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HOMOMORPHISMS OF DIRECT PRODUCTS OF ALGEBRAS

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The aim of this paper is an investigation of homomorphisms of algebras, which are direct products of the so-called algebras without zero-divisors.

Let A be a non-void set and F a set of operations on A. Then (A, F) denotes the algebra with the support A and the set of fundamental operations F. Two algebras (A, F), (B, G) are said to be of the same type if there exists a bijection δ of F onto G such that $\operatorname{ar} \delta(\omega) = \operatorname{ar} \omega$ for each $\omega \in F$, where $\operatorname{ar} \omega$ denotes the arity of ω. For the sake of brevity, by an *operation* of the algebra (A, F) we mean an algebraic operation on (A, F). If there is no danger of misunderstanding, an algebra and its support will be denoted by the same symbol. If the algebras (A, F), (B, G) are of the same type, the corresponding operations from A and B will be denoted by the same symbols. Hence, for (A, F), (B, G) we put $F = G$ if and only if (A, F), (B, G) are of the same type. If h is a maping of a set A into B and k is a mapping of the set B into C, the superposition of h, k is denoted by $h \,.\, k$, i.e. $h \,.\, k(a) = k(h(a))$ for each $a \in A$. Let A_i be algebras of the same type for $i \in T = \{1, \ldots, n\}$. The *direct product of algebras* A_i $(i \in T)$ is the algebra A of the same type as A_i, whose support is the Cartesian product of supports of A_i (for $i \in T$) and the operations on A are performed componentwise. The algebra A_i is called the i-th factor or component of A. By $pr_i\, A$ the *projection* of A onto the i-th factor A_i is denoted. The direct product of algebras A_i will be denoted by $\prod_{i \in T} A_i$ or $\prod_{i=1}^{n} A_i$.

Definition 1. Let (A, F) be an algebra and $\mathscr{A}$ the set of all algebraic operations on (A, F). Let $\mathscr{A} = \{\oplus\} \cup \Omega$, where $\oplus$ is a binary operation on (A, F). If there exists $0 \in A$ such that

(i) $a \oplus 0 = 0 \oplus a = a$ for each $a \in A$,

the element 0 is called a *zero* of the algebra (A, F). An operation $\omega \in \mathscr{A}$ is called *regular* on (A, F), if $\operatorname{ar} \omega = n \geqq 2$ and for each $a_1, \ldots, a_n \in A$ we have

(ii) $a_1, \ldots, a_n \omega = 0$ if and only if $a_i = 0$ for at least one $i \in \{1, \ldots, n\}$.

Definition 2. Let (A, F) be an algebra with card $A \geqq 2$ and let $\mathscr{A}$ be the set of all algebraic operations on (A, F). The algebra (A, F) is said to be *without zero-divisors*, if $\mathscr{A} = \{\oplus\} \cup \Omega$ and

(a) there exists a zero of (A, F),

(b) at least one $\omega \in \Omega$ is regular on (A, F).

Remark 1. From (i) it follows that each (A, F) has at most one zero. Further, if 0 is the zero of (A, F), $a_1, \ldots, a_n \in A$ and $a_i = 0$ for $i = 1, \ldots, k - 1, k + 1, \ldots, n$, then

$$(\ldots(a_1 \oplus a_2) \oplus \ldots) \oplus a_n = a_k \,.$$

It can be easily proved that for these $a_1, \ldots, a_n$ each "sum" (in the sense of $\oplus$) of them is equal to a_k. Without any danger of misunderstanding, the zero of (A, F) will be denoted by 0 for every algebra (A, F) without zero-divisors.

Definition 3. Let $T \neq \emptyset$ and A_τ be algebras of the same type for $\tau \in T$. The algebras A_τ are called *r-similar* if they are without zero-divisors and have the same set of regular operations.

Notation. Let $A_1, \ldots, A_k$ be r-similar algebras and $A = \prod_{i=1}^{k} A_i$. By 0_A we denote an element of A such that $\mathrm{pr}_i\, 0_A = 0$ for each $i = 1, \ldots, k$. Let $j \in \{1, \ldots, k\}$ and $a_j \in A_j$. Denote by $\bar{a}_j$ the element of A such that $\mathrm{pr}_j\, \bar{a}_j = a_j$, $\mathrm{pr}_i\, \bar{a}_j = 0$ for $i \neq j$, i.e. $\bar{a}_j = (0, \ldots, 0, a_j, 0, \ldots, 0)$. By φ_j denote the so called *canonical insertion* of A_j into A, i.e. $\varphi_j(a_j) = \bar{a}_j$ for each $a_j \in A_j$. Further, denote $\bar{A}_j = \{\varphi_j(a_j),\ a_j \in A_j\}$. Clearly, $\bar{A}_j$ is a subalgebra of A and φ_j is an isomorphism of A_j onto $\bar{A}_j$. If $\emptyset \neq T' \subseteq$ $\subseteq T = \{1, \ldots, k\}$, denote

$$\overline{\prod_{i \in T'} A_i} = \{a \in A,\ \mathrm{pr}_i\, a = 0 \text{ for } i \in T - T'\} \,.$$

Evidently, $\overline{\prod_{i \in T'} A_i}$ is a subalgebra of A isomorphic to $\prod_{i \in T'} A_i$ and for $T' = \{i_0\}$ it is equal to $\bar{A}_{i_0}$. For $T' = T$ we have $\overline{\prod_{i \in T} A_i} = \prod_{i \in T} A_i$.

Lemma 1. *Let $A_1, \ldots, A_k$ be r-similar algebras and $A = \prod_{i=1}^{k} A_i$. Then*

(a) *0_A isa zero of A.*

(b) *If ω is regular on A_i, $\mathrm{ar}\, \omega = n$, $b_1, \ldots, b_n \in A$ and for each $i \in \{1, \ldots, k\}$ there exists $j \in \{1, \ldots, n\}$ such that $\mathrm{pr}_i\, b_j = 0$, then $b_1, \ldots, b_n\omega = 0_A$.*

(c) *If ω is regular on A_i, $\mathrm{ar}\, \omega = n$, $i, j \in \{1, \ldots, k\}$, $i \neq j$ and $a_i \in A_i$, $a_j \in A_j$, then $\bar{a}_i\bar{a}_j, \ldots, \bar{a}_j\omega = 0_A$.*

(d) *Let $k \geqq 2$. Then A is not without zero-divisors.*

(e) *Let $a \in A$, $\mathrm{pr}_i\, a = a_i$. Then $a = \bar{a}_1 \oplus \ldots \oplus \bar{a}_k$ and the expression on the right hand side does not depend on any bracketing.*

(f) *Let* $a \in A$, $\mathrm{pr}_i\, a = a_i$ *and let* h *a homomorphism of* A *into* B. *Then* $h(a) = h(\bar{a}_1) \oplus \ldots \oplus h(\bar{a}_k)$ *and this expression does not depend on any bracketing.*

The proof is clear.

Definition 4. Let (A, F) be an algebra without zero-divisors and let ω be a regular operation on (A, F). A unary operation α of (A, F) is called *corresponding with* ω on (A, F), if

(iii) for each $a_1, \ldots, a_n \in A$ (where $n = \mathrm{ar}\, \omega$) there exists $i \in \{1, \ldots, n\}$ such that $a_1, \ldots, a_n\omega = a_i\alpha$.

Lemma 2. *Let* (A, F) *be an algebra without zero-divisors and let* α *be a unary operation corresponding with a regular operation* ω *on* (A, F). *Then*

$$a\alpha = 0 \quad \textit{if and only if} \quad a = 0 \quad \textit{for each} \quad a \in A\,.$$

Proof. If $a \in A$, then, by (iii), $a \ldots a\omega = a\alpha$. For $a = 0$ it follows $0 = 0 \ldots 0\omega = 0\alpha$, for $a \neq 0$ we have $0 \neq a \ldots a\omega = a\alpha$, because ω is regular on (A, F).

