp. right translation in the group G.

Let now g_1, g_2 be fixed elements in G. Then the motion $g_1\, g(t)\, g_2^{-1}$ has the same trajectories as $g(t)$ has because $g_1\, g(t)\, g_2^{-1}(\gamma H) = g_1\, g(t)\, (g_2^{-1}\gamma H)$ and so the difference is only up to transformations from G. From this we can see that to find invariants of motion in G/H means to solve the equivalence problem for a curve in G with respect to the group $G \times G$ acting on G by the rule $(g_1, g_2)\, g = g_1 g g_2^{-1}$ for any $g_1, g_2, g \in G$. The isotropy group of e is the diagonal G' of $G \times G$, consisting of elements of the form (g, g) for all $g \in G$. We then regard G as the homogeneous space $G \times G/G'$. Elements of $G \times G/G'$ are of the form $(g_1, g_2)\, G'$, the multiplication is defined by the direct product structure and we write it in the form $(g_1, g_2)(g_3, g_4) = (g_1 g_3, g_2 g_4)$ for any $g_1, g_2, g_3, g_4 \in G$. If $(g_1 h, g_2 h) \in (g_1, g_2)\, G'$, then the element $g_1 g_2^{-1} = (g_1 h)(g_2 h)^{-1}$ does not depend on the choice of h and so we can identify $(g_1, g_2)\, G'$ with the element $g_1 g_2^{-1} \in G$.

Let $\mathfrak{g} \times \mathfrak{g}$ be the Lie algebra of $G \times G$. Vectors of the form $(X, X) \in \mathfrak{g} \times \mathfrak{g}$ form the Lie algebra of G' and the subspace m of $\mathfrak{g} \times \mathfrak{g}$ formed by elements of the form $(X, -X)$ is invariant by the group ad G'. So $G \times G/G'$ has the natural structure of a reductive homogeneous space, we write $\mathfrak{g} \times \mathfrak{g} = \mathfrak{g}' + m$. We can further identify m with $\mathfrak{g}$ if we let $(X, -X) \in m$ coincide with $X \in \mathfrak{g}$.

Finally we write $\bar{R}(t) = \bar{\Omega}(g'(t))$ resp. $R(t) = \Omega(g'(t))$ for the moving resp. fixed directing cone of the motion $g(t)$. $g'(t)$ denotes the tangent vector $i_t^*(\partial/\partial t)$ of $g(t)$ at the point $g(t)$.

Let us suppose from now on that $\mathfrak{g}$ has commutative Cartan subalgebras, fix one of them and denote it h'. So $[h', h'] = 0$ and $N(h') = h'$ where N means the normalizer. An element X of $\mathfrak{g}$ will be called *regular* if $\dim Z(X) = \dim h'$, here Z means the centralizer of X in $\mathfrak{g}$. We shall call a motion regular (with respect to h') if $R(t)$ is in the orbit of h' under ad G and consists of regular elements only. (If $R(t)$ is regular and in the orbit of h' under ad G, so is $\bar{R}(t)$ because $R(t) = \operatorname{ad} g(t)\, \bar{R}(t)$.)

Let a regular motion $g(t)$ be given and let $(g_1(t), g_2(t))$ be its arbitrary lift into $G \times G$. Then $g_1(t)\, g_2^{-1}(t) = g(t)$. For the canonical (left invariant) form (ω_1, ω_2) on the lift $(g_1, g_2) \in G \times G$ we have

$$(1) \qquad \omega_1 = g_1^{-1}\, \mathrm{d}g_1\,, \quad \omega_2 = g_2^{-1}\, \mathrm{d}g_2\,.$$

The form (ω_1, ω_2) can be written as a sum of two forms

$$(\omega_1, \omega_2) = \left(\frac{\omega_1 + \omega_2}{2}, \frac{\omega_1 + \omega_2}{2}\right) + \left(\frac{\omega_1 - \omega_2}{2}, \frac{\omega_2 - \omega_1}{2}\right)$$

with the second part of the sum in m. From the definition of the lift (g_1, g_2) we get the following formula

$$(2) \qquad R\, \mathrm{d}t = \operatorname{ad} g_1(\omega_1 - \omega_2)\,, \quad \bar{R}\, \mathrm{d}t = \operatorname{ad} g_2(\omega_1 - \omega_2)\,.$$

So $\omega_1 - \omega_2$ is a regular element in the orbit of h'.

If we choose another lift of $g(t)$, say $(g_1\gamma, g_2\gamma)$, we get for the m component $\tilde{\omega}_1 - \tilde{\omega}_2$ of the canonical form for the new lift $\tilde{\omega}_1 - \tilde{\omega}_2 = \operatorname{ad} \gamma^{-1}(\omega_1 - \omega_2)$. So we can choose such a lift (g_1, g_2) of $g(t)$, that $\omega_1 - \omega_2 \in h'$.

We denote now by $\mathscr{H}$ the normalizer of h' in G and by $\mathscr{Z}(X)$ the centralizer of $X \in h'$ in G. (So $\mathscr{H} = \{g \in G \mid \text{ad } h' = h'\}$, $\mathscr{Z}(X) = \{g \in G \mid \text{ad } gX = X\}$). From our assumptions we see that $\mathscr{H}$ and $\mathscr{Z}(\omega_1 - \omega_2)$ have the same Lie algebra, because $N(h') = h' \subseteq \mathscr{Z}(\omega_1 - \omega_2)$ and $\omega_1 - \omega_2$ is regular. This means that we have got tangent lift for $g(t)$ and invariants of the first order. We shall suppose now that we have a lift $(g_1(t), g_2(t))$ of $g(t)$ such that $\omega_1 - \omega_2 \in h'$ for it. This lift depends on the group $\mathscr{H}$. So we can consider the homogeneous space $G/\mathscr{H}$ and we get two curves in it, πg_1 and πg_2, where π is the natural projection from G to $G/\mathscr{H}$. The curve πg_2 is called *the moving centroid* (or *poloid*) of the motion, πg_1 is called *the fixed centroid* (or *poloid*) of the motion $g(t)$. If we denote by $h_2(t)$ resp. $h_1(t)$ the isotropy algebra of $\pi g_2(t)$ resp. $\pi g_1(t)$ in $G/\mathscr{H}$, we get the following

