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# GENERALIZED DEDEKIND COMPLETION OF A LATTICE ORDERED GROUP

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The notions of the generalized Dedekind completion $D_1(G)$ and of the archimedean kernel $A(G)$ of a lattice ordered group $G$ were introduced in [10]. In this paper some further properties of $D_1(G)$ are established.

In § 2 it is shown that to each direct decomposition of $G$ there corresponds a direct decomposition of $D_1(G)$. Namely, if $G$ is a direct product of its $l$-subgroups $G_i$ $(i \in I)$, then $D_1(G)$ is a direct product of its $l$-subgroups $D_1(G_i)$ $(i \in I)$. This generalizes a result from [6] concerning archimedean lattice ordered groups. An analogous assertion is valid for direct sums of lattice ordered groups. If $G$ is epiarchimedean and conditionally orthogonally complete, then $D_1(G)$ is epiarchimedean. If $G$ is strongly projectable, then so is $D_1(G)$. If $G$ is projectable and $A(G)$ is strongly projectable, then $D_1(G)$ is projectable. If $G$ is projectable, then $D_1(G)$ need not be projectable. If $G$ is conditionally orthogonally complete, then so is $D_1(G)$.

Pairwise splitting lattice ordered groups have been studied by MARTINEZ [12]. Generalized Dedekind completions of pairwise splitting lattice ordered groups are dealt with in § 3. It is proved that if $G$ is a pairwise splitting abelian lattice ordered group such that the archimedean kernel $A(G)$ of $G$ is conditionally orthogonally complete, then $D_1(G)$ is pairwise splitting; the assumption of the conditional orthogonal completeness of $A(G)$ cannot be omitted.

In § 4 the relations between higher degrees of distributivity of a lattice ordered group $G$ and those of $D_1(G)$ are investigated. Let $\beta$ be a cardinal. For a lattice ordered group $H$ we write $d(H) = \alpha$ if $H$ is $\gamma$-distributive for each $\gamma < \alpha$ and if $H$ fails to be $\alpha$-distributive. Let $d(G) = \alpha$. If either $A(G)$ is completely distributive or $A(G)$ is projectable, then $d(D_1(G)) = \alpha$. If $A(G)$ is not completely distributive and $d(G_1) = \beta$, where $G_1$ is the Dedekind completion of $A(G)$, then $d(D_1(G)) = \min \{\alpha, \beta\}$.

A lattice ordered group $G$ is called $g$-complete if $D_1(G) = G$. In § 5 it is shown that each lattice ordered group possesses a largest $g$ complete convex $l$-subgroup. This implies that the class of all $g$-complete lattice ordered groups is a radical class [11].

## 1. PRELIMINARIES

The standard terminology for lattices and lattice ordered groups will be used (cf. BIRKHOFF [1], CONRAD [2] and FUCHS [5]). The group operation is written additively, the commutativity of this operation is not assumed.

Let us recall some notions and some results from [10]. Let $G$ be a lattice ordered group. An element $0 < a \in G$ is called *archimedean in* $G$ if for each $0 < x \in G$ there exists a positive integer $n$ such that $nx$ non $\leqq a$. We denote by $A(G)$ the $l$-subgroup of $G$ generated by the set of all archimedean elements of $G$. Then $A(G)$ is a closed $l$-ideal of $G$ and $A(G)$ is archimedean (i.e., each element $0 < a \in A(G)$ is archimedean in $A(G)$). If $H$ is a convex $l$-subgroup of $G$ and if $H$ is archimedean, then $H \subseteq A(G)$. We shall often write $A$ instead of $A(G)$, when no ambiguity can occur.

For any archimedean lattice ordered group $K$ we denote by $D(K)$ the Dedekind closure of $K$ (cf. e.g. [1], Chap. XIII, § 13).

For each lattice ordered group $G$ there exists a lattice ordered group $D_1(G)$ fulfilling the following conditions:

(i) $G$ is an $l$-subgroup of $D_1(G)$;

(ii) $D(A(G))$ is an $l$-ideal of $D_1(G)$;

(iii) if $x \in G$ and if $X$ is a nonempty subset of $x + A(G)$ such that $X$ is upper bounded in $x + A(G)$, then there is $x_0 \in D_1(G)$ with $\sup X = x_0$;

(iv) for each $x_0$ $D_1(G)$ there exist $x \in G$ and $X \subseteq x + A(G)$ such that $X$ is upper bounded in $x + A(G)$ and $x_0 = \sup X$ holds in $D_1(G)$.

The lattice ordered group $D_1(G)$ is determined uniquely up to isomorphisms. More precisely, if $D'$ is a lattice ordered group fulfilling the conditions (i)–(iv) (with $D'$ instead of $D_1(G)$), then there exists an isomorphism $\varphi$ of $D_1(G)$ onto $D'$ such that $\varphi(x) = x$ for each $x \in G$ and each $x \in D(A(G))$.

If $X$ is a subset of $G$ and if $\sup X = x_0$ exists in $G$, then $x_0$ is the least upper bound of $X$ in $D_1(G)$ (and dually). $D_1(G)$ coincides with $D(G)$ if and only if $G$ is archimedean. The lattice ordered group $D_1(G)$ is said to be the generalized Dedekind completion of $G$. We have $A(D_1(G)) = D(A(G))$. The $l$-ideal $D(A(G))$ is closed in $D_1(G)$. If $G$ is abelian, then $D_1(G)$ is abelian as well.

For each $x_0 \in D_1(G)$ there is $x \in G$ and $a \in D(A(G))$ such that $x_0 = x + a$. If $0 \leqq x_0 \in D_1(G)$, then there are elements $0 \leqq x_1 \in G$, $0 \leqq a_1 \in D(A(G))$ with $x_0 = x_1 + a_1$. In fact, if $D(A(G)) = \{0\}$, $x_0 = x + a$, $x \in G$, $a \in D(A(G))$, then $a = 0$ and $x \geqq 0$. Let $D(A(G)) \neq \{0\}$; then $D(A(G))$ has no least element. Hence there is $x' \in x + D(A(G))$ with $x' \in G$, $x' \leqq x_0$ (cf. the condition (iv) above). Put $x_1 = x' \vee 0$, $a_1 = -x_1 + x_0$. Then $x_1 \in x + D(A(G))$, $0 \leqq x_1 \leqq x_0$, $0 \leqq a_1 \in D(A(G))$, $x_0 = x_1 + a_1$.

Let $X \subseteq G$. The set

$$X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called *a polar of* $G$. The set $X^{\delta\delta}$ is said to be *a polar generated by* $X$; if card $X = 1$, then $X^{\delta\delta}$ is called *a principal polar.*

Let us recall some notions concerning direct products and direct sums of lattice ordered groups (cf. e.g. [6]).

Let $I$ be a nonempty set and for each $i \in I$ let $G_i$ be a lattice ordered group. We denote by $G_1 = \prod_{i \in I} G_i$ the direct product of the lattice ordered groups $G_i$. Thus $G_1$ is the set of all mappings $f : I \to \bigcup G_i$ such that $f(i) \in G_i$ for each $i \in I$, the lattice operations and the group operations being defined coordinatewise. For $i \in I$ we denote $G^i = \{f \in G_1 : f(j) = 0 \text{ for all } j \in I, j \neq i\}$.

Let $G$ be a lattice ordered group and let $\varphi$ be an isomorphism of $G$ onto $G_1$. For each $i \in I$ we put $G_i^0 = \varphi^{-1}(G^i)$. Each $G_i^0$ is said to be *a direct factor of* $G$. We write also $G = \prod_{i \in I}^0 G_i^0$. The $l$-subgroup of $G$ generated by the set $\bigcup_{i \in I} G_i^0$ will be denoted by $\sum_{i \in I}^0 G_i^0$ and called *the direct sum of* $G_i^0$ $(i \in I)$. If $I$ is finite, $I = \{1, \ldots, n\}$, then $\prod_{i \in I}^0 G_i^0 = \sum_{i \in I}^0 G_i^0$ and we denote it also by $G_1^0 \oplus \ldots \oplus G_n^0$.

Each direct factor of $G$ is a closed $l$-ideal in $G$. A convex $l$-subgroup $H$ of $G$ is a direct factor of $G$ if and only if it fulfils the following conditions:

(a) For each $0 < g \in G$ the set $S = \{0 \leqq h \in H : h \leqq g\}$ possesses a greatest element.