Definition 5. An algebra (A, F) without zero-divisors is called a *U-algebra*, if there exists a corresponding operation α for at least one ω regular on (A, F). An algebra (A, F) is called a *strong U-algebra*, if it is a U-algebra and $\alpha = id_A$ for at least one α corresponding to ω regular on (A, F).

Definition 6. Let A_τ be U-algebras for $\tau \in T \neq \emptyset$. The algebras A_τ are called *p-similar*, if A_τ are r-similar and, moreover, if $\tau', \tau'' \in T$ and α is corresponding with ω regular on $A_{\tau'}$, then α is also corresponding with ω on $A_{\tau''}$.

Definition 7. Let A_i, B_i be algebras of the same type for $i = 1, \ldots, k$ and $A = \prod_{i=1}^{k} A_i$, $B = \prod_{i=1}^{k} B_i$. Let h_i be a mapping of A_i into B_i. The mapping h of A into B defined by

$$\mathrm{pr}_i\,(h(a)) = h_i(\mathrm{pr}_i\, a)$$

for each $a \in A$ and each $i = 1, \ldots, k$ is called the *direct product of mappings* h_i and is denoted by $h = \prod_{i=1}^{k} h_i$.

This definition is taken from [1]. There it is also proved that the direct product of homomorphisms of similar algebras is also a homomorphism of the algebra, which is the direct product of original algebras. Some sufficient conditions for the converse of this statement will be formulated in this paper.

Theorem 1. *Let A_i, B_j be r-similar algebras for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and let h be a surjective homomorphism of $A = \prod_{i=1}^{m} A_i$ onto $B = \prod_{j=1}^{n} B_j$. Then for each $j \in \{1, \ldots, n\}$ there exists just one $i \in \{1, \ldots, m\}$ such that $\bar{B}_j \subseteq h(\bar{A}_i)$.*

Proof. I. Existence. Denote $T = \{1, \ldots, m\}$.

1° Choose $j \in \{1, \ldots, n\}$ fixed. Let $b \in \bar{B}_j$, $b \neq 0_B$. As h is surjective, there exists $a \in A$, $a \neq 0_A$ with $h(a) = b$. Denote $T_a = \{i \in T, \text{pr}_i\, a \neq 0\}$. As $a \neq 0_A$, we have $T_a \neq \emptyset$. Let $T_a = \{i_1, \ldots, i_k\}$. If $\text{pr}_i\, a = a_i$, then by Lemma 1, $a = \bar{a}_{i_1} \oplus \ldots \oplus \bar{a}_{i_k}$. Choose $t \in \{1, \ldots, n\}$ arbitrarily. Suppose the existence of $i_r, i_s \in T_a$, $i_r \neq i_s$ with $\text{pr}_t\,(h(\bar{a}_{i_r})) \neq 0$, $\text{pr}_t\,(h(\bar{a}_{i_s})) \neq 0$. If ω is regular on A_i, then by Lemma 1 it is

$$0_B = h(0_A) = h(\bar{a}_{i_r}\bar{a}_{i_s} \ldots \bar{a}_{i_s}\omega) = h(\bar{a}_{i_r})\, h(\bar{a}_{i_s}) \ldots h(\bar{a}_{i_s})\, \omega\,,$$

i.e.

$$0 = \text{pr}_t\, 0_B = \text{pr}_t\,(h(\bar{a}_{i_r}))\, \text{pr}_t\,(h(\bar{a}_{i_s})) \ldots \text{pr}_t\,(h(\bar{a}_{i_s}))\, \omega \neq 0\,,$$

a contradiction. Hence for each $t \in \{1, \ldots, n\}$ there exists at most one $i \in T_a$ with $\text{pr}_t\,(h(\bar{a}_i)) \neq 0$. As $h(a) = b \neq 0_B$, such $i \in T_a$ exists for $t = j$.

2° If $h(\bar{a}_{i'}) \notin \bar{B}_j$ for some $i' \in T_a$, then $\text{pr}_{j'}\,(h(\bar{a}_{i'})) \neq 0$ for some $j' \in \{1, \ldots, n\}$, $j' \neq j$. By 1°, $\text{pr}_{j'}\,(h(\bar{a}_i)) = 0$ for each $i \in T_a$, $i \neq i'$, thus

$$0 = \text{pr}_{j'}\, b = \text{pr}_{j'}\,(h(a)) = \text{pr}_{j'}\,(h(\bar{a}_{i_1})) \oplus \ldots \oplus \text{pr}_{j'}\,(h(\bar{a}_{i_k})) = \text{pr}_{j'}\,(h(\bar{a}_{i'})) \neq 0\,,$$

a contradiction. Thus $h(\bar{a}_i) \in \bar{B}_j$ for each $i \in T_a$. By 1°, there exists just one $i \in T_a$ with $\text{pr}_j\,(h(\bar{a}_i)) \neq 0$, i.e. $h(\bar{a}_i) \neq 0_B$. Then

$$b = h(a) = h(\bar{a}_{i_1}) \oplus \ldots \oplus h(\bar{a}_{i_k}) = h(\bar{a}_i)\,.$$

As $b \neq 0_B$, also $\bar{a}_i \neq 0_A$.

3° From 1° and 2° it follows that for each $b \in \bar{B}_j$, $b \neq 0_B$, there exists just one $i \in T$ and $\bar{a}_i \in \bar{A}_i$ with $h(\bar{a}_i) = b$. Prove that this index i is the same for all $b \in \bar{B}_j$, $b \neq 0_B$. Let $b_1, b_2 \in \bar{B}_j$, $b_1 \neq 0_B \neq b_2$. Then there exist $i_1, i_2 \in T$ and $\bar{a}_{i_1} \in \bar{A}_{i_1}$, $\bar{a}_{i_2} \in \bar{A}_{i_2}$ with $h(\bar{a}_{i_1}) = b_1$, $h(\bar{a}_{i_2}) = b_2$. Clearly $\bar{a}_{i_1} \neq 0_A \neq \bar{a}_{i_2}$. Let ω be regular on A_i and $i_1 \neq i_2$, then Lemma 1 yields $0_B = h(0_A) = h(\bar{a}_{i_1}\bar{a}_{i_2} \ldots \bar{a}_{i_2}\omega) = b_1 b_2 \ldots \ldots b_2\omega \neq 0_B$, which is a contradiction. Thus $i_1 = i_2$.

Hence the index $i \in T$ is the same for all $b \in \bar{B}_j$, $b \neq 0_B$. If $b = 0_B$, put $a = 0_A$. Then $h(0_A) = 0_B$ and $0_A \in \bar{A}_i$. Thus $h(\bar{A}_i) \supseteq \bar{B}_j$. As j was chosen arbitrarily, this remains true for each $j \in \{1, \ldots, n\}$.

II. Uniqueness. Suppose that $\bar{B}_j \subseteq h(\bar{A}_{i_1})$, $\bar{B}_j \subseteq h(\bar{A}_{i_2})$ for some $j \in \{1, \ldots, n\}$, $i_1 \neq i_2$, $i_1, i_2 \in T$. Choose $b_j \in B_j$, $b_j \neq 0$ (card $B_j > 1$ by Definition 2). Then there exist $a_1 \in \bar{A}_{i_1}$, $a_2 \in \bar{A}_{i_2}$ with $h(a_1) = \bar{b}_j = h(a_2)$. Clearly $a_1 \neq 0_A \neq a_2$. If ω is regular on A_i, then

$$0_B = h(0_A) = h(a_1 a_2 \ldots a_2\omega) = h(a_1)\, h(a_2) \ldots h(a_2)\, \omega = \bar{b}_j \ldots \bar{b}_j\omega \neq 0_B\,,$$

also a contradiction.