Theorem 1. *Denote $p_2 = \pi g_2$, $p_1 = \pi g_1$. Then*

$$g(t)\, p_2(t) = p_1(t)\,, \quad \bar{R}(t) \in h_2(t)\,, \quad R(t) \in h_1(t)\,, \quad \text{ad } g(t)\, h_2(t) = h_1(t)\,. \tag{3}$$

Let us further suppose that $\mathscr{H}$ acting as a transformation group on h' has a fundamental domain for all regular elements (see [7]) and that the isotropy group of all elements $\omega_1 - \omega_2 \in h$ is the same, denote it by $\mathscr{H}_1$ (all regular elements in h' have the same isotropy algebra h'). In this case we can suppose, that $\omega_1 - \omega_2$ is in the fundamental domain and the lift (g_1, g_2) is then given up to elements in $\mathscr{H}_1$.

Let now e_i, f_α be a base for g such that elements e_i form a base for h'. Then we can write

$$\omega_1 = \omega_1^i e_i + \eta_1^\alpha f_\alpha\,, \quad \omega_2 = \omega_2^i e_i + \eta_2^\alpha f_\alpha\,, \quad i = 1, \ldots, n\,, \quad \alpha = 1, \ldots, m\,, \tag{4}$$

where $\omega_1 - \omega_2 = (\omega_1^i - \omega_2^i)\, e_i + (\eta_1^\alpha - \eta_2^\alpha) f_\alpha$, so for lifts of the first order we have $\eta_1^\alpha = \eta_2^\alpha$.

The remaining isotropy group is now the diagonal of $\mathscr{H}_1 \times \mathscr{H}_1$, a subgroup of G'. So the main components for (ω_1, ω_2) are $\eta_1^\alpha = \eta_2^\alpha$ and $\omega_1^i - \omega_2^i$, secondary components are $\omega_1^i + \omega_2^i$ (in the sense of Cartan's moving frame). From this we see, that to find Frenet's lift for the motion we must find the canonical form of the element $Y = \eta_1^\alpha f_\alpha \in \mathfrak{g}/h'$. This means, that in order to find invariants of the second order of a motion it is enough to find invariants of the first order of a curve in a homogeneous space $G/\mathscr{H}_1$, invariants of the moving and fixed centroid.

If we denote by $v^i = \omega_1^i - \omega_2^i$ the invariants of the first order of the motion, we get formulas connecting invariants of motion with invariants of poloids by formulas

$$\eta_1^\alpha = \eta_2^\alpha\,, \quad v^i = \omega_1^i - \omega_2^i\,. \tag{5}$$

Choosing now a suitable canonical parameter t for the motion and the canonical parameter s for poloids $p_2(s)$, $p_1(s)$, we can write explicit formulas connecting all invariants.

For illustration consider a special case of a one-parametric group $g(t) = \exp tX$ as a motion. If X is regular, we can suppose that $X \in h'$. Taking now an arbitrary lift of $g(t)$, we see, that $g(t)$ has only invariants of the first order v^i and the rest is zero or is not defined (there is no canonical lift in general). This leads to the definition of the instantaneous motion of the given motion $g(t)$ at the point t_0, it is the motion $\gamma(u) = \exp u\, R(t_0) \,.\, g(t_0)$. Directing cones of the instantaneous motion $\gamma(u)$ are $\bar{R}(t_0)$ and $R(t_0)$ respectively and its invariants are numbers $v^i(t_0)$. Trajectories of the instantaneous motion are integral curves of the field of tangent vectors of trajectories of points in any homogeneous space G/K and $v^i(t_0)$ characterize this field. This gives the geometric interpretation of invariants v^i.

A motion is called *harmonic* (according to J. Tölke, see [1]), if invariants v^i are constant during the whole motion. It means geometrically that all instantaneous motions of the given motion are equivalent.

The special case of cyclic motion is also interesting. A motion $g(t)$ is called *cyclic*, if it is product of two one-parametric groups, so $g(t) = \exp(tX)\exp(-tY)$ for some $X, Y \in \mathfrak{g}$ in this case. We see easily, that $(\exp tX, \exp tY)$ is a lift of this cyclic motion and its canonical form is $(\omega_1, \omega_2) = (X\,\mathrm{d}t, Y\,\mathrm{d}t)$. So we have

Theorem 2. *Invariants of a cyclic motion are constant.*

We can also prove the converse, but in the general case only.

Theorem 3. *Let numbers $x_1^i, x_2^i, y_1^\alpha = y_2^\alpha$ be given in such a way, that the vector $y_1^\alpha f_\alpha \in \mathfrak{g}/h'$ is in the fundamental form. Then there is exactly one (up to the equivalence) motion with these numbers as invariants and this motion is cyclic.*

Proof. Let us write $X = x_1^i e_i + y_1^\alpha f_\alpha$, $Y = x_2^i e_i + y_2^\alpha f_\alpha$. Then $(\omega_1, \omega_2) = (X\,\mathrm{d}t, Y\,\mathrm{d}t)$ is the canonical form for $(\exp tX, \exp tY)$. Then $(\exp tX, \exp tY)$ is a lift for a cyclic motion $g(t) = \exp tX \,.\, \exp(-tY)$ and the motion $g(t)$ has the given numbers as invariants because $y_1^\alpha f_\alpha$ is of the fundamental form. The uniqueness is easy.