If $H$ is a direct factor in $G$ and $0 \leqq g \in G$, then the greatest element of the set $S$ will be denoted by $g(H)$ and it is said to be *the component of* $g$ *in* $H$. For any $g_1 \in G$ we put $g_1(H) = g_1^+(H) - g_1^-(H)$. Let $H$ be a direct factor of $G$; then $H^\delta$ is also a direct factor of $G$ and the mapping $\psi(g_1) = (g_1(H), g_1(H^\delta))$ is an isomorphism of $G$ onto $H \times H^\delta$. Let $G = \prod_{i \in I}^0 G_i^0$ and let $\psi$ be a mapping of $G$ into $\prod_{i \in I} G_i^0$ such that $\psi(g_1)(i) = g_i(G_i^0)$ for each $g_1 \in G$ and each $i \in I$. Then $\psi$ is an isomorphism of $G$ onto $\prod_{i \in I} G_i^0$.

The following two assertions are easy to verify.

**2.1. Lemma.** *Let* $G$ *be a lattice ordered group and let* $\{G_j\}_{j \in J}$ *be a system of direct factors of* $G$ *such that*

(i) $G_j \cap G_k = \{0\}$ *whenever* $j$ *and* $k$ *are distinct elements of* $J$;

(ii) $g = \bigvee_{j \in J} g(G_j)$ *for each* $0 \leqq g \in G$;

(iii) *if* $0 \leqq h_j \in G_j$ *for each* $j \in J$, *then* $\bigvee_{j \in J} h_j$ *exists in* $G$.

*Then* $G = \prod_{j \in J}^0 G_j$. *Conversely, if* $G = \prod_{j \in J}^0 G_j$, *then* (i), (ii) *and* (iii) *are valid.*

**2.2. Lemma.** *Let* $G$ *be a lattice ordered group and let* $\{G_j\}_{j \in J}$ *be a system of direct factors of* $G$ *such that the conditions* (i), (ii) *from Lemma* 2.1 *are valid and*

(iv) *for each* $g \in G$, *the set* $\{j \in J : g(G_j) \neq 0\}$ *is finite.*

*Then* $G = \sum_{j \in J}^0 G_j$. *Conversely, if* $G = \sum_{j \in J}^0 G_j$, *then* (i), (ii) *and* (iv) *are valid.*

The condition (a) yields

**2.3. Lemma.** *Let* $G$ *be a lattice ordered group, let* $H$ *be a direct factor of* $G$ *and let* $K$ *be a convex* $l$*-subgroup of* $G$. *Then* $H \cap K$ *is a direct factor of* $K$.

**2.4. Lemma.** *Let* $G = \prod_{i\in I}^{0} G_i$ *and let* $K$ *be a closed convex* $l$*-subgroup of* $G$. *Then* $K = \prod_{i\in I}^{0} (K \cap G_i)$.

This follows from Lemma 2.1 and Lemma 2.3.

Analogously, from Lemma 2.2 and Lemma 2.3 we obtain

**2.5. Lemma.** *Let* $G = \sum_{i\in I}^{0} G_i$ *and let* $K$ *be a convex* $l$*-subgroup of* $G$. *Then* $K = \sum_{i\in I}^{0} (K \cap G_i)$.

Let $G$ be a lattice ordered group and $\emptyset \neq X \subseteq D_1(G)$. We denote by $c_1(X)$ the convex $l$-subgroup of $D_1(G)$ generated by the set $X$. If $X$ is an $l$-subgroup of $D_1(G)$, then $c_1(X)$ is the set of all $y \in D_1(G)$ with the property that there are elements $x_1, x_2 \in X$ with $x_1 \leqq y \leqq x_2$. If $G$ is archimedean and $\emptyset \neq X \subseteq D(G)$, then we denote by $c_0(X)$ the convex $l$-subgroup of $D(G)$ generated by $X$. If we do not suppose that $G$ is archimedean and if $\emptyset \neq X \subseteq A(G)$, then $c_0(X) = c_1(X)$ (here the symbol $c_0$ is taken with respect to $D(A(G))$).

**2.6. Proposition.** *Let* $G$ *be an archimedean lattice ordered group and let* $H$ *be a direct factor of* $G$. *Then* $c_0(H)$ *is a direct factor of* $D(G)$. *The lattice ordered group* $c_0(H)$ *is the Dedekind closure of* $H$. *For each* $g \in G$, $g(H) = g(c_0(H))$.

Proof. Let $0 \leqq d \in D(G)$. There exists $g \in G$ with $d \leqq g$. Put $g_1 = g(H)$ and $g_1 \wedge d = d_1$. Then $d_1 \in c_0(H)$ and $d_1 \leqq d$. Let $0 \leqq x \in c_0(H)$, $x \leqq d$. Hence $x \leqq g$ and there is $g_2 \in H$ with $x \leqq g_2$. Thus $x \leqq g \wedge g_2$. Since $H$ is convex in $G$, we obtain $g \wedge g_2 \in H$ and hence, $H$ being a direct factor of $G$, $g \wedge g_2 \leqq g_1$. Therefore $x \leqq g_1 \wedge d = d_1$. This shows that $c_0(H)$ is a direct factor of $D(G)$.

From the construction of the Dedekind closure it follows immediately that for each convex $l$-subgroup $H_1$ of $G$, $c_0(H_1)$ is the Dedekind closure of $H_1$.

Let $0 \leqq g \in G$. Put $g_1 = g(H)$. Then $g_1 \in c_0(H)$ and $g_1 \leqq g$. Assume that there exists $h \in c_0(H)$ with $g_1 < h \leqq g$. There is $g_0 \in H$ with $h \leqq g_0$. Hence $g_1 < h \leqq g_0 \wedge g \leqq g$ and $g_0 \wedge g \in H$. Since $H$ is a direct factor of $G$, we have a contradiction. Thus $g_1 = g(c_0(H))$. Since each element $g_2 \in G$ can be written as $g_2 = g_3 - g_4$ with $g_3, g_4 \in G^+$, we get $g_2(H) = g_2(c_0(H))$.

**2.7. Proposition.** *Let* $G$ *be an archimedean lattice ordered group,* $G = \prod_{i\in I}^{0} G_i$. *Then* $D(G) = \prod_{i\in I}^{0} c_0(G_i)$.

Proof. According to 2.6, each $c_0(G_i)$ is a direct factor of $D(G)$. We have to verify that the conditions (i) (ii) and (iii) from 2.1 are fulfilled with $G_i$, $G$, $J$ replaced by $c_0(G_i)$, $D(G)$, $I$.

Let $j, k \in I$, $j \neq k$. From $G_j \cap G_k = \{0\}$ we obtain $c_0(G_j) \cap c_0(G_k) = \{0\}$, hence (i) is valid. Let $0 \leqq g \in D(G)$. For $j \in I$ denote $g_j = g(c_0(G_j))$. There exists $h \in G$ with $g \leqq h$. Put $h_j = h(c_0(G_j))$. Hence $h_j \geqq g_j$ for each $j \in I$. By 2.6, $h_j = h(G_j)$ for each $j \in I$. Thus by 2.1, $h = \bigvee h_j$. Since $g \geqq g \wedge h_j \geqq g_j$ and since $g \wedge h_j \in c_0(G_j)$, we have $g \wedge h_j = g_j$.

Therefore

$$g = g \wedge h = g \wedge (\bigvee h_j) = \bigvee(g \wedge h_j) = \bigvee g_j .$$

Hence (ii) holds.

Let $0 \leqq g_j \in c_0(G_j)$ for each $j \in I$. Then for each $j \in I$ there is $h_j \in G_j$ with $g_j \leqq h_j$. According to 2.1 there exists $\bigvee h_j = h$ in $G$. Hence the set $\{g_j\}_{j \in I}$ is upper bounded in $D(G)$ and so $\bigvee g_j$ exists in $D(G)$. Therefore (iii) is fulfilled.

Remark 1. According to 2.6 we can also write $D(G) = \prod^0_{i \in I} D(G_i)$.

Remark 2. In [6] it was shown that if $G_i$ $(i \in I)$ are integrally closed directed groups, then $D(\prod G_i)$ is isomorphic with $\prod(D(G_i))$.

The proof of the following proposition is similar to that of Prop. 2.7.