Corollary. *Let A_i, B_j be r-similar algebras for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and let $\prod_{i=1}^{m} A_i$ be isomorphic to $\prod_{j=1}^{n} B_j$. Then $m = n$ and there exists a permutation π of $\{1, \ldots, n\}$ such that A_i is isomorphic to $B_{\pi(i)}$ for each $i \in \{1, \ldots, n\}$.*

Proof. Let h be an isomorphism of $A = \prod_{i=1}^{m} A_i$ onto $B = \prod_{j=1}^{n} B_j$. Then h^{-1} is an isomorphism of B onto A and, by Theorem 1, there exists just one A_i for each B_j with $h(\bar{A}_i) \supseteq \bar{B}_j$ and just one $B_{j'}$ for each A_i with $h^{-1}(\bar{B}_{j'}) \supseteq \bar{A}_i$. Thus

$$\bar{B}_{j'} = h(h^{-1}(\bar{B}_{j'})) \supseteq h(\bar{A}_i) \supseteq \bar{B}_j .$$

As $\bar{B}_{j'} \cap \bar{B}_j = \{0_B\}$ for $j' \neq j$, we have $j' = j$ and $h(\bar{A}_i) = \bar{B}_j$. Put $\pi(i) = j$ for $h(\bar{A}_i) = \bar{B}_j$, thus π is a bijection of $\{1, \ldots, m\}$ onto $\{1, \ldots, n\}$ and $\bar{A}_i$ is isomorphic to $\bar{B}_j$. From this we obtain the assertion.

Theorem 2. *Let A_i, B_j be p-similar U-algebras for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and let h a surjective homomorphism of $A = \prod_{i=1}^{m} A_i$ onto $B = \prod_{j=1}^{n} B_j$. Then for each $j \in \{1, \ldots, n\} = S$ there exists just one $i_j \in \{1, \ldots, m\} = T$ such that $h(\bar{A}_{i_j}) = \bar{B}_j$ and the mapping $j \to i_j$ is an injection of S into T.*

Proof. By Theorem 1, for each $j \in S$ there exists just one $i_j \in T$ with $\bar{B}_j \subseteq h(\bar{A}_{i_j})$. Thus $j \to i_j$ is a mapping of S into T.

I. First we prove the injectivity of the mapping $j \to i_j$. Let there exist $j_1, j_2 \in S$, $j_1 \neq j_2$, $i \in T$ with $\bar{B}_{j_1} \subseteq h(\bar{A}_i)$, $\bar{B}_{j_2} \subseteq h(\bar{A}_i)$. As each B_j has at least two elements, there exist $b_1 \in \bar{B}_{j_1}$, $b_2 \in \bar{B}_{j_2}$, $b_1 \neq 0_B \neq b_2$. Choose $a_1, a_2 \in \bar{A}_i$ with $h(a_1) = b_1$, $h(a_2) = b_2$. Clearly $a_1 \neq 0_A \neq a_2$. If ω is regular on A_i and α is corresponding with ω, then by Lemma 2

$$0_B = b_1 b_2 \ldots b_2 \omega = h(a_1)\, h(a_2) \ldots h(a_2)\, \omega = h(a_1 a_2 \ldots a_2 \omega) =$$
$$= h(a_s \alpha) = b_s \alpha \neq 0_B , \quad \text{where} \quad s \in \{1, 2\} ,$$

a contradiction. Hence, $j \to i_j$ is an injection of S into T.

II. It remains to prove $h(\bar{A}_{i_j}) = \bar{B}_j$. Let $\bar{B}_j \neq h(\bar{A}_{i_j})$. By Theorem 1 we have $\bar{B}_j \subseteq h(\bar{A}_{i_j})$, thus there exists $c \in h(\bar{A}_{i_j}) - \bar{B}_j$, $c \neq 0_B$ such that $\mathrm{pr}_{j'}\, c = c_1 \neq 0$ for some $j' \in S$, $j' \neq j$. Denote by $\bar{c}_1 \in \bar{B}_{j'}$ an element fulfilling $\mathrm{pr}_{j'}\, \bar{c}_1 = c_1$. As $c \in h(\bar{A}_{i_j})$, there exists $d \in \bar{A}_{i_j}$ with $h(d) = c$. Further, $\bar{c}_1 \in \bar{B}_{j'}$, thus by Theorem 1 there exists $d_1 \in \bar{A}_{i_{j'}}$ with $h(d_1) = \bar{c}_1$. As $j \to i_j$ is an injection, we have $i_j \neq i_{j'}$. Let ω be regular on A_i. By Lemma 1 we obtain $d d_1 \ldots d_1 \omega = 0_A$. However,

$$\mathrm{pr}_{j'}\,(h(d d_1 \ldots d_1 \omega)) = \mathrm{pr}_{j'}\,(c \bar{c}_1 \ldots \bar{c}_1 \omega) = c_1 c_1 \ldots c_1 \omega \neq 0$$

because $c_1 \neq 0$, a contradiction. Thus $\bar{B}_j = h(\bar{A}_{i_j})$.

Corollary. *Let A_i, B_j be p-similar U-algebras for $i = 1, \ldots, m$, $j = 1, \ldots, n$. If h is a surjective homomorphism of $\prod_{i=1}^{m} A_i$ onto $\prod_{j=1}^{n} B_j$, then $m \geqq n$.*

Notation. Let $A_1, \ldots, A_n$ be algebras of the same type and π a permutation of the index set $\{1, \ldots, n\}$. Clearly $\prod_{j=1}^{n} A_j$ is isomorphic to $\prod_{j=1}^{n} A_{\pi(j)}$. Denote by $\imath_\pi$ the isomorphism of these algebras given by the rule

$$(a_1, \ldots, a_n) \to (a_{\pi(1)}, \ldots, a_{\pi(n)}) .$$

Definition 7. Let A_j, B_j be algebras of the same type for $j = 1, \ldots, n$ and let h be a homomorphism of $A = \prod_{j=1}^{n} A_j$ into $B = \prod_{j=1}^{n} B_j$. We call h *directly decomposable*, if there exist a permutation π of the index set $\{1, \ldots, n\}$ and a homomorphism h_j of A_j into $B_{\pi(j)}$ for each $j = 1, \ldots, n$ such that $h \,.\, \imath_\pi = \prod_{j=1}^{n} h_j$.

Theorem 3. *Let A_j, B_j be p-similar U-algebras for $j = 1, \ldots, n$ and h a surjective homomorphism of $A = \prod_{j=1}^{n} A_j$ onto $B = \prod_{j=1}^{n} B_j$. Then h is directly decomposable.*

Proof. By Theorem 2, there exists an injection π of $\{1, \ldots, n\}$ into itself with $h(\bar{A}_{\pi(j)}) = \bar{B}_j$ for each $j \in \{1, \ldots, n\}$. As $\{1, \ldots, n\}$ is finite, π is a permutation. Then $h \,.\, \imath_\pi(\bar{A}_{\pi(j)}) = \bar{B}_{\pi(j)}$ for each $j \in \{1, \ldots, n\}$. Denote $h_j = \varphi_j \,.\, h \,.\, \imath_\pi \,.\, \mathrm{pr}_j$, where φ_j is a canonical insertion. Then h_j is a homomorphism of A_j onto B_j and $\mathrm{pr}_j\,(h \,.\, \imath_\pi(a)) = h_j(\mathrm{pr}_j\, a)$ for each $a \in A$, thus $h \,.\, \imath_\pi = \prod_{j=1}^{n} h_j$, which completes the proof.

Lemma 3. *Every at least two-element chain with the least or the greatest element (considered as a lattice) is a strong U-algebra.*

Proof. Let A be an at least two-element chain with the least element 0. Put $a \oplus b = a \vee b = \max(a, b)$, ω binary and $ab\omega = a \wedge b = \min(a, b)$. Then clearly 0 is a zero of A, $\oplus$ fulfils (i), ω fulfils (ii), (iii) for $\alpha = \mathrm{id}$, thus (A, F) is a strong U-algebra for $F = \{\oplus, \omega\}$. For a chain with the greatest element, the proof is dual.