For illustration let us consider now the motion in the homogeneous space $G/\mathscr{H}$, where $\mathscr{H}$ is the normalizer of h'. For any regular motion $g(t)$ in G we have two poloids $p_2(t)$ and $p_1(t)$ in $G/\mathscr{H}$. Their properties are characterized as follows.

Theorem 4. *$p_1(t_0)$ is the only stationary point at the instant t_0.*

Proof. The tangent vector of the trajectory of any point t is given by the Killing vector field defined by $R(t)$. Formula (3) now means that the corresponding vector of the right invariant vector field on G defined by R is vertical at $g_1(t)$ and so its projection is zero.

Conversely, if the projection is zero at some point $\gamma\mathscr{H}$ for some $\gamma \in G$, then $R \in \operatorname{ad}\gamma h_1$. This means that $\operatorname{ad}\gamma h_1 = \operatorname{ad} g_1 h_1$ and so $\operatorname{ad}(\gamma^{-1} g_1) h_1 = h_1$, $\gamma^{-1} g_1 \in \mathscr{H} = \mathscr{N}(h_1)$ and $\pi\gamma = \pi g_1$.

Theorem 5. *The moving centroid p_2 is rolling on the fixed centroid during the motion. This means that $g^*(t)\, p_2'(t) = p_1'(t)$ and the invariants of the first order are the same.*

Proof. We have $g(\pi g_2) = \pi g_1$, $g = g_1 g_2^{-1}$ and so $g^* = g_1^*(g_2^{-1})^*$. Further

$$g^*(p_2') = g^*(\pi g_2)' = g^*(\pi g_2') = \pi^*(g^* g_2') = \pi^*(g_1^*(g_2^{-1})^* g_2') = \pi^*(g_1^* \omega_1) =$$

$$= \pi^*(g_1^*(\omega_1 + h)) = \pi^* g_1^* \omega_1 + \pi^* g_1^* h = \pi^* g^* \omega_1 + g_1^* \pi^* h = \pi^* g_1' = (\pi g_1)' = p_1'$$

for some $h \in h_1$. The rest is obvious from the definition of ω_1, ω_2.

Theorem 6. *Let us be given two curves $p_2(t)$, $p_1(t)$ in homogeneous space $G/\mathscr{H}$ with t a canonical parameter. Let p_2, p_1 have the same invariants of the first order with $\omega_1 - \omega_2$ regular. Then there is exactly one regular motion (up to equivalence) having p_2 and p_1 as its poloids.*

Proof. Let $\omega_1 = \omega_1^i e_i + \eta_1^\alpha f_\alpha$, $\omega_2 = \omega_2^i e_i + \eta_2^\alpha f_\alpha$. Then there are curves $g_1(t)$, $g_2(t)$ such that (ω_1, ω_2) is the canonical form of the curve $(g_1(t), g_2(t))$ in $G \times G$. $g_1 g_2^{-1}$ is the required motion. The rest is obvious.

Theorem 7. *Poloids of a regular cyclic motion are trajectories of one-parametric motions.*

Proof. Let $g(t) = \exp tX \exp(-tY)$. Then $(\exp tX, \exp tY)$ is a lift of $g(t)$. We can suppose that $X - Y \in h_1$ and then $(\exp tX, \exp tY)$ is a first order lift of $g(t)$ and so poloids are $p_2 = \pi \exp tX$, $p_1 = \pi \exp tY$ and those are trajectories of one parametric motions.

3. INVARIANTS OF A CURVE IN $G/\mathscr{H}$

Let $\mathfrak{g}$ be a semisimple Lie algebra over C. Let us denote by G the real Lie group of all inner automorphisms of $\mathfrak{g}$, $\mathscr{H}$ be the normalizer in G of a fixed Cartan subalgebra $\mathfrak{h}$. Let further a curve $x(t)$ in the homogeneous space $G/\mathscr{H}$ be given and let $g(t)$ be its arbitrary lift. Denote by $\omega(t)$ the values of the canonical form on the given lift, let $X(t)$ be the projection of $\omega(t)$ in $\mathfrak{g}/\mathfrak{h}$. To find invariants of $x(t)$ means to find the canonical form of $X(t)$ with respect to $\mathscr{H}$. From this we see, that $\mathfrak{g}$ can be supposed to be simple. Let us now fix a Weyl base E_α in $\mathfrak{g}/\mathfrak{h}$ and an ordered system of simple roots $\pi = \{\alpha_1, \ldots, \alpha_n\}$ in $\mathfrak{h}$. Denote $-\pi = \{-\alpha_1, \ldots, -\alpha_n\}$. From [8] we know that the group $\mathscr{H}/\mathrm{Exp}\ \mathrm{ad}\ \mathfrak{h}$ is isomorphic to the Weyl group W of $\mathfrak{g}$.

Definition 1. Let us denote by ϑ the automorphism of $\mathfrak{g}$ given by relations $\vartheta(E_\alpha) = E_{-\alpha}$, $\vartheta(h) = -h$ for $h \in \mathfrak{h}$. g will be called *of the first type* if ϑ is inner and *of the second type* if ϑ is an outer automorphism.

From [4] we know that ϑ is outer iff $g = A_n$ for $n \geqq 2$, $g = D_{2n+1}$ for $n \geqq 1$ or $g = E_6$.

The set $\exp \operatorname{ad} \mathfrak{h} . \{1, \vartheta\}$ is a group, as we see from the formula $\vartheta \exp \operatorname{ad} h \vartheta^{-1} = \exp \operatorname{ad}(-h)$ for $h \in \mathfrak{h}$.

