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THE RANK OF EXTREME POSITIVE OPERATORS ON POLYHEDRAL CONES

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In a recent paper on diagonals of convex sets [1] the authors have obtained a number of results describing the properties of this notion. It turned out that there is an intimate connection between diagonals of a polyhedral cone and linear dependence relations between the extreme rays of the cone.

One of the main results stated that an indecomposable polyhedral cone (of dimension n greater than one) has at least two diagonals; if it has exactly two diagonals then it is generated by $n + 1$ extreme vectors which satisfy exactly one relation of linear dependence. Such cones are called minimal.

In the present paper we intend to describe completely the extreme rays of the cone $P(C_1, C_2)$ of all operators A such that $AC_1 \subset C_2$ in the case that both C_1 and C_2 are minimal cones.

The main result (Thm. (2,6)) says – roughly speaking – that the rank of such an extreme operator may assume any of the possible values within the natural boundaries with the exception of rank two.

1. Preliminaries. We shall use the notation and terminology of the paper [1]. In particular, Hom (E, F) will denote the set of all linear operators of the linear space E into the linear space F.

We shall need the following technical result.

(1,1) Suppose we are given two spaces $E^{(1)}$ and $E^{(2)}$ and two sets of vectors

$$p(1,1), \ldots, p(1, v_1) \in E^{(1)}, \quad p(2,1), \ldots, p(2, v_2) \in E^{(2)}.$$

Denote by R_1 the linear space of all column vectors r of length v_1 with coordinates $r^T = (r(1,1), \ldots, r(1, v_1))$ such that

$$\sum_{j=1}^{v_1} r(1, j)\, p(1, j) = 0 .$$

Let R_2 have a similar meaning with respect to the space $E^{(2)}$ and the set of vectors $p(2,1), \ldots, p(2, v_2)$.

Denote by $D(\psi)$ the linear space of all v_2 by v_1 matrices B such that $BR_1 \subset R_2$. For each $B \in D(\psi)$, there exists an operator $\psi(B) \in \mathrm{Hom}\,(E^{(1)}, E^{(2)})$ such that

$$\psi(B)\, p(1, i) = \sum_{k=1}^{v_2} b_{ik}\, p(2, k)\,.$$

The mapping ψ is a linear mapping of $D(\psi)$ onto $\mathrm{Hom}\,(E^{(1)}, E^{(2)})$. Its kernel consists of all v_2 by v_1 matrices all columns of which belong to R_2.

Suppose now we have chosen bases in $E^{(1)}$ and $E^{(2)}$ and that the vectors $p(1, i)$ and $p(2, j)$ are represented in these bases as column vectors respectively, of lengths n_1 and n_2. Denote by P_1 the n_1 by v_1 matrix

$$P_1 = (p(1,1), p(1,2), \ldots, p(1, v_1))\,,$$

and similarly, by P_2 the n_2 by v_2 matrix

$$P_2 = (p(2,1), p(2,2), \ldots, p(2, v_2))\,.$$

If $B \in D(\psi)$ is given and if $\psi(B)$ is represented in these bases by the matrix $M(\psi(B))$, we have

$$(*) \qquad MP_1 = P_2 B\,.$$

Conversely, if M is an n_2 by n_1 matrix for which there exists a v_2 by v_1 matrix B satisfying $(*)$, then $B \in D(\psi)$ and $\psi(B)$ is the operator whose matrix in the given bases is M.

Denote by C_1 and C_2 the convex cones generated respectively by the sets of vectors $p(1, i)$ and $p(2, j)$. Then $T \in P(C_1, C_2)$ if and only if $T = \psi(B)$ for some nonnegative $B \in D(\psi)$.

Proof. May be left to the reader.

2. Results. We shall be dealing with minimal cones generated by relations of a particularly simple type. Set

$$N_1 = \{1, \ldots, n_1 + 1\}\,, \quad N_2 = \{1, \ldots, n_2 + 1\}$$

and consider two decompositions

$$N_1 = A_1 \cup B_1\,, \quad N_2 = A_2 \cup B_2$$

such that each of the four sets A_1, B_1, A_2, B_2 contains at least two elements.