Corollary. *Let A_j, B_j be at least two element chains and for each $j = 1, \ldots, n$ at least one of the following conditions let be true:*

(a) *Each A_j, B_j has the greatest element.*

(b) *Each A_j, B_j has the least element.*

Then each surjective homomorphism of the lattice $A = \prod_{j=1}^{n} A_j$ onto $B = \prod_{j=1}^{n} B_j$ is directly decomposable.

Lemma 4. *Let G be a linearly ordered additive group with* card $G \geqq 2$. *Denote* $a\alpha = \sup(a, -a)$, $ab\omega = \inf(a\alpha, b\alpha)$. *Then α, ω are operations on the support of G and every Ω-group G' with G as the additive group and $\{\alpha, \omega\} \subseteq \Omega$ is a U-algebra. Moreover, the group zero is a zero of this U-algebra G', ω is regular on G' and α is corresponding with ω.*

The proof is clear.

Let G be an ℓ-group. Denote by $\vee, \wedge$ the lattice operations on G. A homomorphism h of G is called an ℓ-homomorphism, if

$$h(a \vee b) = h(a) \vee h(b), \quad h(a \wedge b) = h(a) \wedge h(b)$$

for each $a, b \in G$.

Lemma 5. *Let A_j, B_j be linearly ordered groups for $j = 1, \ldots, n$, $A = \prod_{j=1}^{n} A_j$, $B = \prod_{j=1}^{n} B_j$. Let A_j' or B_j' be Ω-groups with A_j or B_j as additive groups, respectively, and $\Omega = \{\alpha, \omega\}$ for the operations α, ω introduced in Lemma 4. Let $A' = \prod_{j=1}^{n} A_j'$, $B' = \prod_{j=1}^{n} B_j'$. Then each ℓ-homomorphism h of the ℓ-group A into B is a homomorphism of the Ω-group A' into B'.*

Proof. Let $a, b \in A$, $h(a) = c$. Denote $a = (a_1, \ldots, a_n)$, $c = (c_1, \ldots, c_n)$, where $\mathrm{pr}_j\, a = a_j$, $\mathrm{pr}_j\, c = c_j$. Then

$$h(a\alpha) = h((a_1\alpha, \ldots, a_n\alpha)) = h((\max(a_1, -a_1), \ldots, \max(a_n, -a_n))) =$$
$$= h(a \vee -a) = h(a) \vee -h(a) = (c_1, \ldots, c_n) \vee (-c_1, \ldots, -c_n) =$$
$$= (\max(c_1, -c_1), \ldots, \max(c_n, -c_n)) = c\alpha = h(a)\,\alpha\,.$$

From this we obtain

$$h(ab\omega) = h(a\alpha \wedge b\alpha) = h(a)\,\alpha \wedge h(b)\,\alpha = h(a)\,h(b)\,\omega\,,$$

thus each ℓ-homomorphism of A into B is a homomorphism of A' into B'.

Corollary. *Let A_j, B_j be at least two-element linearly ordered groups for $j = 1, \ldots, n$ and let $A = \prod_{j=1}^{n} A_j$, $B = \prod_{j=1}^{n} B_j$ be ℓ-groups with the induced orderings. Then each surjective ℓ-homomorphism of A onto B is directly decomposable.*

The proof follows directly from Theorem 3, Lemmas 4 and 5.

Theorem 4. *Let A_j, B_k be r-similar algebras for $j \in \{1, \ldots, m\} = T$, $k \in \{1, \ldots, n\} = S$ and let h be a surjective homomorphism of $A = \prod_{j=1}^{m} A_j$ onto $B = \prod_{k=1}^{n} B_k$. Then* *there exist a partition $\{S_\alpha, \alpha \in I\}$ of S and an injection $\alpha \to j_\alpha$ of I into T such that:*

(1) *If* $T^* = \{j_\alpha, \alpha \in I\}$, $A^* = \overline{\prod_{j \in T^*} A_j}$, *then* $h(A) = h(A^*)$.

(2) *There exist a permutation π of S and a surjective homomorphism f_α of A_{j_α} onto $\prod_{k \in S_\alpha} B_k$ for each $\alpha \in I$ such that* $h | A^* . \iota_\pi = \prod_{\alpha \in I} f_\alpha$.

Proof. By Theorem 1, for each $k \in S$ there exists just one $j_k \in T$ with $h(\bar{A}_{j_k}) \supseteq \bar{B}_k$. Denote by T^* the set of all these j_k (without repetitions) and choose a new indexing $T^* = \{j_\alpha, \alpha \in I\}$ such that I is linearly ordered and $j_{\alpha'} < j_{\alpha''}$ for $\alpha' < \alpha''$. Thus the map $\alpha \to j_\alpha$ is an injection of I into T.

1° First we prove the following implication:

$$\text{if} \quad \Gamma = \{k_1, \ldots, k_p\} \subseteq S \quad \text{and} \quad h(\bar{A}_{r_s}) \supseteq \bar{B}_s \quad \text{for each} \quad s \in \Gamma ,$$

$$\text{then} \quad h(\overline{\prod_{s \in \Gamma} A_{r_s}}) \supseteq \overline{\prod_{s \in \Gamma} B_s} .$$

If $b \in \overline{\prod_{s \in \Gamma} B_s}$, then $b = \bar{b}_{k_1} \oplus \ldots \oplus \bar{b}_{k_p}$. Suppose $h(\bar{A}_{r_s}) \supseteq \bar{B}_s$, then there exists $\bar{a}_{r_s} \in \bar{A}_{r_s}$ with $h(\bar{a}_{r_s}) = \bar{b}_s$ for each $\bar{b}_s \in \bar{B}_s$. Put $a = \bar{a}_{r_{k_1}} \oplus \ldots \oplus \bar{a}_{r_{k_p}}$, then $a \in \overline{\prod_{s \in \Gamma} A_{r_s}}$ and, by Lemma 1(f), we have $h(a) = h(\bar{a}_{r_{k_1}}) \oplus \ldots \oplus h(\bar{a}_{r_{k_p}}) = \bar{b}_{k_1} \oplus \ldots \oplus \bar{b}_{k_p} = b$. The implication is proved.

2° By 1° for $\Gamma = S$ we obtain:

$$h(A^*) = h(\overline{\prod_{j \in T^*} A_j}) = h(\overline{\prod_{k \in S} A_{j_k}}) \supseteq \overline{\prod_{k \in S} B_k} = \prod_{k \in S} B_k = B .$$

However, $A^* \subseteq A$ implies $h(A^*) \subseteq h(A) = B$, thus the first assertion of the theorem is proved.

3° For $\alpha \in I$ fixed, denote $S_\alpha = \{k \in S, \bar{B}_k \subseteq h(\bar{A}_{j_\alpha})\}$. By Theorem 1, S_α's are mutually disjoint and $S = \bigcup_{\alpha \in I} S_\alpha$, thus $\{S_\alpha, \alpha \in I\}$ forms a partition of S. By 1° we obtain

$$\overline{\prod_{k \in S_\alpha} B_k} \subseteq h(\bar{A}_{j_\alpha}) \quad \text{for each} \quad \alpha \in I .$$

Let $\alpha \in I$ and $\overline{\prod_{k \in S_\alpha} B_k} \neq h(\bar{A}_{j_\alpha})$. Then there exists $c \in h(\bar{A}_{j_\alpha}) - \overline{\prod_{k \in S_\alpha} B_k}$, $c \neq 0_B$, i.e. $\mathrm{pr}_{k'} c = c_1 \neq 0$ for some $k' \in S - S_\alpha$. Denote by $\bar{c}_1$ an element of $\bar{B}_{k'}$ with $\mathrm{pr}_{k'} \bar{c}_1 = c_1$. As $c \in h(\bar{A}_{j_\alpha})$, there exists $d \in \bar{A}_{j_\alpha}$ with $h(d) = c$ and $\alpha' \in I$, $\alpha' \neq \alpha$ with $k' \in S_{\alpha'}$. As $\alpha \to j_\alpha$ is a bijection, it is $j_{\alpha'} \neq j_\alpha$. However, $h(\bar{A}_{j_{\alpha'}}) \supseteq \overline{\prod_{k \in S_{\alpha'}} B_k}$, thus there exists $d_1 \in \bar{A}_{j_{\alpha'}}$ with $h(d_1) = \bar{c}_1$. If ω is regular on A_j, then

$$dd_1 \ldots d_1 \omega = 0_A .$$

However,

$$\mathrm{pr}_{k'}\left(h(dd_1 \ldots d_1\omega)\right) = \mathrm{pr}_{k'}\left(c\bar{c}_1 \ldots \bar{c}_1\omega\right) = c_1c_1 \ldots c_1\omega \neq 0\,,$$

a contradiction with $h(0_A) = 0_B$.