**2.8. Proposition.** *Let $G$ be an archimedean lattice ordered group and let $G = \sum^0_{i \in I} G_i$. Then $D(G) = \sum^0_{i \in I} c_0(G_i)$.*

**2.9. Lemma.** *Let $G$ be a lattice ordered group and let $H$ be a direct factor of $G$. Then $c_1(H)$ is a direct factor of $D_1(G)$.*

Proof. Let $0 \leqq x_0 \in D_1(G)$. There are elements $x' \in G^+$, $a \in D(A(G))^+$ with $x_0 = x' + a$. Denote $x_1 = x'(H)$, $x_2 = x'(H^\delta)$. Then $x' = x_1 + x_2$, $x_1 \wedge x_2 = 0$.

According to [10], Thm. 2.18, $A$ is a closed convex $l$-subgroup of $G$ and hence by Lemma 2.4 we have

$$A = (A \cap H) \oplus (A \cap H^\delta),$$

thus according to Prop. 2.7

$$D(A) = c_0(A \cap H) \oplus c_0(A \cap H^\delta), \tag{1}$$

where $c_0(A \cap H)$ is the convex $l$-subgroup of $D(A)$ generated by the set $A \cap H$, and analogously for $c_0(A \cap H^\delta)$. Clearly

$$c_0(A \cap H) \subseteq c_1(H), \quad c_0(A \cap H^\delta) \subseteq c_1(H^\delta). \tag{2}$$

From (1) it follows that $a = a_1 + a_2$ with $0 \leqq a_1 \in c_0(A \cap H)$, $0 \leqq a_2 \in c_0(A \cap H^\delta)$, thus $a_1 \wedge a_2 = 0$. By (2), $a_1 \in c_1(H)$, $a_2 \in c_1(H^\delta)$, hence $x_1 \wedge a_2 = 0$, $x_2 \wedge a_1 = 0$. Thus $x_2 + a_1 = a_1 + x_2$ and so $x_0 = x_1 + a_1 + x_2 + a_2$. Because $(x_1 + a_1) \wedge (x_2 + a_2) = 0$, we get

$$x_0 = (x_1 + a_1) \vee (x_2 + a_2). \tag{3}$$

Clearly $x_1 + a_1 \in c_1(H)$, $x_2 + a_2 \in c_1(H^\delta)$. Let $0 \leqq x'' \in c_1(H)$, $x'' \leqq x_0$. Then $x'' \wedge (x_2 + a_2) = 0$, thus from (3) we obtain

$$x'' = x'' \wedge x_0 = x'' \wedge (x_1 + a_1).$$

Hence $x_1 + a_1$ is the greatest element of the set $\{0 \leqq h_1 \in c_1(H) : h_1 \leqq x_0\}$. Therefore in view of (a), $c_1(H)$ is a direct factor of $D_1(G)$.

**2.10. Proposition.** *Let $G$ be a lattice ordered group, $G = \prod_{i\in I}^0 G_i$. Then $D_1(G) = \prod_{i\in I}^0 c_1(G_i)$.*

Proof. According to Lemma 2.9, each $c_1(G_i)$ is a direct factor of $D_1(G)$. We have to verify that the conditions (i), (ii) and (iii) from Lemma 2.1 are satisfied for the system $\{c_1(G_i)\}_{i\in I}$ in $D_1(G)$. If $j, k \in I$ are distinct, then $G_j \cap G_k = \{0\}$ and hence $c_1(G_j) \cap c_1(G_k) = \{0\}$. Thus (i) holds. Let $0 \leqq g \in D_1(G)$. Suppose that $g = \bigvee_{j\in J} g(c_1(G_j))$ does not hold. Then there is $x_1 \in D_1(G)$ with $0 \leqq x_1 < g$ such that $g(c_1(G_j)) \leqq x_1$ is valid for each $j \in J$. Put $x_0 = -x_1 + g$. There is $0 < x \in G$ with $x \leqq x_0$. Since $G = \prod_{i\in I}^0 G_i$, there is $i \in I$ such that $x(G_i) > 0$. Hence $x(c_1(G_i)) > 0$ and thus

$$g(c_1(G_i)) < g(c_1(G_i)) + x(c_1(G_i)) \leqq x_1 + x \leqq g .$$

Since $g(c_1(G_i)) + x(c_1(G_i)) \in c_1(G_i)$, in view of (a) we must have $g(c_1(G_i)) + x(c_1(G_i)) \leqq g(c_1(G_i))$, which is a contradiction. Therefore (ii) is valid.

Let $0 \leqq g_i \in c_1(G_i)$ for each $i \in I$. There are elements $0 \leqq x_i \in G$, $0 \leqq a_i \in D(A)$ with $g_i = x_i + a_i$. Further, there are elements $y_i \in G_i$ with $g_i \leqq y_i$. Hence $x_i, a_i \in G_i$ for each $i \in I$. From $G = \prod_{i\in I}^0 G_i$ and from (iii) it follows that there exists $x = \bigvee_{i\in I} x_i$ in $G$. Let $i \in I$ be fixed. According to Lemma 2.4 and Prop. 2.7 we have

$$D(A) = \prod_{j\in I}^0 c_0(A \cap G_j) , \tag{4}$$

where the symbol $c_0$ has the same meaning as in the proof of Lemma 2.9. Hence

$$a_i = \bigvee_{j\in I} a_i(c_0(A \cap G_j)) .$$

Since $a_i \in c_1(G_i)$ we have $a_i(c_0(A \cap G_j)) \in c_1(G_i)$. From this and from the relations $a_i(c_0(A \cap G_j)) \in c_1(G_j)$, $c_1(G_i) \cap c_1(G_j) = \{0\}$ we obtain $a_i(c_0(A \cap G_j)) = 0$ for each $j \in I$, $j \neq i$. This implies that $a_i = a_i(c_0(A \cap G_i)) \in c_0(A \cap G_i)$. Thus according to (4) and Lemma 2.1 there exists $a \in D(A)$ with $a = \bigvee_{i\in I} a_i$.

We have

$$x + a = (\bigvee_{i\in I} x_i) + a = \bigvee_{i\in I} (x_i + a) = \bigvee_{i\in I} \bigvee_{j\in I} (x_i + a_j) .$$

If $i, j \in I$, $i \neq j$, then $x_i \wedge a_j = 0$, thus

$$x_i + a_j = x_i \vee a_j \leqq (x_i + a_i) \vee (x_j + a_j) .$$

Therefore

$$x + a = \bigvee_{i\in I} (x_i + a_i) = \bigvee_{i\in I} g_i .$$

Hence (iii) is valid and the proof is complete.

**2.11. Proposition.** *Let $G$ be a lattice ordered group, $G = \sum_{i\in I}^0 G_i$. Then $D_1(G) = \sum_{i\in I}^0 c_1(G_i)$.*

Proof. According to Lemma 2.9, each $c_1(G_i)$ is a direct factor of $D_1(G)$. Analogously as in the proof of Prop. 2.10 we can verify that the conditions (i) and (ii) hold (we use Prop. 2.8 instead of Prop. 2.7). Let $x_0 \in D_1(G)$. There are elements $x \in G$, $a \in D(A)$ with $x_0 = x + a$. Both the sets $\{i \in I : x(G_i) \neq 0\}$ and $\{i \in I : a(c_0(A \cap G_i)) \neq 0\}$ are finite. For each $i \in I$ we have $x(G_i) = x(c_1(G_i))$, $a(c_1(G_i)) = a(c_0(A \cap G_i))$. Hence $x_0(c_1(G_i)) = x(G_i) + a(c_0(A \cap G_i))$ and thus the set $\{i \in I : x_0(c_1(G_i)) \neq 0\}$ is finite as well.

A lattice ordered group $G$ is said to be epiarchimedean, if each homomorphic image of $G$ is archimedean. Epiarchimedean lattice ordered groups were investigated by CONRAD [4].

Let $x \in G$. The least convex $l$-subgroup of $G$ containing the element $x$ will be denoted by $c(x)$; it is said to be a principal convex $l$-subgroup of $G$. Similarly, if $x_0 \in D_1(G)$, then we put $c_1(x_0) = c_1(\{x_0\})$. The following result has been proved in [3]:

**2.12. Theorem.** *A lattice ordered group G is epiarchimedean if and only if each principal convex l-subgroup of G is a direct factor of G.*

For $Y \subseteq D_1(G)$ we denote

$$Y^\beta = \{g \in D_1(G) : |g| \wedge |y| = 0 \text{ for each } y \in Y\}\,.$$

A set $S \neq \emptyset$ of strictly positive elements of $G$ will be said to be *disjoint* if $s_1 \wedge s_2 = 0$ for each pair of distinct elements $s_1, s_2$ of $S$. The lattice ordered group $G$ is called (*conditionally*) *orthogonally complete* if each (upper bounded) disjoint subset of $G$ possesses the least upper bound in $G$.