Assume that the vectors $p(1, i)$, $i \in N_1$, satisfy exactly one (up to a factor) linear relation

$$\sum_{i \in A_1} p(1, i) - \sum_{i \in B_1} p(1, i) = 0\,.$$

Similarly, assume that the vectors $p(2, i)$, $i \in N_2$, satisfy exactly one (up to a factor) linear relation

$$\sum_{i \in A_2} p(2, i) - \sum_{i \in B_2} p(2, i) = 0 .$$

(2,1) Proposition. (i) *Let A be a matrix of type $n_2 + 1$ by $n_1 + 1$. Then $A \in D(\psi)$ if and only if there exists a number $\lambda(A)$ such that*

$$\begin{aligned} \sum_{\sigma \in A_1} a_{i\sigma} - \sum_{\sigma \in B_1} a_{i\sigma} &= \lambda(A) \quad \textit{for} \quad i \in A_2 , \\ &= -\lambda(A) \quad \textit{for} \quad i \in B_2 . \end{aligned}$$

(ii) *Given $T \in P(C_1, C_2)$ there exists exactly one $A \geqq 0$ with $\psi(A) = T$ which satisfies the following postulate: for each σ there exists a $j \in A_1$ such that $a_{j\sigma} = 0$. The corresponding $\lambda(A)$ will be denoted by $\mu(T)$.*

(iii) *According to* (ii), *there exists a one-to-one correspondence between operators in $P(C_1, C_2)$ and $n_1 + 1$ by $n_2 + 1$ matrices which satisfy the conditions* sub (i). *The matrix corresponding in this way to the operator T will be called the canonical form of T.*

Proof. According to (1,1) the inclusion $A \in D(\psi)$ is equivalent to the inclusion $BR_1 \subset R_2$, in other words,

$$Br_1 = \lambda(B)\, r_2$$

where r_1 and r_2 are respectively the column vectors with coordinates $r_{1i} = 1$ for $i \in A_1$, $r_{1i} = -1$ for $i \in B_1$, $r_{2j} = 1$ for $j \in A_2$, $r_{2j} = -1$ for $j \in B_2$. This proves (i).

Now consider an $A \in P(C_1, C_2)$; there exist, for each $\sigma \in N_1$, nonnegative numbers $\tilde{a}_{i\sigma}$ such that

$$T\,p(1, \sigma) = \sum_{j \in A_2} \tilde{a}_{j\sigma}\, p(2, j) + \sum_{j \in B_2} \tilde{a}_{j\sigma}\, p(2, j) .$$

For each $\sigma \in N_1$, denote by ξ_σ the minimum of all $\tilde{a}_{j\sigma}$. For each $\sigma \in N_1$ and each $j \in A_2$, set

$$a_{j\sigma} = \tilde{a}_{j\sigma} - \xi_\sigma ;$$

for $j \in B_2$, set

$$a_{j\sigma} = \tilde{a}_{j\sigma} + \xi_\sigma .$$

Clearly, all $a_{j\sigma}$ are nonnegative and, for each $\sigma \in N_1$, at least one $a_{j\sigma}$, $j \in A_2$, is zero.

Since

$$\sum_{\sigma \in A_1} p(1, \sigma) - \sum_{\tau \in B_1} p(1, \tau) = 0 ,$$

it follows that

$$\sum_{\sigma \in A_1} \sum_{j \in A_2} a_{j\sigma}\, p(2, j) + \sum_{\sigma \in A_1} \sum_{j \in B_2} a_{j\sigma}\, p(2, j) - \sum_{\tau \in B_1} \sum_{j \in A_2} a_{j\tau}\, p(2, j) - \sum_{\tau \in B_1} \sum_{j \in B_2} a_{j\tau}\, p(2, j) = 0 .$$

Rearranging this sum, we obtain

$$\sum_{j\in A_2}\Big(\sum_{\sigma\in A_1} a_{j\sigma} - \sum_{\tau\in B_1} a_{j\tau}\Big)\, p(2,j) + \sum_{j\in B_2}\Big(\sum_{\sigma\in A_1} a_{j\sigma} - \sum_{\tau\in B_1} a_{j\tau}\Big)\, p(2,j) = 0\,.$$

Since there exists only one relation for the $p(2, j)$, the relation just obtained must be a multiple of the relation

$$\sum_{j\in A_2} p(2,j) - \sum_{j\in B_2} p(2,j) = 0\,.$$

Hence there exists a number λ such that

$$\sum_{\sigma\in A_1} a_{j\sigma} - \sum_{\tau\in B_1} a_{j\tau} = \lambda \quad \text{if} \quad j \in A_2\,, \qquad \sum_{\sigma\in A_1} a_{j\sigma} - \sum_{\tau\in B_1} a_{j\tau} = -\lambda \quad \text{if} \quad j\in B_2\,.$$

This proves the conditions sub (i).