So let us denote $V = \exp \operatorname{ad} \mathfrak{h} . \{1, \vartheta\} \cap \operatorname{Int} g$.

Lemma 1. *V is an invariant subgroup of $\mathscr{H}$.*

Proof. *Let $A \in \mathscr{H}$, $\exp \operatorname{ad} h . \varepsilon \in V$, where ε is 1 or ϑ. Then $(A \exp \operatorname{ad} h . \varepsilon A^{-1}) \varepsilon = \exp \operatorname{ad} h'$ for some $h' \in \mathfrak{h}$, as $(A \exp \operatorname{ad} h . \varepsilon A^{-1}) \varepsilon$ induces the identical transformation in $\mathfrak{h}$. So we have $A \exp \operatorname{ad} h . \varepsilon A^{-1} = \exp \operatorname{ad} h' . \varepsilon$.*

Let now a vector $X = x_\alpha E_\alpha$ in $\mathfrak{g}/\mathfrak{h}$ be given. Let us define the numbers $\xi_\alpha = x_\alpha x_{-\alpha}$. Then $\xi_\alpha = \xi_{-\alpha}$ and we have the following lemma:

Lemma 2. *The numbers ξ_α do not depend on the group V.*

Proof. Let $A \in V$. Then $AE_\alpha = v_\alpha E_{\pm\alpha}$, $AE_{-\alpha} = v_{-\alpha} E_{\mp\alpha}$, $v_\alpha v_{-\alpha} = 1$ and $\tilde{X} = AX = \sum x_\alpha A E_\alpha = \sum x_\alpha v_\alpha E_{\pm\alpha} = \tilde{x}_{\pm\alpha} E_{\pm\alpha}$. We get $\tilde{x}_\alpha = x_{\pm\alpha} v_{\pm\alpha}$ and $\tilde{\xi}_\alpha = \tilde{x}_\alpha \tilde{x}_{-\alpha} = x_{\pm\alpha} v_{\pm\alpha} x_{\mp\alpha} v_{\mp\alpha} = \xi_\alpha$.

Let us suppose from now on that X satisfies the following condition: if $\alpha \neq \pm\beta$, then $|\xi_\alpha| \neq |\xi_\beta|$. As we have chosen an ordered system π of simple roots, we have determined one Cartan matrix of $\mathfrak{g}$ – the Cartan matrix $A_{ij} = 2(\alpha_{ij}\alpha_j)(\alpha_{ij}\alpha_i)^{-1}$ of π. With every sequence of n roots we can associate its Cartan matrix. Let us denote by Π the set of all sequences of n roots having the same Cartan matrix, as π has. To every $\varphi = \{\beta_1, \ldots, \beta_n\} \in \Pi$ we now get a sequence $\varphi(\xi) = \{|\xi_{\beta_1}|, \ldots, |\xi_{\beta_n}|\}$ of n real numbers. We have also $(-\varphi)(\xi) = \varphi(\xi)$. The system $\Pi_0 \supset \Pi_1 \ldots \supset \Pi_n$ of subsets of Π will be defined as follows: Put $\Pi_0 = \Pi$. Let us suppose, that we have already defined Π_k and consider the number $\eta_{k+1} = \max_{\varphi \in \Pi_k} |\xi_{\beta_{k+1}}|$, where $\varphi = \{\beta_1, \ldots, \beta_n\}$. Then we define $\Pi_{k+1} = \{\psi \in \Pi_k \mid |\xi_{\gamma_{k+1}}| = \eta_{k+1}\}$, where $\psi = \{\gamma_1, \ldots, \gamma_n\}$.

Lemma 3. *Π_n contains only two elements, φ_1 and $-\varphi_1$.*

Proof. Sets Π_k are nonempty for $k = 1, \ldots, n$. If we have $\varphi = \{\beta_1, \ldots, \beta_n\} \in \Pi_k$ and $\psi = \{\gamma_1, \ldots, \gamma_k\} \in \Pi_k$, we get $\gamma_i = \pm\beta_i$ for $i = 1, \ldots, k$. If now $\varphi \in \Pi_{k+1}$ and $\psi \in \Pi_{k+1}$, then $\gamma_k = \beta_k$ or $\gamma_k = -\beta_k$. From $\gamma_k = \beta_k$ we get $\gamma_{k+1} = \beta_{k+1}$, because the Dynkin diagram of a simple Lie algebra is connected and so $(\gamma_k, \gamma_{k+1}) < 0$ and $(\beta_k, \beta_{k+1}) < 0$. The case $\gamma_k = -\beta_k$ is similar. The converse is easy because $\varphi \in \Pi_n$ implies $-\varphi \in \Pi_n$.

Using the construction mentioned above we get two systems of simple roots, say φ and $-\varphi$ from Π. As we know that there exists an inner automorphism mapping φ onto π or $-\pi$, we are led to the following definition:

Definition 2. Let the notation be as above and let $X \in \mathfrak{g}/\mathfrak{h}$. We say that X is *in the fundamental form*, if it satisfies the following condition:

Let $(\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_n)$ resp. $(-\alpha_1, \ldots, -\alpha_k, \beta_{k+1}, \ldots, \beta_n)$ be a system of simple roots, where $\alpha_1, \ldots, \alpha_k \in \pi$ and $\beta_{k+1} = \pm\alpha_{k+1}$. Then $|\xi_{\alpha_k}| > |\xi_{\beta_k}|$.

The automorphism A which maps φ on π or $-\pi$ maps the vector X on the vector $A(X)$ in the fundamental form. If $A \in \mathscr{H}$ and X is in the fundamental form, then $A(X)$ is in the fundamental form if and only if $A \in V$.