**2.13. Theorem.** *Let G be an epiarchimedean lattice ordered group. Suppose that G is conditionally orthogonally complete. Then D(G) is epiarchimedean as well.*

Proof. The lattice ordered group $G$ is archimedean and hence $D(G)$ exists. Moreover, $D(G) = D_1(G)$. Let $0 < g_0 \in D(G)$. There exists $g_1 \in G$ with $g_0 \leqq g_1$. By using the Axiom of Choice we infer that there exists a disjoint subset $S$ of $G$ such that (i) $s \leqq g_0$ for each $s \in S$, and (ii) if $0 < h_3 \in G$, $h_3 \wedge s = 0$ for each $s \in S$, then $h_3 \wedge g_0 = 0$. The least upper bound of $S$ in $G$ will be denoted by $g_2$. Clearly $g_2 \leqq g_0$. From the construction of $g_2$ it follows that

$$\{g_2\}^\beta = \{g_0\}^\beta\,. \tag{5}$$

Since $G$ is epiarchimedean, $c(g_2)$ is a direct factor of $G$. Thus according to Prop. 2.6, $c_1(c(g_2))$ is a direct factor of $D(G)$. Clearly $c_1(c(g_2)) = c_1(g_2)$. Further, we have

$$c_1(g_2)^\beta = \{g_2\}^\beta\,. \tag{6}$$

From (5) and (6) we obtain

$$c_1(g_2) = \{g_2\}^{\beta\beta} = \{g_0\}^{\beta\beta}\,.$$

Since $g_0 \in \{g_0\}^{\beta\beta}$, we have $c_1(g_0) \subseteq c_1(g_2)$. On the other hand, $g_2 \leqq g_0$ yields $c_1(g_2) \subseteq c_1(g_0)$. Hence $c_1(g_0) = c_1(g_2)$. Therefore $c_1(g_0)$ is a direct factor of $D(G)$. Thus according to Thm. 2.12, $D(G)$ is epiarchimedean.

Remark. It can be shown by examples that if $G$ is epiarchimedean, then $D(G)$ need not be epiarchimedean. (Cf. Example 6.4 below.)

The following remark will be useful in the sequel: if $X$ is a lattice ordered group and if $Y_1$, $Y_2$ are $l$-subgroups of $X$ with $Y_1^+ \subseteq Y_2^+$, then $Y_1 \subseteq Y_2$.

**2.14. Lemma.** *Let $G$ be a lattice ordered group, $G = \prod_{i\in I}^0 G_i$. Then $c_1(G_i) \cap G = G_i$ for each $i \in I$.*

Proof. Let $i \in I$. We have to verify that $c_1(G_i) \cap G \subseteq G_i$. Let $0 < x \in c_1(G_i) \cap G$. Then $x = \bigvee_{j\in I} x(G_j)$, $x(G_j) \geqq 0$, hence $x(G_j) \in c_1(G_i)$ for each $j \in I$. If $j \neq i$, then $x(G_j) \in G_j \subseteq c_1(G_j)$; according to Prop. 2.11 we have $c_1(G_i) \cap c_1(G_j) = \{0\}$, thus $x(G_j) = 0$. Therefore $x = x(G_i) \in G_i$.

**2.15. Lemma.** *Let $G$ be a lattice ordered group, $G = \prod_{i\in I}^0 G_i$. Then $D(A(G_i)) = c_1(G_i) \cap D(A)$.*

Proof. Clearly $A(G_i) = A \cap G_i$. According to 2.4 we have $A = \prod_{i\in I}^0 (A \cap G_i)$, hence $A = \prod_{i\in I}^0 A(G_i)$. In view of Prop. 2.7 we obtain $D(A) = \prod_{i\in I}^0 D(A(G_i))$. Thus $D(A(G_i)) \subseteq D(A)$. Let $0 \leqq x \in D(A(G_i))$. There exists $y \in A(G_i)$ with $x \leqq y$. Then $y \in G_i$, hence $x \in c_1(G_i)$ and therefore

$$D(A(G_i)) \subseteq c_1(G_i) \cap D(A).$$

Let $0 < x \in c_1(G_i) \cap D(A)$. There exists a subset $\{a_k\} \subseteq A$ and an element $a \in A$ such that $0 \leqq a_k \leqq a$ holds for each $a_k$, and $\bigvee a_k = x$ is valid in $D(A)$. From the convexity of $c_1(G_i)$ we obtain $a_k \in c_1(G_i)$ and hence, in view of Lemma 2.14, $a_k \in G_i$ for each $a_k$. Moreover, $a_k = a_k(G_i) \leqq a(G_i)$, hence $\{a_k\}$ is an upper bounded subset of $A(G_i)$. Thus $\{a_k\}$ is an upper bounded subset of $D(A(G_i))$. Since $D(A(G_i))$ is a direct factor of $D(A)$, it is a closed $l$-subgroup of $D(A)$ and hence $x \in D(A(G_i))$. Therefore

$$c_1(G_i) \cap D(A) \subseteq D(A(G_i)).$$

**2.16. Lemma.** *Let $G$ be a lattice ordered group, $G = \prod_{i\in I}^0 G_i$. Then $c_1(G_i) = D_1(G_i)$ for each $i \in I$.*

Proof. Let $i \in I$ be fixed. We have to verify that the conditions (i)–(iv) from the definition of $D_1(G)$ (cf. § 1) are fulfilled with $G$ and $D_1(G)$ replaced by $G_i$ and $c_1(G_i)$, respectively. The validity of (i) is obvious. From Lemma 2.15 it follows that (ii) holds.

Let $0 < y_0 \in c_1(G_i)$. There are elements $0 \leqq y \in G$, $0 \leqq a \in D(A)$ with $y_0 = y + a$. By the convexity of $c_1(G_i)$, both $y$ and a belong to $c_1(G_i)$. According to Lemma 2.14 we have $y \in G_i$. Further, from Lemma 2.15 we obtain $a \in D(A(G_i))$.

Now let $x_0 \in c_1(G_i)$. There are elements $y_0, z_0 \in (c_1(G_i))^+$ with $x_0 = y_0 - z_0$. Let $y, a$ be as above. Analogously, there are elements $z \in G_i$ and $a_1 \in D(A(G_i))$ with $z_0 = z + a_1$. Further, there is $a_2 \in A(G_i)$ such that $a_1 \leqq a_2$. Put $a_3 = a_2 - a_1$. Then we have $a_3 \in D(A(G_i))$, $a_3 \geqq 0$, $z_0 = z_1 - a_3$, $z_1 = z + a_2 \in G_i$. Hence

$$x_0 = y + a + a_3 - z_1 = y - z_1 + a_4$$

with $0 \leqq a_4 \in D(A(G_i))$. Hence there exists an upper bounded subset $\{a_k\}$ of $A(G_i)$ with $a_4 = \bigvee a_k$ (holding in $D(A(G_i))$, and hence also in $D_1(G)$). Thus $\{y - z_2 + a_k\}$ is an upper bounded subset of $y - z_1 + A(G_i)$ and $x_0 = \bigvee(y - z_1 + a_k)$. Therefore (iv) is valid.

Let $x \in G_i$ and let $\{x_k\}$ be an upper bounded subset of $x + A(G_i)$. Hence $\{x_k\}$ is an upper bounded subset in $x + A$. Thus the least upper bound $x_0$ of $\{x_k\}$ in $D_1(G)$ exists. Since $c_1(G_i)$ is convex in $D_1(G)$, the element $x_0$ must belong to $c_1(G_i)$. Hence the condition (iii) holds.

From Prop. 2.10 and Lemma 2.16 we obtain

**2.17. Theorem.** *Let $G$ be a lattice ordered group, $G = \prod_{i \in I}^0 G_i$. Then $D_1(G) = \prod_{i \in I}^0 D_1(G_i)$.*

Analogously we can verify the following assertion:

**2.18. Proposition.** *Let $G$ be a lattice ordered group, $G = \sum_{i \in I}^0 G_i$. Then $D_1(G) = \sum_{i \in I}^0 D_1(G_i)$.*

A lattice ordered group $G$ is said to be *projectable* (*strongly projectable*) if each principal polar (each polar) of $G$ is a direct factor of $G$.