Accordingly, we have $\overline{\prod_{k\in S_\alpha} B_k} = h(\bar{A}_{j_\alpha})$ for each $\alpha \in I$.

4° It is evident that we must only find a suitable permutation guaranteeing the direct decomposability of $h|A^*$. Let us introduce the following mapping π of S into itself. Denote $S_\alpha = \{k_{\alpha_1}, \ldots, k_{\alpha r_\alpha}\}$ for each $\alpha \in I$ and put $\pi(k_{\alpha s}) < \pi(k_{\alpha' t})$ for $\alpha < \alpha'$ or $\alpha = \alpha'$, $s < t$. As S_α's are mutually disjoint, this can be satisfied and π is a permutation of S. Denote

$$f_\alpha = \varphi_{j_\alpha} \,.\, h \mid A^* \,.\, p_\alpha\,,$$

where p_α is a projection of $\overline{\prod_{k\in S_\alpha} B_k}$ onto $\prod_{k\in S_\alpha} B_k$. Then f_α is a homomorphism of A_{j_α} onto $\prod_{k\in S_\alpha} B_k$ and clearly $h|A^* \,.\, \iota_\pi = \prod_{\alpha\in I} f_\alpha$.

Corollary 1. *Let A_j, B_j be r-similar algebras for $j = 1, \ldots, n$ and let h be a surjective homomorphism of $\prod_{j=1}^{n} A_j$ onto $\prod_{j=1}^{n} B_j$ such that $h(\bar{A}_j)$ is without zero-divisors for each $j \in \{1, \ldots, n\}$. Then h is directly decomposable.*

Proof. In the notation of Theorem 4, put $S = T = \{1, \ldots, n\}$. As $h(\bar{A}_{j_\alpha}) = \overline{\prod_{k\in S_\alpha} B_k}$ is without zero-divisors, then, by Lemma 1, card $S_\alpha = 1$ for each $\alpha \in I$. Thus card $I =$ = card $S = n$ and $\alpha \to j_\alpha$ is a bijection. Then $A^* = A$. Put $S_\alpha = \{s_\alpha\}$, then f_α is a homomorphism of A_{j_α} onto B_{s_α}. By Theorem 4, $h \,.\, \iota_\pi = \prod_{\alpha\in I} f_\alpha$, i.e. h is directly decomposable.

Corollary 2. *Let A_j, B_j be non-zero rings without zero-divisors for $j = 1, \ldots, n$ and let h be a surjective homomorphism of the ring $\prod_{j=1}^{n} A_j$ onto $\prod_{j=1}^{n} B_j$ such that $(\bar{A}_j \cap \ker h)$ is a prime ideal of $\bar{A}_j$ for each $j = 1, \ldots, n$. Then h is directly decomposable.*

Proof. Let Θ be a congruence relation on A induced by h. Denote $\Theta_j = \Theta \mid \bar{A}_j$. As $(\bar{A}_j \cap \ker h)$ is a prime ideal of $\bar{A}_j$, $\bar{A}_j/\Theta_j$ is a factor-ring without zero-divisors isomorphic to $h(\bar{A}_j)$. By Corollary 1 we obtain the assertion.

Corollary 3. *Let A_j, B_j be simple rings for $j = 1, \ldots, n$. Then each surjective homomorphism of the ring $\prod_{j=1}^{n} A_j$ onto $\prod_{j=1}^{n} B_j$ is directly decomposable.*

This follows directly from Corollary 2, because a simple ring has improper ideals only and these are prime.

Definition 8. Let A_i, B_j be algebras without zero-divisors for $i = 1, \ldots, m$, $j = 1 \ldots, n$ and h a homomorphism of $A = \prod_{i=1}^{m} A_i$ into $B = \prod_{j=1}^{n} B_j$. We say that A, B, h *satisfy* (P), if at least one of the two following conditions is valid:

(1) $h(0_A) = 0_B$.

(2) A_i, B_j are p-similar strong U-algebras.

Theorem 5. *Let A_i, B_j be r-similar algebras for $i = 1, \ldots, m$, $j = 1, \ldots, n$, let h be a homomorphism of $A = \prod_{i=1}^{m} A_i$ into $B = \prod_{j=1}^{n} B_j$ and let A, B, h satisfy* (P). *Let $j \in \{1, \ldots, n\}$. If*

$$\mathrm{pr}_j\,(h(A)) \neq \mathrm{pr}_j\,(h(0_A))\,,$$

then there exists just one $i \in \{1, \ldots, m\}$ such that

$$\mathrm{pr}_j\,(h(A)) = \mathrm{pr}_j\,(h(\bar{A}_i))\,.$$

Proof. I. Existence. Put $T = \{1, \ldots, m\}$. Let $j \in \{1, \ldots, n\}$ and

$$\mathrm{pr}_j\,(h(A)) \neq \mathrm{pr}_j\,(h(0_A))\,.$$

1° First we prove that for each $b \in h(A)$ there exist $i \in T$ and $\bar{a}_i \in \bar{A}_i$ with $\mathrm{pr}_j\,(h(\bar{a}_i)) = \mathrm{pr}_j\,b$. Let $b \in h(A)$. If $\mathrm{pr}_j\,b = \mathrm{pr}_j\,(h(0_A))$, put $\bar{a}_i = 0_A$, because $0_A \in \bar{A}_i$ for each $i \in T$. Suppose $\mathrm{pr}_j\,b = b_j \neq \mathrm{pr}_j\,(h(0_A))$. Then there exists $a \in A$ with $h(a) = b$ and $a \neq 0_A$. Put $a_i = \mathrm{pr}_i\,a$.

(a) Let $\mathrm{pr}_j\,(h(\bar{a}_i)) = \mathrm{pr}_j\,(h(0_A))$ for each $i \in T$. As $h(0_A)$ is a zero of $h(A)$, by Lemma 1 we obtain a contradiction:

$$\mathrm{pr}_j\,(h(a)) = \mathrm{pr}_j\,(h(\bar{a}_1 \oplus \ldots \oplus \bar{a}_m)) = \mathrm{pr}_j\,(h(\bar{a}_1)) \oplus \ldots \oplus \mathrm{pr}_j\,(h(\bar{a}_m)) =$$
$$= \mathrm{pr}_j\,(h(0_A)) \oplus \ldots \oplus \mathrm{pr}_j\,(h(0_A)) = \mathrm{pr}_j\,(h(0_A))\,.$$

(b) Let $i_1, i_2 \in T$, $i_1 \neq i_2$ and $\mathrm{pr}_j\,(h(\bar{a}_{i_1})) \neq \mathrm{pr}_j\,(h(0_A)) \neq \mathrm{pr}_j\,(\mathrm{h}(\bar{a}_{i_2}))$. If (1) of (P) is true and ω is regular on A_i, then

$$0 = \mathrm{pr}_j\,0_B = \mathrm{pr}_j\,(h(0_A)) = \mathrm{pr}_j\,(h(\bar{a}_{i_1}\bar{a}_{i_2} \ldots \bar{a}_{i_2}\omega)) =$$
$$= \mathrm{pr}_j\,(h(\bar{a}_{i_1}))\,\mathrm{pr}_j\,(h(\bar{a}_{i_2})) \ldots \mathrm{pr}_j\,(h(\bar{a}_{i_2}))\,\omega \neq 0\,,$$

because by (1) of (P) it is $\mathrm{pr}_j\,(h(0_A)) = \mathrm{pr}_j\,(0_B) = 0$. If (2) of (P) is true and ω is regular on A_i with the corresponding operation $\alpha = id$, then

$$\mathrm{pr}_j\,(h(0_A)) = \mathrm{pr}_j\,(h(\bar{a}_{i_1}\bar{a}_{i_2} \ldots \bar{a}_{i_2}\omega)) =$$
$$= \mathrm{pr}_j\,(h(\bar{a}_{i_1}))\,\mathrm{pr}_j\,(h(\bar{a}_{i_2})) \ldots \mathrm{pr}_j\,(h(\bar{a}_{i_2}))\,\omega =$$
$$= \mathrm{pr}_j\,(h(\bar{a}_{i_s})) \neq \mathrm{pr}_j\,(h(0_A))\,,$$

where $s \in \{1, 2\}$.