Lemma 4. *Let $X \in \mathfrak{g}/\mathfrak{h}$, $\mathfrak{g} \neq D_4$. Then there exists only one $A \in \mathscr{H}/V$ such that $A(X)$ is in the fundamental form.*

Proof. If $\mathfrak{g}$ is of the second type, then $V = \exp \operatorname{ad} \mathfrak{h}$ and $\mathscr{H}/V$ is isomorphic to W. Let $\pi = (\alpha_1, \ldots, \alpha_n)$ be fixed and let $\varphi = (\beta_1, \ldots, \beta_n) \in \Pi_n$. As the Cartan matrices of φ and π are the same, there exists an automorphism $\mu \in \mathscr{H}$ resp. $\gamma \in \mathscr{H}$ such that $\mu(\varphi) = \pi$ or $\gamma(\varphi) = -\pi$ (it means that $\mu(\beta_i) = \alpha_i$ or $\gamma(\beta_i) = -\alpha_i$ for $i = 1, \ldots, n$). As $\gamma = \vartheta\mu \exp \operatorname{ad} h$ for some $h \in \mathfrak{h}$, we see, that exactly one of the automorphisms μ and γ is inner. (We know, that the group $\operatorname{Aut} \mathfrak{g}/\operatorname{Int} \mathfrak{g}$ is trivial or Z_2 except for D_4.) If all automorphisms of $\mathfrak{g}$ are inner, the lemma is obvious. In the case of D_{2k}, $k \geqq 3$ we shall take the group $\operatorname{Aut} \mathfrak{g}$ instead of $\operatorname{Int} \mathfrak{g}$ for the group G, $\mathscr{H}$ is in this case the normalizer of $\mathfrak{h}$ in $\operatorname{Aut} \mathfrak{g}$ and the lemma is again true. For D_4 the conclusion of lemma 4 is not valid. In the following we shall take $\operatorname{Aut} \mathfrak{g}$ for G in case of D_{2k}, $k \geqq 3$, case D_4 will be left out.

Let now $X \in \mathfrak{g}/\mathfrak{h}$ in the fundamental form be given with $X = x_\alpha E_\alpha$. Let $A \in V$. Then A leaves invariant every subspace $U_\alpha = CE_\alpha + CE_{-\alpha}$ and the matrix of the restriction of A to U_α is of the form

$$\begin{pmatrix} c_\alpha, & 0 \\ 0, & c_\alpha^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0, & c_\alpha \\ c_\alpha^{-1}, & 0 \end{pmatrix}$$

for the 1. type and

$$\begin{pmatrix} c_\alpha, & 0 \\ 0, & c_\alpha^{-1} \end{pmatrix}$$

for the 2. type.

The automorphism $A \in V$ is uniquely determined, if we give the numbers c_{α_i}, where $\alpha_i \in \pi$, $c_{\alpha_i} \neq 0$ for the 2. type resp. $c_{\alpha_i}, \varepsilon$, where $\alpha_i \in \pi$, $c_{\alpha_i} \neq 0$, $\varepsilon = \pm 1$ for the 1. type, $i = 1, \ldots, n$. Let us write $\tilde{X} = A(X)$, $\tilde{X} = \tilde{x}_\alpha E_\alpha$.

a) For the 2. type: We get $\tilde{x}_\alpha = c_\alpha x_\alpha$, $\tilde{x}_{-\alpha} = c_\alpha^{-1} x_{-\alpha}$. Let us choose $c_\alpha = \pm(x_{-\alpha} x_\alpha^{-1})^{1/2}$. Then $\tilde{x}_{\alpha_i} = (x_{-\alpha_i} x_{\alpha_i}^{-1})^{1/2} x_{\alpha_i} = (x_{\alpha_i} x_{-\alpha_i})^{1/2} = (x_{\alpha_i} x_{-\alpha_i}^{-1})\, x_{-\alpha_i} = \tilde{x}_{-\alpha_i}$. If $\tilde{x}_{\alpha_i} = \tilde{x}_{-\alpha_i}$ and $x_{\alpha_i} = x_{-\alpha_i}$, then $\tilde{x}_{\alpha_i}(\tilde{x}_{-\alpha_i})^{-1} = c_{\alpha_i}^2 x_{\alpha_i}(x_{-\alpha_i})^{-1}$, $c_{\alpha_i}^2 = 1$, $c_{\alpha_i} = \pm 1$. This suggests the following

Definition 3. Let w be a complex number. We shall say that $w > 0$ iff $\operatorname{Re} w > 0$ or $\operatorname{Re} w = 0$ and $\operatorname{Im} w > 0$.

We can see easily, that $w \neq 0$ and $w \ngtr 0$ implies $-w > 0$. With definition 3 in mind we see, that the numbers c_{α_i} can be chosen in such a way, that $x_{\alpha_i} = x_{-\alpha_i} > 0$.

b) For the 1. type: The consideration is the same as in a) only we get the group $\{1, \vartheta\}$ as a rest. We can choose $\delta \in \{1, \vartheta\}$ so, that $x_{\alpha_1+\alpha_2} - x_{-\alpha_1-\alpha_2} > 0$. This can be done, as $\alpha_1 + \alpha_2$ is a root (if the rank of g is at least 2) and $x_{\alpha_1+\alpha_2} \neq x_{-\alpha_1-\alpha_2}$ (in general).

Let us choose now the arc s of the curve $x(t)$ in such a way that X satisfies the relation $\sum_{\alpha\in\Sigma} |\xi_\alpha| = 1$.