**2.19. Theorem.** *Let $G$ be a strongly projectable lattice ordered group. Then $D_1(G)$ is strongly projectable.*

Proof. Let $X_0 \subseteq D_1(G)$. We have to verify that $X_0^\beta$ is a direct factor of $D_1(G)$. Without loss of generality we can assume that $X_0 \subseteq (D_1(G))^+$. Put $X = \{x \in G : 0 \leqq x \leqq x_0$ for some $x_0 \in X_0\}$. In [10] (Proof of 3.4) it has been shown that $X_0^\beta = X^\beta$ is the set of all $y \in D_1(G)$ with the property that there is a subset $\{y_i\} \subseteq (X^\delta)^+$ with $|y| = \bigvee y_i$.

Since $G$ is strongly projectable, we have

$$G = X^{\delta\delta} \oplus X^\delta$$

and hence, in view of Prop. 2.10,

$$D_1(G) = c_1(X^{\delta\delta}) \oplus c_1(X^\delta)\,.$$

It suffices to verify that $X_0^\beta = c_1(X^\delta)$.

Because $X^\delta \subseteq X^\beta = X_0^\beta$, we have $c_1(X^\delta) \subseteq c_1(X_0^\beta) = X_0^\beta$, hence $c_1(X^\delta) \subseteq X_0^\beta$.

Let $0 \leqq z \in X_0^\beta$. There is a subset $\{z_i\} \subseteq (X^\delta)^+$ such that $z = \bigvee z_i$ holds in $D_1(G)$.

We have $\{z_i\} \subseteq c_1(X^\delta)$ and since $c_1(X^\delta)$ is a direct factor of $D_1(G)$, it is a closed $l$-subgroup of $D_1(G)$. Thus $z \in c_1(X^\delta)$ and hence $X_0^\beta \subseteq c_1(X^\delta)$.

**2.20. Theorem.** *Let $G$ be a projectable lattice ordered group. Suppose that $A(G)$ is strongly projectable. Then $D_1(G)$ is projectable.*

Proof. Let $g_0 \in D_1(G)$. We have to verify that $\{g_0\}^{\beta\beta}$ is a direct factor of $D_1(G)$. Since $\{g_0\}^{\beta\beta} = \{|g_0|\}^{\beta\beta}$, we may assume that $g_0 \geqq 0$. There are elements $0 \leqq g \in G$, $0 \leqq a \in D(A)$ with $g_0 = g + a$. The $l$-subgroup $\{g_0\}^{\beta\beta}$ is a direct factor of $D_1(G)$ if and only if $\{g_0\}^\beta$ is a direct factor of $D_1(G)$.

Since $G$ is projectable, we have

$$G = \{g\}^{\delta\delta} \oplus \{g\}^\delta .$$

This implies by 2.10

$$D_1(G) = c_1(\{g\}^{\delta\delta}) \oplus c_1(\{g\}^\delta) .$$

Denote $c_1(\{g\}^{\delta\delta}) = F_1$, $c_1(\{g\}^\delta) = F_2$. Then $g$ is a weak unit in $F_1$. Put $a_i = a(F_i)$ $(i = 1, 2)$, $g_1 = g + a_1$. Clearly $g_1 \in F_1$, $a_2 \in D(A)$.

There is $a_3 \in A$ with $a_2 \leqq a_3$. Put $X = [0, a_2] \cap A$ (the interval $[0, a_2]$ being taken with respect to $D(A)$). Because $A$ is strongly projectable, we obtain

$$A = X^{\delta\delta} \oplus X^\delta .$$

Denote $a_3(X^{\delta\delta}) = a_4$. Since $a_4 \in A \subseteq G$, $\{a_4\}^{\delta\delta}$ is a direct factor of $G$ and thus $F_3 = c_1(\{a_4\}^{\delta\delta})$ is a direct factor of $D_1(G)$. It is not difficult to verify that $a_2$ is a weak unit in $F_3$. From this we infer that $F_3 \subseteq F_2$. Hence there is a direct factor $F_4$ of $D_1(G)$ such that $F_2 = F_3 \oplus F_4$; thus

$$D_1(G) = F_1 \oplus F_3 \oplus F_4 .$$

Let $0 \leqq h \in \{g_0\}^\beta$. Hence $g_0 \wedge h = 0$ and thus $g \wedge h = 0$, $a_2 \wedge h = 0$. Since $g$ and $a_2$ are weak units in $F_1$ and $F_3$, respectively, we have $h(F_1) = 0 = h(F_3)$. Thus $h = h(F_4) \in F_4$ Therefore $\{g_0\}^\beta \subseteq F_4$.

Conversely, let $0 \leqq h \in F_4$. Then $t \wedge h = 0$ for each $0 \leqq t \in F_1 \oplus F_3$. By putting $t = g_1 + a_2 = g_0$ we obtain $g_0 \wedge h = 0$ and hence $h \in \{g_0\}^\beta$. Thus $F_4 \subseteq \{g_0\}^\beta$. Therefore $\{g_0\}^\beta = F_4$ is a direct factor of $D_1(G)$.

If both $G$ and $A(G)$ are projectable lattice ordered groups, then $D_1(G)$ need not be projectable (cf. Example 6.2 below).

The following result has been obtained by Rotkovič [13].

(*) *Let $G$ be a conditionally orthogonally complete archimedean lattice ordered group. Then $G$ is projectable.*

**2.21. Theorem.** *Let $G$ be a conditionally orthogonally complete lattice ordered group. Then $D_1(G)$ is conditionally orthogonally complete.*

Proof. Let $Z = \{z_i\}_{i \in I}$ be a bounded disjoint subset of $D_1(G)$. Let $z_1 \in D_1(G)$ be an upper bound of $Z$. There exists $z \in G$ with $z_1 \leqq z$. For each $i \in I$ there are elements $x_i \in G$, $a_i \in D(A)$ such that $0 \leqq x_i$, $0 \leqq a_i$, $z_i = x_i + a_i$. If $i, j \in I$, $i \neq j$, then $x_i \wedge x_j = 0 = a_i \wedge a_j$. Hence there exists $x = \bigvee x_i$ in $G$.

Let $i \in I$ be fixed. If $a_i = 0$, we put $X_i = \{0\}$. If $a_i > 0$, then we choose a maximal disjoint subset $X_i$ of the set $[0, a_i] \cap A$. The set $X_i$ is upper bounded in $G$, hence there exists $c_i = \sup X_i$ in $G$. Since $A$ is a closed $l$-subgroup of $G$, we have $c_1 \in A$. If $i, j$ are distinct elements of $I$, then $c_i \wedge c_j = 0$. Because $A$ is a convex $l$-subgroup of $G$, it is conditionally orthogonally complete and hence, according to (*), it is projectable. Let $D_i$ be the principal polar in $A$ generated by the element $c_i$; thus $D_i$ is a direct factor of $A$. We denote by $E_i$ the convex $l$-subgroup of $D(A)$ generated by the set $D_i$. By 2.10, $E_i$ is a direct factor of $D(A)$.

For each $i \in I$ there is $b_i \in A$ with $a_i \leqq b_i \leqq z$. Denote $d_i = b_i(D_i)$. Then $0 \leqq d_i \leqq z$ for each $i \in I$ and $d_i \wedge d_j = 0$ whenever $i, j$ are distinct elements of $I$. Hence there is $d = \bigvee d_i$ in $G$; since $A$ is closed in $G$, we have $d \in A$.

If $a_i(E_i) < a_i$, then there is $0 < a \in A$ with $a \leqq a_i - a_i(E_i)$; but then $0 < a' = x \wedge a$ for some $x \in X_i$ and hence $a' \in D_i \subseteq E_i$, thus $a_i(E_i) < a_i(E_i) + a' \leqq a_i$ and $a_i(E_i) + a' \in E_i$, which is a contradiction. Hence $a_i = a_i(E_i) \in E_i$. We have $t(E_i) = t(D_i)$ for each $t \in A$. Thus

$$a_i = a_i(E_i) \leqq b_i(E_i) = b_i(D_i) = d_i \leqq d\,.$$

Hence it follows that $a = \bigvee a_i$ exists in $D(A)$. Put $z_0 = x + a$. Clearly $z_i \leqq z_0$ for each $i \in I$. In the same way as in the proof of 2.10 we can now verify that $z_0 = \bigvee z_i$. Hence $D_1(G)$ is conditionally orthogonally complete.

**2.22. Theorem.** *Let $G$ be an orthogonally complete lattice ordered group. Then $D_1(G)$ is orthogonally complete.*

The proof is analogous to that of 2.22.