The contradiction is obtained for both possibilities of (P).

(c) By (a) and (b), there exists just one $i_0 \in T$ such that $\mathrm{pr}_j (h(\bar{a}_{i_0})) \neq \mathrm{pr}_j (h(0_A))$. As $\mathrm{pr}_j (h(0_A))$ is a zero of $\mathrm{pr}_j (h(A))$, we obtain

$$b_j = \mathrm{pr}_j b = \mathrm{pr}_j (h(a)) = \mathrm{pr}_j (h(\bar{a}_1) \oplus \dots \oplus h(\bar{a}_m)) = \mathrm{pr}_j (h(\bar{a}_{i_0})) .$$

2° Now we prove that this index $i_0 \in T$ is the same for all $b \in h(A)$ and a fixed $j \in \{1, \dots, n\}$ such that $\mathrm{pr}_j b \neq \mathrm{pr}_j (h(0_A))$. Let $b_1, b_2 \in h(A)$, $\mathrm{pr}_j b_1 = b'_1 \neq \mathrm{pr}_j (h(0_A)) \neq b'_2 = \mathrm{pr}_j b_2$. By 1° there exist $i_1, i_2 \in T$ and $\bar{a}_{i_1} \in \bar{A}_{i_1}$, $\bar{a}_{i_2} \in \bar{A}_{i_2}$ such that $b'_1 = \mathrm{pr}_j (h(\bar{a}_{i_1}))$, $b'_2 = \mathrm{pr}_j (h(\bar{a}_{i_2}))$. Suppose $i_1 \neq i_2$.

If (1) of (P) is valid and ω is regular on A_i, then, by Lemma 1, we have

$$0 = \mathrm{pr}_j (h(0_A)) = \mathrm{pr}_j (h(\bar{a}_{i_1}\bar{a}_{i_2} \dots \bar{a}_{i_2}\omega)) = b'_1 b'_2 \dots b'_2 \omega \neq 0 .$$

If (2) of (P) is valid and ω is regular on A_i with the corresponding $\alpha = id$, then

$$\mathrm{pr}_j (h(0_A)) = \mathrm{pr}_j (h(\bar{a}_{i_1}\bar{a}_{i_2} \dots \bar{a}_{i_2}\omega)) = b'_1 b'_2 \dots b'_2 \omega = b'_s \neq \mathrm{pr}_j (h(0_A))$$

where $s \in \{1, 2\}$. We have again a contradiction in both cases.

3° By 1° and 2°, there exists $i_0 \in T$ for $j \in \{1, \dots, n\}$ fixed such that for each $b \in h(A)$, $\mathrm{pr}_j b \neq \mathrm{pr}_j (h(0_A))$, there exists $\bar{a}_{i_0} \in \bar{A}_{i_0}$ with $\mathrm{pr}_j (h(\bar{a}_{i_0})) = \mathrm{pr}_j b$. If $\mathrm{pr}_j b = \mathrm{pr}_j (h(0_A))$, then also $0_A \in \bar{A}_{i_0}$. Hence $\mathrm{pr}_j (h(\bar{A}_{i_0})) \supseteq \mathrm{pr}_j (h(A))$. The converse inclusion is evident, thus $\mathrm{pr}_j (h(\bar{A}_{i_0})) = \mathrm{pr}_j (h(A))$.

II. Uniqueness. Suppose $\mathrm{pr}_j (h(\bar{A}_{i_1})) = \mathrm{pr}_j (h(A)) = \mathrm{pr}_j (h(\bar{A}_{i_2}))$, $\mathrm{pr}_j (h(A)) \neq \mathrm{pr}_j (h(0_A))$ for some $i_1, i_2 \in T$, $i_1 \neq i_2$ and $j \in \{1, \dots, n\}$. Choose $b \in h(A)$ such that $\mathrm{pr}_j b = b_j \neq \mathrm{pr}_j (h(0_A))$. Then there exist $a_{i_1} \in A_{i_1}$, $a_{i_2} \in A_{i_2}$ with $\mathrm{pr}_j (h(\bar{a}_{i_1})) = b_j = \mathrm{pr}_j (h(\bar{a}_{i_2}))$.

For (1) of (P) and ω regular on A_i we have

$$0 = \mathrm{pr}_j 0_B = \mathrm{pr}_j (h(0_A)) = \mathrm{pr}_j (h(\bar{a}_{i_1}\bar{a}_{i_2} \dots \bar{a}_{i_2}\omega)) = b_j b_j \dots b_j \omega \neq 0 .$$

For (2) of (P) and ω regular on A_i with the corresponding $\alpha = id$ it is

$$\mathrm{pr}_j (h(0_A)) = \mathrm{pr}_j (h(\bar{a}_{i_1}\bar{a}_{i_2} \dots \bar{a}_{i_2}\omega)) = b_j b_j \dots b_j \omega = b_j \neq \mathrm{pr}_j (h(0_A)) .$$

From these contradictions we obtain $i_1 = i_2$ and the uniqueness is proved.

Definition 9. Let (A, F) be an algebra with the set $\mathscr{A} = \{\oplus\} \cup \Omega$ of algebraic operations and let 0 be a zero of (A, F). The algebra (A, F) is called *normal*, if for each $\omega \in F$, $\mathrm{ar}\, \omega = n \geqq 1$, each $i \in \{1, \dots, n\}$ and arbitrary $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$ it holds

$$a_1 \dots a_{i-1} 0 a_{i+1} \dots a_n \omega = 0 .$$

Definition 10. Let (A, F) be a normal algebra and $B \subseteq A$. We cal! B an *ideal* of (A, F), if

(I) $a, b \in B \Rightarrow a \oplus b \in B$;

(II) $\omega \in F$, ar $\omega = n$, $a_i \in B$ for at least one $i \in \{1, \ldots, n\}$ imply $a_1 \ldots a_n\omega \in B$.

If

(III) $\omega \in \Omega$, ar $\omega = n$, ω is regular on (A, F) and $a_1 \ldots a_n\omega \in B$ for $a_1, \ldots, a_n \in A$ imply $a_j \in B$ for at least one $j \in \{1, \ldots, n\}$,

the ideal B of (A, F) is called *prime*.

It is clear that the set of all ideals of a normal algebra (A, F) forms a complete lattice with respect to the set inclusion as the lattice order. Further, $\{0\}$ is the least and A the greatest element in this lattice.

If h is a homomorphism of A into an algebra B with a zero 0, denote $\ker h = \{a \in A, h(a) = 0\}$.

Theorem 6. *Let (A, F) be a normal algebra without zero-divisors and let h be a homomorphism of (A, F) into (B, F). Then the following conditions are equivalent:*

(a) $\ker h$ *is a prime ideal of* (A, F).