Theorem 8. *Let $x(t)$ be a curve in $G/\mathscr{H}$ such that its arbitrary lift satisfies $|\xi_\alpha| \neq |\xi_\beta|$ for $\alpha \neq \pm\beta$ at $t = 0$. Then there exists a parameter s ($s(0) = 0$), an interval $I = (-\varepsilon, \varepsilon)$ and a lift $g(s)$ of $x(s)$ such that for $X = x_\alpha(s) E_\alpha$ and $s \in I$ we have*

a) *$X(0)$ is in the fundamental form,*
b) *$x_{\alpha_i} = x_{-\alpha_i} > 0$ for $i = 1, \ldots, n$,*
c) $\sum|\xi_\alpha| = 1$,
d) $x_{\alpha_1+\alpha_2} - x_{-\alpha_1-\alpha_2} > 0$ (for the 1. type).

Definition 4. Let us write for the lift $g(s)$ from theorem 8:

$$g^{-1} \frac{dg}{ds} = \sum x_\alpha E_\alpha + \sum_{i=1}^{n} y_i H_{\alpha_i}.$$

Then the functions $x_\alpha(s)$, $y_i(s)$ are called *the invariants of the curve $x(t)$.*

Theorem 9. *Let $x_\alpha(s)$, $y_i(s)$ be complex functions of a real parameter s, let I be an interval. Let us suppose, ιhat the following are satisfied:*

e) *$x_\alpha(s)$, $y_{\alpha_i}(s)$ are defined on I,*
ii) *$|\xi_{\alpha_i}| = x_{\alpha_i} \cdot x_{-\alpha_i} \neq 0$ on I,*
iii) *conditions* a)–d) *from theorem* 8.

Then there exists only one (up to transformations from G) curve $x(s)$ in $G/\mathscr{H}$ defined on I such that $x_\alpha(s)$ and $y_i(s)$ are its invariants.

Remark. If we consider $G/\mathscr{H}$ as a complex manifold with a curve as a function of a complex parameter on it, we may proceed analogously, only the arc must be defined in another way. We can choose it so, that $(\sum\xi_\alpha)^{1/2} = 1$ for example. Then we get similar two theorems as above.

Now we shall give the parallel theory for a compact real semisimple Lie algebra. Let the notation be the same as above. Then the vectors $\sqrt{(-1)}\, H_{\alpha_i}$, $E_\alpha + E_{-\alpha}$, $\sqrt{(-1)}\,(E_\alpha - E_{-\alpha})$ generate over R a real compact form g_n of $\mathfrak{g}$. Every automorphism of g_n can be uniquely extended to an automorphism of $\mathfrak{g}$, which leaves g_n invariant. Groups Aut g_n and Int g_n can be then considered as subgroups of Aut $\mathfrak{g}$ and Int $\mathfrak{g}$

respectively. Let now $\mathscr{H}$ be the normalizer of $\sqrt{(-1)}\,h$, where $h = \sum RH_{\alpha_i}$. Then $\mathscr{H}/\text{Exp ad}\,(\sqrt{(-1)}\,h)$ is isomorphic to the Weyl group W of $\mathfrak{g}$. The automorphism ϑ, also leaves g_n invariant, as is seen from its definition. So we can suppose that we have a simple compact real Lie algebra g_n situated in its complexification as described above. Let us identity every automorphism of g_n with its complexification and let A be an automorphism of g_n leaving invariant $\sqrt{(-1)}\,h$. Then $A(E_\alpha) = a_\alpha E_{A(\alpha)}$, where $a_\alpha a_{-\alpha} = 1$ and $|a_\alpha| = 1$ (see [6]). From this we see, that everything is similar to the complex case and the result is as follows:

Theorem 10. *Let $x(t)$ be a curve in $G/\mathscr{H}$, where $G = \text{Int}\,g_n$ and $\mathscr{H}$ is the same as above. Let us have $|\xi_\alpha| \neq |\xi_\beta|$ for $\alpha \neq \pm\beta$ at $t = 0$ for an arbitrary lift of $x(t)$. Then the conclusion of theorem 8 is valid and $x_\alpha = \bar{x}_{-\alpha}$ for all $\alpha \in \Sigma$.*

Proof. The proof is similar to that of theorem 8, only now $|c_\alpha| = 1$ and $\bar{x}_\alpha = x_{-\alpha}$.

Theorem 11. *Let Σ^+ be the set of all positive roots. Let real functions $x_\alpha(s)$, $y_\alpha(s)$, $z_{\alpha_i}(s)$ on I be given, where $\alpha \in \Sigma^+$, $\alpha_i \in \pi$. Let us suppose that $X = \sum_{\alpha\in\Sigma^+} [(x_\alpha(s) + \sqrt{(-1)}\,y_\alpha(s))\,E_\alpha + (x_\alpha(s) - \sqrt{(-1)}\,y_\alpha(s))\,E_{-\alpha}]$ and $\xi_\alpha = \xi_{-\alpha} = x_\alpha^2 + y_\alpha^2$ satisfy conditions* a)–d) *from theorem 8. Then there exists only one curve $x(s)$ in $G/\mathscr{H}$ such that x_α, y_α, z_{α_i} are its invariants.*

4. $SU(n+1)$ AS AN EXAMPLE

Let us consider the group $G = SU(n+1)/Z$, where Z is the center of $SU(n+1)$. Z is the group of $(n+1)$st roots from one. G is a simple compact Lie group of type $\mathscr{A}_n$, it is isomorphic with the group of all inner automorphisms of a simple real compact Lie algebra of the type A_n. This algebra can be realized as the algebra of all skew-hermitian matrices with trace zero, its complexification is the algebra of all complex matrices of type $(n+1) \times (n+1)$ with trace zero – let us denote it $\tilde{A}_n$. Let us denote by $E_{i,j}$ the matrix with entries $a_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$. Vectors $E_{ii} \bmod \sum_{i=1}^{n+1} E_{ii}$ form a base of a Cartan subalgebra of $\tilde{A}_n$, say h. Matrices E_{ij} for $i < j$ and $-E_{ij}$ for $i > j$ form a Weyl base of $\tilde{A}_n/h$ up to a constant factor. As usual write λ_i for $\sqrt{(-1)}\,E_{ii}$. Roots of A_n are then $\lambda_i - \lambda_j$. $X \in A_n$ iff $\bar{x}_{ij} + x_{ji} = 0$, $\sum x_{ii} = 0$. For π we usually choose the system $\pi = \{\lambda_1 - \lambda_2, \ldots, \lambda_n - \lambda_{n+1}\}$. Denote further by $\mathscr{H}$ the normalizer of $\sqrt{(-1)}\,h$ in G.