## 3. PAIRWISE SPLITTING LATTICE ORDERED GROUPS

Let $G$ be a lattice ordered group, $0 \leqq x$, $y \in G$. We write $x \ll y$ if $nx \leqq y$ for each positive integer $n$. We say that $x$ splits by $y$ if there are elements $x_1, x_2 \in G$ such that $x = x_1 + x_2$, $x_1 \wedge x_2 = 0$, $x_1 \in c(y)$ and $x_2 \wedge y \ll x_2$.

Let us consider the following condition for $G$:

(p) For each pair $0 \leqq x$, $y \in G$, the element $x$ splits by $y$.

A lattice ordered group $G$ fulfilling (p) is said to be *pairwise splitting*; lattice ordered groups with this property were investigated by Martinez [12]. It is easy to verify that an archimedean lattice ordered group is pairwise splitting if and only if it is epiarchimedean. Let $\mathscr{P}$ be the class of all pairwise splitting lattice ordered groups. If $G$ is pairwise splitting, then each convex $l$-subgroup of $G$ is pairwise splitting.

**3.1. Lemma.** *Let G be a pairwise splitting abelian lattice ordered group. Suppose that* $A = A(G)$ *is conditionally orthogonally complete. Let* $0 \leqq x \in G$, $0 \leqq y_0 \in D_1(G)$. *Then x splits by* $y_0$ *in* $D_1(G)$.

Proof. There are elements $y \in G$, $b \in D(A)$ such that $0 \leqq y$, $0 \leqq b$, $y_0 = y + b$. If $b = 0$, then $x$ splits by $y_0$. Suppose that $b > 0$. There exists $b_1 \in A$ with $b \leqq b_1$. From the Axiom of Choice it follows that there exists a disjoint subset $\{b_i\}$ of $A$ such that

(i) $b_i \leqq b$ for each $b_i$,

(ii) if $0 < a_1 \in A$, $a_1 \leqq b$, then $a_1 \wedge b_i > 0$ for some $b_i$.

The set $\{b_i\}$ is upper bounded in $A$, hence there exists $\bigvee b_i = b_2$ in $A$ and by (i), $b_2 \leqq b$.

Since $A$ is a convex $l$-subgroup of $G$ and because $\mathscr{P}$ is a torsion class, $A$ must be pairwise splitting and hence $A$ is epiarchimedean. Thus by 2.13, $D(A)$ is epiarchimedean. Hence $c(b_2)$ is a direct factor of $A$ and $c_1(c(b_2)) = c_1(b_2)$ is a direct factor of $D(A)$. From (i) and (ii) it follows that $c_1(b)^\beta = c_1(b_2)^\beta$. Hence we obtain (because $D(A)$ is epiarchimedean)

$$c_1(b_2) = c_1(b)\,.$$

Put $b_3 = b_1(c_1(b_2))$. From the construction of the convex $l$-subgroup $c_1(b_2)$ it follows that $b_3 = \bigvee_{m \geqq 0} (mb_2 \wedge b_1)$ and that there exists a positive integer $n$ with $b_3 \leqq nb_2 \wedge b_1$. Thus $b_3 = nb_2 \wedge b_1$, hence $b_3 \in G$. We have $b \leqq b_1$ and thus

$$b = b(c_1(b)) = b(c_1(b_2)) \leqq b_1(c_1(b_2)) = b_3\,.$$

This implies that $c_1(b) = c_1(b_3)$. From this and from the commutativity of $G$ we get $c_1(y_0) = c_1(y + b) = c_1(y + b_3)$.

Since $y + b_3 \in G$, the element $x$ splits by $y + b_3$. Thus there are elements $x_1, x_2 \in G$ such that $x = x_1 + x_2$, $x_1 \wedge x_2 = 0$, $x_1 \in c(y + b_3)$, $(y + b_3) \wedge x_2 \ll x_2$. Therefore $x_1 \in c_1(y_0)$, $y_0 \wedge x_2 \ll x_2$. Hence $x$ splits by $y_0$ in $D_1(G)$.

**3.1.1. Corollary.** *Let G be as in* 3.1. *Let* $0 \leqq a \in D(A)$, $0 \leqq y_0 \in D_1(G)$. *Then a splits by* $y_0$.

Proof. There is $b \in A$ with $a \leqq b$. According to Lemma 3.1, $b$ splits by $y_0$. Hence there are elements $b_1, b_2 \in D_1(G)$ such that $b = b_1 + b_2$, $b_1 \wedge b_2 = 0$, $b_1 \in c(y_0)$ and $b_2 \wedge y_0 \ll b_2$. Since $D(A)$ is archimedean, $b_2 \wedge y_0 = 0$. Put $a_1 = b_1 \wedge a$, $a_2 = b_2 \wedge a$. Then $a = a_1 + a_2$, $a_1 \wedge a_2 = 0$, $a_1 \in c(y_0)$, $a_2 \wedge y_0 = 0$. Hence $a$ splits by $y_0$.

**3.2. Lemma.** *Let G be a pairwise splitting abelian lattice ordered group. Suppose that* $A(G)$ *is conditionally orthogonally complete. Let* $0 \leqq x \in D_1(G)$, $0 \leqq z \in D_1(G)$, $0 \leqq a \in D(A)$, $z \ll x + a$. *Then* $z \ll x$.

Proof. According to Corollary 3.1.1, the element $a$ splits by $z$. Thus there are elements $a_1, a_2 \in D_1(G)$ such that $a = a_1 + a_2$, $a_1 \wedge a_2 = 0$, $a_1 \in c_1(z)$, $a_2 \wedge z \ll$

$\ll a_2$. Then $a_1, a_2 \in D(A)$ and since $D(A)$ is archimedean, we have $a_2 \wedge z = 0$. Hence $a_2 \wedge nz = 0$ for each positive integer $n$.

Since $nz \leqq x + a_1 + a_2$, there are elements $0 \leqq z_1$, $z_2 \in D_1(G)$ with $nz = z_1 + z_2$, $0 \leqq z_1 \leqq x + a_1$, $0 \leqq z_2 \leqq a_2$. If $z_2 > 0$, then $a_2 \wedge nz \geqq z_2 > 0$, a contradiction. Thus $z_2 = 0$, $nz \leqq x + a_1$ for each positive integer $n$. If $a_1 = 0$, then the assertion of the lemma is valid; suppose that $a_1 > 0$.

There exists a maximal disjoint subset $\{a_i\} \subset A$ with $a_i \leqq a_1$. The set $\{a_i\}$ is upper bounded in $A$, hence there exists $\bigvee a_i = a_3$ in $A$ and $a_3 \leqq a_1$. From the construction of $a_3$ it follows that $c_1(a_3)^\beta = c_1(a_1)^\beta$; from this and from the fact that $D(A)$ is epiarchimedean we obtain $c_1(a_3) = c_1(a_1)$. Hence there is a positive integer $n_1$ with $n_1 a_3 \geqq a_1$.

Since $a_1 \in c_1(z)$, there is a positive integer $m$ with $a_1 \leqq mz$. Then for each positive integer $n$ we have

$$(n + m) z \leqq x + a_1 ,$$

$$nz \leqq x + a_1 - mz \leqq x .$$

**3.3. Theorem.** *Let $G$ be a pairwise splitting abelian lattice ordered group. Suppose that $A(G)$ is conditionally orthogonally complete. Then $D_1(G)$ is pairwise splitting.*

Proof. Let $0 \leqq x_0$, $y_0 \in D_1(G)$. There are elements $0 \leqq x \in G$, $0 \leqq a_1 \in D(A)$ with $x_0 = x + a_1$. Further, there is $a \in A$ such that $a_1 \leqq a$. Then $x + a \in G$ and hence, according to Lemma 3.1, $x + a$ splits by $y_0$. Hence there are elements $x_1, x_2 \in G$ with $x + a = x_1 + x_2$, $x_1 \wedge x_2 = 0$, $x_1 \in c_1(y_0)$, $x_2 \wedge y_0 \ll x_2$. Denote $x_1' = x_1 \wedge x_0$, $x_2' = x_2 \wedge x_0$. We have $x_1' \wedge x_2' = 0$ and

$$x_0 = x_0 \wedge (x_1 + x_2) = x_0 \wedge (x_1 \vee x_2) = (x_0 \wedge x_1) \vee (x_0 \wedge x_2) =$$

$$= x_1' \vee x_2' = x_1' + x_2' , \quad x_1' \in c_1(y_0) , \quad x_2' \wedge y_0 \leqq x_2 \wedge y_0 \ll x_2 .$$

Since $x + a \in x_0 + D(A)$, we have $x_2 = (x + a) \wedge x_2 \in x_0 \wedge x_2 + D(A)$. Hence there is $0 \leqq a_2 \in D(A)$ such that $x_2 = x_2' + a_2$. We have $x_2' \wedge y_0 \ll x_2' + a_2$, thus according to Lemma 3.2, $x_2' \wedge y_0 \ll x_2'$. Therefore $x_0$ splits by $y_0$.