(b) *If ω is regular on (A, F), then ω is regular on $(h(A), F)$.*

Proof. 1°. Let (a) be true and let ω be regular on (A, F), ar $\omega = n$. Suppose that ω is not regular on $(h(A), F)$. Then there exist $h(a_1), \ldots, h(a_n) \in h(A)$, $h(a_i) \neq h(0)$ for each $i = 1, \ldots, n$ such that $h(a_1) \ldots h(a_n)\, \omega = h(0)$, because $h(0)$ is clearly a zero of $(h(A), F)$. As h is a homomorphism, we have $h(0) = h(a_1) \ldots h(a_n)\, \omega = h(a_1 \ldots a_n\omega)$, thus $a_1 \ldots a_n\omega \in \ker h$. As $\ker h$ is a prime ideal, $a_j \in \ker h$ for some $j \in \{1, \ldots, n\}$, thus $h(a_j) = h(0)$, which is a contradiction. Thus ω is regular also on $(h(A), F)$.

2°. Let (b) be true and let $\ker h$ be no prime ideal of (A, F). It is clear that $\ker h$ is an ideal of (A, F). If this ideal is not prime, there exists an ω regular on (A, F) and by (b) also on $(h(A), F)$ and elements $a_1, \ldots, a_n \in A$ such that $a_1 \ldots a_n\omega \in \ker h$ and $a_i \notin \ker h$ for some $i \in \{1, \ldots, n\}$. Thus

$$h(0) = h(a_1 \ldots a_n\omega) = h(a_1) \ldots h(a_n)\, \omega\,,$$

but ω is regular on $(h(A), F)$ and $h(0)$ is its zero, thus $h(a_j) = h(0)$ for at least one $j \in \{1, \ldots, n\}$. Hence $a_j \in \ker h$, which is a contradiction.

Corollary. *Let A_j, B_j be normal r-similar algebras for $j = 1, \ldots, n$ and h a surjective homomorphism of $A = \prod_{j=1}^{n} A_j$ onto $B = \prod_{j=1}^{n} B_j$. If $(\bar{A}_j \cap \ker h)$ is a prime ideal of $\bar{A}_j$ for each $j = 1, \ldots, n$, then h is directly decomposable.*

Proof follows directly from Theorem 6 and Corollary 1 of Theorem 4.

Notation. Let A, B be algebras of the same type. Denote by $\mathrm{Hom}(A, B)$ the set of all homomorphisms of A into B. If A, B are r-similar, then $\mathrm{Hom}(A, B) \neq \emptyset$, because the mapping $o : A \to \{0\}$ is a homomorphism of A into B. This mapping o is called the *zero-homomorphism*. Let $A_1, \ldots, A_m$ be r-similar, $A = \prod_{i=1}^{m} A_i$ and $a_1, \ldots, a_k \in A$. We introduce the following notation:

$$\circ\sum_{j=1}^{k} a_j = (\ldots((a_1 \oplus a_2) \oplus a_3) \oplus \ldots) \oplus a_k\,.$$

By Lemma 1, if $a \in A$, then $a = \circ\sum_{i=1}^{m} \bar{a}_i$, where $a_i = \mathrm{pr}_i\, a$.

Definition 11. Algebras $A_1, \ldots, A_m$ are called *super similar*, if they are r-similar and $f(0) = 0$ for each $f \in \mathrm{Hom}(A_i, A_j)$ and each $i, j \in \{1, \ldots, m\}$.

Clearly rings or Ω-groups without zero-divisors are super similar contrary to chains with the least element.

Definition 12. Let A_i, B_j be super similar algebras for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $A = \prod_{i=1}^{n} A_i$, $B = \prod_{j=1}^{m} B_j$. Let $\mathrm{F} = \|f_{ij}\|$ be a matrix of the type n/m with elements $f_{ij} \in \mathrm{Hom}(A_i, B_j)$. The mapping f of A into B defined by

$$\mathrm{pr}_j\,(f(a)) = \circ\sum_{i=1}^{n} f_{ij}(\mathrm{pr}_i\, a) \quad \text{for each} \quad j = 1, \ldots, m$$

and each $a \in A$ is said to be *represented by the matrix* F.

Theorem 7. *Let A_i, B_j be super similar algebras for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and h a homomorphism of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{j=1}^{m} B_j$. If there exists a matrix* F *representing h, then all elements in the j-th column of* F *except at most one are zero-homomorphisms for each $j = 1, \ldots, m$.*

Proof. Let h be represented by a matrix F and for some $j \in \{1, \ldots, m\}$ let there exist two $f_{kj}, f_{k'j}$ for $k \neq k'$ which are not zero-homomorphisms. Then there exist $a_k \in A_k$, $a_{k'} \in A_{k'}$ with $f_{kj}(a_k) \neq 0$, $f_{k'j}(a_{k'}) \neq 0$. Hence

$$\mathrm{pr}_j\,(h(\bar{a}_k)) = \circ\sum_{i=1}^{n} f_{ij}(\mathrm{pr}_i\, \bar{a}_k) = f_{kj}(a_k)\,,$$

because $\mathrm{pr}_i\, \bar{a}_k = 0$ for $i \neq k$ and $f_{kj}(0) = 0$. Analogously, $\mathrm{pr}_j\,(h(\bar{a}_{k'})) = f_{k'j}(a_{k'})$. If ω is an n-ary regular operation on A_i, then

$$0 \neq f_{kj}(a_k)\, f_{k'j}(a_{k'}) \ldots f_{k'j}(a_{k'})\, \omega = \mathrm{pr}_j\,(h(a_k \bar{a}_{k'} \ldots \bar{a}_{k'} \omega)) =$$
$$= \mathrm{pr}_j(h(0_A)) = \mathrm{pr}_j\, 0_B = 0\,,$$

which is a contradiction.

Theorem 8. *Let A_i, B_j be super similar algebras for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and let* $\mathrm{F} = \|f_{ij}\|$ *be a matrix of the type n/m with $f_{ij} \in \mathrm{Hom}(A_i, B_j)$. Let all elements except at most one in the j-th column be zero-homomorphisms for each $j = 1, \ldots, m$. Then the mapping f of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{j=1}^{m} B_j$ represented by* F *is a homomorphism fulfilling $f(0_A) = 0_B$.*

Proof. Let $j \in \{1, \ldots, m\}$ and let all elements in the j-th column be zero-homomorphisms. Then

$$f \cdot \mathrm{pr}_j(a) = \mathrm{pr}_j(f(a)) = {\circ}\sum_{i=1}^{n} o(a) = {\circ}\sum_{i=1}^{n} 0 = 0$$

for each $a \in A$, thus $f \cdot \mathrm{pr}_j$ is a zero-homomorphism.

Let $j \in \{1, \ldots, m\}$ and f_{kj} be the one non-zero-homomorphism in the j-th column. Then

$$f \cdot \mathrm{pr}_j(a) = \mathrm{pr}_j(f(a)) = {\circ}\sum_{i=1}^{n} f_{ij}(\mathrm{pr}_i\, a) = f_{kj}(\mathrm{pr}_k\, a),$$

because $f_{ij} = o$ for $i \neq k$, thus also $f \cdot \mathrm{pr}_j$ is a homomorphism fulfilling $f \cdot \mathrm{pr}_j(0_A) = 0$.

Since $f \cdot \mathrm{pr}_j$ is a homomorphism fulfilling $f \cdot \mathrm{pr}_j(0_A) = 0$ for each $j \in \{1, \ldots, m\}$, f is also a homomorphism of A into B and $f(0_A) = 0_B$.

Theorem 9. *Let A_i, B_j be super similar algebras for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and let h be a homomorphism of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{j=1}^{m} B_j$. Then there exists just one matrix* $\mathrm{F} = \|f_{ij}\|$ *of the type m/n with $f_{ij} \in \mathrm{Hom}(A_i, B_j)$ representing h.*

Proof. As A_i, B_j are super similar, clearly $f(0_A) = 0_B$ for an arbitrary homomorphism of A into B. Put $S = \{1, \ldots, m\}$, $S' = \{j \in S, \mathrm{pr}_j(h(A)) \neq 0\}$.