Let now $X = (x_{ij})$ be a matrix from $A_n/\sqrt{(-1)}\,h$, where $x_{ij} = y_{ij} + \sqrt{(-1)}\,z_{ij}$, $x_{ij} = -\bar{x}_{ji}$, $x_{ii} = 0$. Following theorem 10 we define the matrix $\xi_{ij} = y_{ij}^2 + z_{ij}^2$ for $i \neq j$ and $\xi_{ii} = 0$. Then ξ_{ij} is a symmetrical matrix. Let us suppose that $\xi_{ij} \neq \xi_{km}$ for $\{i, j\} \neq \{k, m\}$. With the use of the algorithm described above we now choose a system of simple roots. At the first step let $\xi_{i_1 i_2} = \max \xi_{ij}$. Then we get two roots, $\lambda_{i_1} - \lambda_{i_2}$ and $\lambda_{i_2} - \lambda_{i_1}$, and any of them can be the first in the needed system. At the second step let us take the maximum of all $\xi_{i_1 i_\alpha}$ and $\xi_{i_2 i_\alpha}$ where $i_\alpha \neq i_1, i_2$. We get either $\xi_{i_2 i_r}$ or $\xi_{i_1 i_r}$. In the first case we take the roots $\lambda_{i_1} - \lambda_{i_2}, \lambda_{i_2} - \lambda_{i_r}$ at the first two places of our system, in the second case we take $\lambda_{i_2} - \lambda_{i_1}, \lambda_{i_1} - \lambda_{i_r}$. Then we take the maximum of $\xi_{i_r i_\alpha}$, where $i_\alpha \neq i_1, i_2$, get the third root and so on. At the n-th step we get two systems of simple roots, say $\pm\{\lambda_{i_1} - \lambda_{i_2}, \ldots, \lambda_{i_n} - \lambda_{i_{n+1}}\}$ and there is only one element of the Weyl group, which maps one of these systems on π (we take them as ordered). We then see, that X is in the fundamental form if for $\xi_{ij} = x_{ij}^2 + z_{ij}^2$ the following is true:

$$\begin{aligned}
&\xi_{12} > \xi_{ij} \quad \text{where} \quad 1, 2 \neq i, j\,,\\
&\xi_{23} > \xi_{\alpha k} \quad \text{where} \quad \alpha = 1, 2\,, \quad k > 3\,,\\
&\xi_{23} > \xi_{13}\,,\\
&\xi_{l,l+1} > \xi_{lk}\,, \quad k > l + 1\,, \quad l = 1, \ldots, n\,.
\end{aligned}$$

For the matrix $\omega = (x_{ij})$, where $x_{ij} = y_{ij} + \sqrt{(-1)}\, z_{ij}$, of invariants of a curve in $G/\mathscr{H}$ we have:

(6) $$X = \pi\omega \quad \text{is the fundamental form}\,, \quad x_{i,i+1} = -x_{i+1,i} > 0$$

is real, ω is skew-hermitian, $\sum_{i \neq j} x_{ij} \,.\, \bar{x}_{ij} = 1$ and $\operatorname{Tr} \omega = 0$. Specially for $n = 2$ we get

$$\omega = \begin{pmatrix} \sqrt{(-1)}\, z_{11}, & y_{12}, & y_{13} + \sqrt{(-1)}\, z_{13} \\ -y_{12}, & \sqrt{(-1)}\, z_{22}, & y_{23} \\ -y_{13} + \sqrt{(-1)}\, z_{13}, & -y_{23}, & \sqrt{(-1)}\, z_{33} \end{pmatrix},$$

where $y_{12}^2 > y_{23}^2 > y_{13}^2 + z_{13}^2$, $y_{12} > 0$, $y_{23} > 0$, $y_{12}^2 + y_{23}^2 + y_{13}^2 + z_{13}^2 = 2^{-1}$, $z_{11} + z_{22} + z_{33} = 0$.

Let now V^{n+1} be a unitary vector space of dimension $n + 1$ with a unitary scalar product (x, y) and a determinant function $|x_1, \ldots, x_{n+1}|$. By a frame $\mathscr{F} = \{x_1, \ldots, x_{n+1}\}$ we shall mean an orthonormal and unimodular base ($(x_i, x_j) = \delta_{ij}$, $|x_1, \ldots, x_{n+1}| = 1$). If we fix a frame then $SU(n + 1)$ acts naturally on V^{n+1}. Elements of $SU(n + 1)$ can be identified with frames in V^{n+1}. Further consider the unitary sphere S^{2n+1} in V^{n+1} given by the equation $(x, x) = 1$. Then the group Z of $(n + 1)$st roots of one acts naturally on S^{2n+1} and we can form the quotient manifold $S^{2n+1}/Z = M$. The group $SU(n + 1)/Z = G$ acts then transitively on M.