**Problem.** Does the assertion of Thm. 3.3 remain valid without the assumption of commutativity of $G$?

## 4. THE $\alpha$-DISTRIBUTIVITY

Let $\alpha$ be a cardinal and let $L$ be a lattice. Consider the following condition for $L$:

($\alpha$) If $\{x_{t,s}\}_{t\in T, s\in S}$ is a subset of $L$ such that both $\bigwedge_{t\in T} \bigvee_{s\in S} x_{t,s}$ and $\bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x_{t,\varphi(t)}$ exist in $L$ and if card $T \leqq \alpha$, card $S \leqq \alpha$, then

$$\bigwedge_{t\in T} \bigvee_{s\in S} x_{t,s} = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x_{t,\varphi(t)} .$$

If $L$ fulfils the condition $(\alpha)$ and the condition dual to $(\alpha)$, then it is said to be *$\alpha$-distributive.* $L$ is called *completely distributive* if it is $\alpha$-distributive for each cardinal $\alpha$.

Let $\beta$ be a cardinal. If $L$ is $\beta_1$-distributive for each cardinal $\beta_1 < \beta$ and if $L$ fails to be $\beta$-distributive, then we write $d(L) = \beta$.

Let $G$ be a lattice ordered group. It is easy to verify that $G$ is $\alpha$-distributive if it fulfils $(\alpha)$.

The following assertion is easy to verify.

**4.1. Lemma.** *Let $G$ be a lattice ordered group and let $\alpha$ be an infinite cardinal. Suppose that $G$ fails to be $\alpha$-distributive. Then there is $0 < v \in G$ such that for each $0 < v_1 \in G$ with $v_1 \leqq v$, the interval $[0, v_1]$ of $G$ fails to be $\alpha$-distributive.*

We need the following result:

(A) (Cf. [7].) Let $\alpha$ be an infinite cardinal and let $G$ be an archimedean lattice ordered group. Suppose that card $[0, v] \leqq \alpha$ for each strictly positive element $v$ of $G$. Assume that $G$ is $\alpha$-distributive. Then $D(G)$ is $\alpha$-distributive.

**4.2. Proposition.** *Let $G$ be a lattice ordered group. Suppose that $G$ is completely distributive. Then $D_1(G)$ is completely distributive.*

Proof. This follows from Prop. 1.7 of the paper [15] and from the fact that each element of $D_1(G)$ is the supreum of a certain family of elements of $G$ (cf. the condition (iv) in § 1).

**4.3. Theorem.** *Let $\alpha$ be an infinite cardinal and let $G$ be a lattice ordered group. Suppose that* card $[0, v] \leqq \alpha$ *for each strictly positive element of $A(G)$. Assume that $G$ is $\alpha$-distributive. Then $D_1(G)$ is $\alpha$-distributive.*

Proof. Since $G$ is $\alpha$-distributive, $A(G) = A$ must be $\alpha$-distributive as well. Thus (A) implies that $D(A)$ is $\alpha$-distributive.

Assume that $D_1(G)$ is not $\alpha$-distributive. By Lemma 4.1 there is $0 < v \in D_1(G)$ such that the interval $[0, v_1]$ of $D_1(G)$ fails to be $\alpha$-distributive for each $0 < v_1 \in D_1(G)$ with $v_1 \leqq v$.

We distinguish two cases. First suppose that there exists $0 < a \in A$ with $a \leqq v$. Then the interval $[0, a]$ of $D_1(G)$ is a sublattice of $D(A)$ and hence it is $\alpha$-distributive, which is a contradiction. Now suppose that no $0 < a \in A$ with $a \leqq v$ exists. Each element $0 < v_1 \in D_1(G)$ with $v_1 \leqq v$ can be written as $v_1 = x + a_1$, $0 \leqq x \in G$, $0 \leqq a_1 \in D(A)$. Then we have $a_1 \leqq v_1 \leqq v$, hence $a_1 = 0$ and thus $v_1 = x \in G$. Hence the interval $[0, v]$ of $D_1(G)$ is a sublattice of $G$ and so it is $\alpha$-distributive, which is a contradiction.

**4.4. Theorem.** *Let $G$ be a lattice ordered group that is not completely distributive, $d(G) = \alpha$. If $A(G)$ is completely distributive, then $d(D_1(G)) = \alpha$. If $A(G)$ is not completely distributive, $d(D(A(G))) = \beta$, then $d(D_1(G)) = \min(\alpha, \beta)$.*

Proof. From [10], Prop. 2.20 it follows that $D_1(G)$ is not $\alpha$-distributive, hence $d(D_1(G)) \leqq \alpha$. If $A(G)$ is not completely distributive, then $D(A(G))$ cannot be completely distributive; if $d(D(A(G))) = \beta$, then according to [10], Prop. 2.16, $d(D_1(G)) \leqq \beta$. Now it suffices to verify that if $\gamma$ is a cardinal and both $G$ and $D(A(G))$ are $\gamma$-distributive, then $D_1(G)$ is $\gamma$-distributive as well. To prove it we can use the same method as in the proof of 4.2.

**4.5. Theorem.** *Let $G$ be a lattice ordered group that is not completely distributive, $d(G) = \alpha$. Suppose that $A(G)$ is projectable. Then $d(D_1(G)) = \alpha$.*

Proof. If $A(G)$ is completely distributive, then the assertion is valid according to 4.4. Suppose that $d(A(G)) = \beta$. Hence $\beta \geqq \alpha$. Since $A(G)$ is projectable, from [8] we obtain $d(D(A(G))) = \beta$. Hence $d(D_1(G)) = \alpha$ by 4.4.

## 5. $g$-COMPLETE LATTICE ORDERED GROUPS

An archimedean lattice ordered group $G$ is complete if and only if $D(G) = G$. A lattice ordered group $H$ will be called *$g$-complete* (generalized complete) if $D_1(H) = H$. It was remarked in [11] that $D_1(H) = H$ if and only if $A(H)$ is complete.

The following assertion has been proved in [9]:

(A) Let $G$ be a lattice ordered group. Then there exists a convex $l$-subgroup $C(G)$ of $G$ such that

(a) $C(G)$ is complete;

(b) if $H$ is a convex $l$-subgroup of $G$ and if $H$ is complete, then $H \subseteq C(G)$.

A class $\mathscr{K}$ of lattice ordered groups is said to be *a radical class* [11] if it fulfils the following conditions:

(i) $\mathscr{K}$ is closed with respect to isomorphisms.

(ii) If $H_1$ is a convex $l$-subgroup of a lattice ordered group $H$ and if $H \in \mathscr{K}$, then $H_1 \in \mathscr{K}$.

(iii) If $H_i$ is a system of convex $l$-subgroups of a lattice ordered group $H$ and if each $H_i$ belongs to $\mathscr{K}$. then $\bigvee H_i \in \mathscr{K}$.

In this paragraph it will be shown that the class of all $g$-complete lattice ordered groups is a radical class.

**5.1. Theorem.** *Let $G$ be a lattice ordered group. There exists a convex $l$-subgroup $B_0(G)$ of $G$ such that*

(a) *$B_0(G)$ is $g$-complete,*

(b) *if $B_1$ is a convex $l$-subgroup of $G$ and if $B_1$ is $g$-complete, then $B_1 \subseteq B_0(G)$.*

Proof. Let $\{B_i\}$ be the set of all convex $l$-subgroups of $G$ fulfilling

$$B_i \cap A(G) \subseteq C(G)\,.$$

Put $B_0(G) = \bigvee B_i$. Then $B_0(G) \cap A(G) \subseteq C(G)$. Hence $A(B_0(G)) = A(G) \cap B_0(G) \subseteq \subseteq C(G)$. Since $A(B_0(G))$ is complete, $B_0(G)$ is $g$-complete.

Let $B_1$ be a convex $l$-subgroup of $G$ and suppose that $B_1$ is $g$-complete. Then $A(B_1) = A(G) \cap B_1$ is complete, hence $A(G) \cap B_1 \subseteq C(G)$ and thus $B_1 \in \{B_i\}$. Therefore $B_1 \subseteq B_0(G)$.