1°. Let $j \in S'$. By Theorem 5, there exist just one $i_0 \in \{1, \ldots, n\}$ with $\mathrm{pr}_j(h(A)) = \mathrm{pr}_j(h(\bar{A}_{i_0}))$. Denote $f_{i_0 j} = \varphi_{i_0} \cdot h \cdot \mathrm{pr}_j$, where φ_{i_0} is the canonical insertion. For $i' \in \{1, \ldots, n\}$, $i' \neq i_0$ we put $f_{i'j} = o \in \mathrm{Hom}(A_i, B_j)$. If $a_i = \mathrm{pr}_i\, a$ for $a \in A$ then, by Theorem 5,

$$\mathrm{pr}_j(h(a)) = \mathrm{pr}_j(h(\bar{a}_{i_0})) = \mathrm{pr}_j(h(\varphi_{i_0}(a_{i_0})))$$

and

$$\mathrm{pr}_j(h(a)) = \varphi_{i_0} \cdot h \cdot \mathrm{pr}_j(a_{i_0}) = f_{i_0 j}(a_{i_0}) = {\circ}\sum_{i=1}^{n} f_{ij}(\mathrm{pr}_i\, a),$$

because $f_{ij}(\mathrm{pr}_i\, a) = o(\mathrm{pr}_i\, a) = 0$ for $i \neq i_0$. These f_{ij} form the j-th column of the matrix F.

2°. Let $j \in S - S'$, put $f_{ij} = o \in \mathrm{Hom}(A_i, B_j)$ for each $i = 1, \ldots, n$. Thus $\mathrm{pr}_j(h(a)) = 0 = {}_\circ\sum_{i=1}^{n} f_{ij}(\mathrm{pr}_i\, a)$ for each $a \in A$. Also these f_{ij} form the j-th column of F for this j.

3°. The matrix F thus obtained is of the type n/m, $f_{ij} \in \mathrm{Hom}(A_i, B_j)$ and $\mathrm{pr}_j(h(a)) = {}_\circ\sum_{i=1}^{n} f_{ij}(\mathrm{pr}_i\, a)$ for each $j \in S$ and each $a \in A$. Hence h is represented by F, which completes the proof.

Corollary. *Let A_i, B_j be super similar algebras for $i = 1, \ldots, n$, $j = 1, \ldots, m$, $A = \prod_{i=1}^{n} A_i$, $B = \prod_{j=1}^{m} B_j$. If $p_{ij} = \mathrm{card\,Hom}(A_i, B_j)$ is a natural number for each $i = 1, \ldots, n$, $j = 1, \ldots, m$, then there exist precisely $s = \prod_{j=1}^{m}(1 + \sum_{i=1}^{n}(p_{ij} - 1))$ homomorphisms of A into B.*

Proof. By Theorems 7, 8, 9 the number is equal to the number of matrices $\mathrm{F} = \|f_{ij}\|$ of the type n/m with $f_{ij} \in \mathrm{Hom}(A_i, B_j)$, which have at most one non-zero-homomorphism in each column. If $p_{ij} = \mathrm{card\,Hom}(A_i, B_j)$, then the j-th column can be constructed in $1 + \sum_{i=1}^{n}(p_{ij} - 1)$ different ways for each $j \in \{1, \ldots, n\}$. However, F has just m columns, thus $s = \prod_{j=1}^{m}(1 + \sum_{i=1}^{n}(p_{ij} - 1))$.

Theorem 10. *Let A_i, B_i be super similar algebras for $i = 1, \ldots, n$ and let h be a surjective homomorphism of $A = \prod_{i=1}^{n} A_i$ onto $B = \prod_{i=1}^{n} B_i$. If the matrix* H *representing h has just one non-zero-homomorphism in each row, then h is directly decomposable.*

Proof. Clearly H is a square matrix of the type n/n. Denote it by $\mathrm{H} = \|h_{ij}\|$. By Theorem 9, such a matrix H representing h exists. If H has just one non-zero-homomorphism in each row, by Theorem 7 it has just one non-zero-homomorphism also in each column, because H is square. Accordingly, there exists just one $j \in \{1, \ldots, n\}$ for each $i \in \{1, \ldots, n\}$ such that $h_{ij}(A_i) = B_j$. Thus $h(\bar{A}_i) = \bar{B}_j$ and by Corollary 1 of Theorem 4, h is directly decomposable.

Definition 13. Let A_i, B_j, C_k be super similar algebras for $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, p$. Let $\mathrm{F} = \|f_{ij}\|$, $\mathrm{G} = \|g_{jk}\|$ be matrices of the types n/m, m/p, respectively, and $f_{ij} \in \mathrm{Hom}(A_i, B_j)$, $g_{jk} \in \mathrm{Hom}(B_j, C_k)$. *The matrix product of* F, G is the matrix $\mathrm{H} = \|h_{ik}\|$ of the type n/p such that $h_{ik}(\mathrm{pr}_i\, a) = {}_\circ\sum_{j=1}^{m} f_{ij} \,.\, g_{jk}(\mathrm{pr}_i\, a)$ for each $a \in A$. Symbolically, $\mathrm{H} = \mathrm{F} \,.\, \mathrm{G}$.

Theorem 11. *Let A_i, B_j, C_k be super similar algebras for $i = 1, \ldots, n, j = 1, \ldots, m$, $k = 1, \ldots, p$ and let f be a homomorphism of $A = \prod_{i=1}^{n} A_i$ into $B = \prod_{k=1}^{m} B_j$ and g a homomorphism of B into $C = \prod_{j=1}^{p} C_k$. If f is represented by* F *and g by* G, *then the mapping $h = f \,.\, g$ of A into C is represented by the matrix* H = F . G.

Proof. By Theorem 9, there exist F, G of the types n/m, m/p representing f, g, respectively. Put H = F . G. Then H is of the type n/p. Denote it by $\mathrm{H} = \|h_{ik}\|$. By Theorem 7, in each column of F and G there is at most one non-zero-homomorphism. Let $j \in \{1, \ldots, m\}$. Choose $i_j \in \{1, \ldots, n\}$ as follows: if there exists a non-zero-homomorphism $f_{i'j}$ in the j-th column of F, put $i_j = i'$, in the other case put $i_j = 1$. Analogously we choose j_k from $\{1, \ldots, m\}$ for each $k \in \{1, \ldots, p\}$. Then

$$h_{ik}(\mathrm{pr}_i\, a) = \circ\sum_{j=1}^{m} f_{ij} \,.\, g_{jk}(\mathrm{pr}_i\, a) = \circ\sum_{j=1}^{m} g_{jk}(f_{ij}(\mathrm{pr}_i\, a)) =$$
$$= g_{j_k j}(f_{ij_k}(\mathrm{pr}_i\, a)) = f_{ij_k} \,.\, g_{j_k k}(\mathrm{pr}_i\, a)\,.$$

Hence $h_{ik} \in \mathrm{Hom}\,(A_i, C_k)$. Let h be represented by H. Then

$$\mathrm{pr}_k(h(a)) = \circ\sum_{i=1}^{n} h_{ik}(\mathrm{pr}_i\, a) = \circ\sum_{i=1}^{n} g_{j_k k}(f_{ij_k}(\mathrm{pr}_i\, a)) = g_{j_k k}(f_{i_{jk} j_k}(\mathrm{pr}_{i_{jk}}\, a))\,.$$

Also

$$\mathrm{pr}_k\,(f \,.\, g(a)) = \mathrm{pr}_k\,(g(f(a))) = g_{j_k k}(\mathrm{pr}_{j_k}\,(f(a))) = g_{j_k k}(f_{i_{jk} j_k}(\mathrm{pr}_{i_{jk}}\, a))\,,$$

thus $h = f \,.\, g$.

References

[1] *G. Grätzer:* Universal Algebra (N.Y., Van Nostrand, 1968).

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