Let us suppose that a unitary motion $g(t)$ is given. We consider it as the motion of V^{n+1} or of S^{2n+1}. The fixed centroid consists of a system of $n + 1$ mutually orthogonal directions in V^{n+1} or of $n + 1$ mutually orthogonal great circles in S^{2n+1}. These systems are determined by the eigenvectors of the fixed directing cone R. The situation for the moving centroid is similar. If we denote the Frenet frame of the fixed centroid by $\mathscr{F} = \{U_1, \ldots, U_{n+1}\}$, we have the fixed centroid $\{\lambda_1 U_1, \ldots \ldots, \lambda_{n+1} U_{n+1}\}$ where $\lambda_i \in C$ in V^{n+1} or $\lambda_i = e^{\sqrt{(-1)}\varphi_i}$, $\sum \varphi_i = 0$ in S^{2n+1}. Denote now by $\omega_1 = m_{ij}\, dt$, $\omega_2 = n_{ij}\, dt$ the invariants of the motion $g(t)$ with t the canonical parameter (defined by the condition that R is a unit vector with respect to the Killing form on $\mathfrak{g}$). Similarly as in (6) $\pi\omega_1 = \pi\omega_2$ is in the fundamental form, $x_{i,i+1} = -x_{i+1,i} > 0$ is real, ω_1 and ω_2 are skew-hermitian and $\operatorname{Tr}\omega_1 = \operatorname{Tr}\omega_2 = 0$. The only difference is, that now $\sum_{i=1}^{n+1} (m_{ii} - n_{ii})^2 = -1$. Numbers $v_i = m_{ii} - n_{ii}$ are eigenvalues of the directing cone R.

Let s be the common arc for the fixed and the moving centroid. Then we have

$$m_{ij} = x_{ij} \frac{\mathrm{d}s}{\mathrm{d}t}$$

and so

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \sqrt{\left[-\sum_{i=1}^{n+1} (x_{ii} - \tilde{x}_{ii})^2\right]},$$

where x_{ij} resp. $\tilde{x}_{ij}$ are invariants of the fixed resp. the moving centroid. Similarly we get

$$\sqrt{\left(\sum_{\substack{i \neq j \\ i,j=1}}^{n+1} m_{ij}\overline{m}_{ij}\right)} = \frac{\mathrm{d}s}{\mathrm{d}t}.$$

The Frenet formulas for the fixed centroid can now be written in the form

$$U_i' = x_{ij} U_j, \quad i, j = 1, \ldots, n+1, \quad \text{where the matrix } x_{ij} \text{ satisfies (6)}.$$

Let us find out, how the Frenet's frame is defined. Let the fixed centroid be given by vectors $X_1, \ldots, X_{n+1}$, where $X_1, \ldots, X_{n+1}$ are given as eigenvectors of the fixed directing cone R.

For Frenet's frame $\mathscr{F} = \{U_1, \ldots, U_{n+1}\}$ we have

$$X_i = U_i e^{\sqrt{(-1)}\varphi_i} \quad \text{with} \quad \sum_{i=1}^{n+1} \varphi_i = 0$$

as we can suppose $|X_i| = 1$, $|X_1, \ldots, X_{n+1}| = 1$. After differentiation and scalar multiplication we get

$$(X_i', X_{i+1}) = y_{i,i+1} e^{\sqrt{(-1)}(\varphi_i - \varphi_{i+1})} \quad \text{with} \quad x_{ij} = y_{ij} + \sqrt{(-1)}\, z_{ij}.$$

This gives (together with $\sum_{i=0}^{n+1} \varphi_i = 0$) $n + 1$ linear equations of rank $n + 1$ for unknown φ_i and this defines Frenet's frame uniquely. The other invariants can be now expressed using derivatives of Frenet's frame and scalar multiplication. In the end we shall illustrate the difference by the nature of problems in the 3-dimensional kinematic geometry and in that of n dimensions. Consider the unitary motion as a motion in S^{2n+1}. The fixed axoid consists then of $n + 1$ mutually orthogonal ruled surfaces (in spherical space). Let us find out what is the connection between the invariants of the fixed axoidand the invariants of the mentioned ruled surfaces. Let x be a unit vector in V^{n+1}. Then all points in S^{2n+1} of the form $y = e^{\sqrt{(-1)}\varphi}x$ determine a direct line in the spherical space S^{2n+1} (it is a great circle of the sphere). The homogeneous space of all direct lines of the described form is the space $SU(n+1)/U(n)$. Using the method of specialisation of the frames we get Frenet's formulas for such a ruled surface in the form

$$V_k' = -\eta_{k,k-1}V_{k-1} + \sqrt{(-1)}\,\eta_{kk}V_k + \eta_{k+1,k}V_{k+1}$$

with the arc determined by η_{21} and with $\eta_{10} = \eta_{n+2,n+1} = 0$, η_{ij} real.

Let the ruled surface be given by a vector X with $(X, X) = 1$. Then $X = V_1 e^{\sqrt{(-1)}\varphi}$. Taking the derivative n-times and using Frenet's formulas, we get

$$\det |X, X', \ldots, X^{(n)}| = e^{(n+1)\sqrt{(-1)}\varphi}\eta_{21} \cdot \ldots \cdot \eta_{n+1,n}|V_1, \ldots, V_{n+1}|$$

and so

$$e^{(n+1)\sqrt{(-1)}\varphi}\eta_{21} \cdot \ldots \cdot \eta_{n+1,n} = \det |X, X', \ldots, X^{(n)}| .$$

This determines V_1. From

$$V_1' = \sqrt{(-1)}\,\eta_{11}V_1 + V_2$$

we get ds/dt for the arc s and the rest follows if we use the orthogonalisation process and Frenet's formulas.

From this consideration we see that Frenet's frame of the ruled surface of our type is of the $(n+1)$st order, but Frenet's frame for the axoid is of the second order in coefficients of the matrix of the motion. This implies that the connection between them becomes very complicated for n general, we can get reasonable formulas only in cases of low dimensions or for the invariants of low order.

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