Remark. It is easy to verify that $B_0(G)$ is a characteristic $l$-subgroup of $G$. It can be shown by examples that $B_0(G)$ need not be a closed $l$-subgroup of $G$ (cf. Example 6.5 below).

**5.2. Theorem.** *The class $K_g$ of all g-complete lattice ordered groups is a radical class.*

Proof. $K_g$ obviously fulfils (i). Let $H \in K_g$ and let $H_1$ be a convex $l$-subgroup of $H$. Then $A(H)$ is complete and since $A(H_1) = H_1 \cap A(H)$, $A(H_1)$ is complete as well. Thus (ii) holds. Let $G$ be a lattice ordered group and let $\{G_i\}$ be a system of convex $l$-subgroups of $G$ such that each $G_i$ belongs to $K_g$. Let $B_0(G)$ be as in 5.1. Then each $G_i$ is a subset of $B_0(G)$, hence $\bigvee G_i \subseteq B_0(G)$; in view of (ii) we have $\bigvee G_i \in K_g$ and hence (iii) is valid.

**5.3. Proposition.** *Let $G$ be a lattice ordered group, $G = \prod_{i \in I}^0 G_i$. Then $G$ is g-complete if and only if all $G_i$ are g-complete.*

Proof. Assume that all $G_i$ are $g$-complete. Then according to Prop. 2.17, $G$ is $g$-complete. Conversely, suppose that $G$ is $g$-complete. Hence $A(G)$ is complete. We have $A(G) = \prod_{i \in I}^0 (A(G) \cap G_i)$ and $A(G) \cap G_i = A(G_i)$ for each $i \in I$. Since each direct factor of a complete lattice ordered group is complete, all $G_i$'s are $g$-complete.

An analogous proposition is valid for direct sums.

Let $H$ be an abelian lattice ordered group. Consider the following condition for $H$:

(I) it is possible to define a multiplication of elements of $H$ by reals so that $H$ turns out to be a vector lattice.

We denote by $\mathscr{V}_1$ the class of all archimedean lattice ordered groups fulfilling the condition (I). Further, let $\mathscr{V}_2$ be the class of all $G \in \mathscr{V}_1$ that are complete. Lattice ordered groups belonging to $\mathscr{V}_2$ are called *complete vector lattices* [1] *or K-spaces* [14]. Let us denote by $\mathscr{V}_3$ the class of all $G \in K_g$ fulfilling (I).

**5.4. Proposition.** *Both $\mathscr{V}_1$ and $\mathscr{V}_2$ are radical classes.*

Proof. $\mathscr{V}_1$ obviously fulfils the conditions (i) and (ii). Let $G$ be a lattice ordered group and let $\{H_i\}$ be a system of convex $l$-subgroups of $G$ such that each $H_i$ belongs to $\mathscr{V}_1$. Put $H = \bigvee H_i$. Each $H_i$ is a convex $l$-subgroup of $A(G)$, hence $H$ is a convex $l$-subgroup of $A(G)$ as well. From Thm. 1.3, [4] it follows that each archimedean lattice ordered group possesses a largest convex $l$-subgroup fulfilling the condition (I). We denote by $H_0$ the largest convex $l$-subgroup of $A(G)$ fulfilling (I). Since all $H_i$

are convex $l$-subgroups of $H_0$, we obtain that $H$ is a convex $l$-subgroup of $H_0$. Thus $H$ belongs to $\mathscr{V}_1$. Therefore $\mathscr{V}_1$ is a radical class. Let $\mathscr{C}$ be the class of all complete lattice ordered groups. $\mathscr{C}$ is a radical class [12] and $\mathscr{V}_2 = \mathscr{V}_1 \cap \mathscr{C}$. The intersection of two radical classes being again a radical class, $\mathscr{V}_2$ is a radical class as well.

**5.5. Corollary.** *Let $G$ be a lattice ordered group. Then $G$ possesses a largest convex $l$-subgroup $V_i(G)$ belonging to $\mathscr{V}_i$ $(i = 1, 2)$.*

**Problem.** *Is $\mathscr{V}_3$ a radical class?*

## 6. EXAMPLES

**6.1.** If a lattice ordered group $G$ is complete, then each polar of $G$ is a direct factor of $G$. A polar of a $g$-complete lattice ordered group $H$ need not be a direct factor of $H$.

Let $H$ be the set of all triples $(x, y, z)$ of reals, the operation $+$ in $H$ being defined componentwise. For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in H$ we put $(x_1, y_1, z_1) \leqq (x_2, y_2, z_2)$, if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 \leqq y_2$, $z_1 \leqq z_2$. Then $H$ is a $g$-complete lattice ordered group. The set $P$ consisting of all $(x, y, z) \in H$ with $x = z = 0$ is a polar of $H$ and $P$ fails to be a direct factor of $H$.

**6.2.** If a lattice ordered group $G$ is projectable and if $A(G)$ is projectable, then $D_1(G)$ need not be projectable.

Let $I = [0, 1]$ be the interval of reals and let $F$ be the set of all real functions $f$ defined on $I$ with the following property: there is a finite set $M(f) \subseteq I$ such that $f(x_1) = f(x_2)$ whenever $x_1, x_2 \in I \setminus M(f)$. The partial order and the operation $+$ on the set $F$ are defined in the natural way. Let $G$ be the set of all pairs $(f, g)$ with $f, g \in F$. For $(f_i, g_i) \in G$ $(i = 1, 2)$ we put $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$ and we set $(f_1, g_1) \leqq (f_2, g_2)$ if for each $x \in I$ we have either $f_1(x) < f_2(x)$, or $f_1(x) = f_2(x)$ and $g_1(x) \leqq g_2(x)$. Then $G$ is a projectable lattice ordered group. $A(G)$ consists of all elements $(0, g)$ with $g \in F$; $A(G)$ is projectable as well. Let $I_1$ be an infinite subset of $I$, $I_1 \neq I$. For each $t \in I_1$ let $f_t \in F$ such that $f_t(t) = 1$ and $f_t(x) = 0$ for $x \in I$, $x \neq t$. The least upper bound $h$ of the set $\{(0, f_t) : t \in I_1\}$ in $D_1(G)$ exists. Let $f \in F$, $f(x) = 1$ for each $x \in I$. Let $P$ be the principal polar of $D_1(G)$ generated by the element $h$. Then the set

$$\{h_1 \in P : 0 \leqq h_1 \leqq (f, 0)\}$$

has no greatest element. Hence $P$ fails to be a direct factor of $G$.

**6.3.** If a lattice ordered group $G$ is pairwise splitting, then $D_1(G)$ need not be pairwise splitting.

Let $F$ be as in 6.2. Then $F$ is pairwise splitting lattice ordered group. Put $t_n = 1/n$ $(n = 1, 2, \ldots)$. For each positive integer $n$ let $f_n \in F$ with $f_n(t_n) = 1/n$, $f_n(x) = 0$

for each $x \neq t_n$. Since $G$ is archimedean, $D_1(F) = D(F)$. The least upper bound $h$ of the set $\{f_n\}$ $(n = 1, 2, \ldots)$ in $D_1(F)$ exists. Let $f \in F$, $f(x) = 1$ for each $x \in I$. The element $f$ does not split by $h$ in $D_1(F)$. Hence $D_1(F)$ is not pairwise splitting.

**6.4.** There exists an epiarchimedean lattice ordered group $G$ such that $D(G)$ fails to be epiarchimedean.

Let $F$, $h$ be as in 6.3. The lattice ordered group $F$ is epiarchimedean and the principal convex $l$-subgroup of $D_1(F)$ generated by the element $h$ fails to be a direct factor of $D_1(G)$. Hence $D_1(F)$ is not epiarchimedean.

**6.5.** The largest $g$-complete $l$-ideal $B_0(G)$ of a lattice ordered group $G$ need not be a closed $l$-subgroup of $G$.

Let $F$ be as in 6.2. Let $F_1$ be the set of all $f \in F$ such that (i) $f(x)$ is an integer for each $x \in I$, and (ii) if $x \in I \setminus M(f)$, then $f(x)$ is even. $F_1$ is an archimedean lattice ordered group. Hence $B_0(F_1) = C(F_1)$. Thus $B_0(F_1)$ consists of all $f \in F_1$ such that $f(x) = 0$ for each $x \in I \setminus M(f)$. Let $f \in F_1$ with $f(x) = 2$ for each $x \in I$. There is a subset $X \subseteq B_0(F_1)$ such that $\sup X = f$ holds in $F_1$; hence $B_0(F_1)$ fails to be closed in $F_1$.

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