Suppose now that $a'_{j\sigma}$, $(j \in N_2, \sigma \in N_1)$ and λ' satisfy the conditions sub (i). Let σ be fixed. Then

$$\sum_{j\in A_2} (a_{j\sigma} - a'_{j\sigma})\, p(2,j) + \sum_{j\in B_2} (a_{j\sigma} - a'_{j\sigma})\, p(2,j) = 0\,.$$

Consequently, there exists an η_σ such that

$$a_{j\sigma} - a'_{j\sigma} = \eta_\sigma \quad \text{for each} \quad j\in A_2\,, \qquad a_{j\sigma} - a'_{j\sigma} = -\eta_\sigma \quad \text{for each} \quad j \in B_2\,.$$

By (ii), there exist indices $k(\sigma)$ and $l(\sigma)$ such that

$$a_{k(\sigma),\sigma} = a_{l(\sigma),\sigma} = 0\,.$$

Since $a_{k(\sigma),\sigma} \geqq 0$, it follows that $\eta_\sigma \leqq 0$. Since $a_{l(\sigma),\sigma} \geqq 0$, we have $\eta_\sigma \geqq 0$. This proves the uniqueness of the numbers $a_{i\sigma}$. Uniqueness of λ follows from that of $a_{i\sigma}$.

To complete the proof, we shall apply Theorem (1,1) to the matrix $B = (a_{\sigma j})$. It is easy to verify that the relation $BR_1 \subset R_2$ is fulfilled.

(2,2) *Let $A, B, C \in P(C_1, C_2)$ and suppose that $A = B + C$. Suppose that for some $i \in N_1$*

$$A\, p(1,i) = \sum_{j\in N_2} \alpha(j)\, p(2,j)$$

with nonnegative $\alpha(j)$,

$$Bp(1,i) = \sum_{j\in N_2} \beta(j)\, p(2,j)\,, \quad C\, p(1,i) = \sum_{j\in N_2} \gamma(j)\, p(2,j)\,.$$

Suppose further that there exist indices $k \in A_2$ and $j \in B_2$ such that $\alpha(k) = \alpha(j) = 0$. Then $\alpha(s) = \beta(s) + \gamma(s)$ for all $s \in N_2$.

Proof. Denote by r the vector $r = (r(1), \ldots, r(n_1))$ with $r(i) = -1$ for $i \in A_2$ and $r(i) = 1$ for $i \in B_2$. If we denote by α, β, γ the vectors $\alpha = (\alpha(1), \ldots, \alpha(n_1))$,

$\beta = \ldots, \gamma = \ldots$ it follows from the relation $A = B + C$ that

$$\beta + \gamma = \alpha + \xi r$$

for some ξ; in particular, taking the j-th coordinate, we obtain

$$0 \leqq \beta(j) + \gamma(j) = \xi .$$

For the k-th coordinate we obtain

$$0 \leqq \beta(k) + \gamma(k) = -\xi = -\beta(j) + \gamma(j) \leqq 0 .$$

Thus $\xi = 0$. The rest is obvious.

Now we are able to formulate the main result of this section, a complete description of all extreme rays of $P(C_1, C_2)$.

(2,3) Theorem. *Let $E^{(1)}$ and $E^{(2)}$ be two linear spaces of dimensions n_1 and n_2 respectively. Let C_1 and C_2 be minimal cones in $E^{(1)}$ and $E^{(2)}$ respectively. Suppose that C_1 is generated by the vectors*

$$p(1, 1), \ldots, p(1, n_1 + 1)$$

and that C_2 is generated by

$$p(2, 1), \ldots, p(2, n_2 + 1) .$$

Let us introduce the following abbreviations

$$N_i = \{1, 2, \ldots, n_i + 1\} \quad for \quad i = 1, 2 .$$

Consider two decompositions

$$N_i = A_i \cup B_i , \quad i = 1, 2$$

such that each of the four sets A_1, A_2, B_1, B_2 has at least two elements and the vectors p satisfy the following two relations

$$\sum_{j \in A_i} p(i, j) - \sum_{j \in B_i} p(i, j) = 0 .$$

1° *For each triple $i \in A_1, j \in B_1, s \in N_2$ there exists an operator T such that*

$$T p(1, i) = p(2, s) , \quad T p(1, j) = p(2, s) ,$$
$$T p(1, k) = 0 \quad \text{for all} \quad k \in N_1 , \quad i \neq k \neq j .$$

Such operators will be called operators of type 1. The operator described above will be denoted by T_{sij}.

2° *For each pair of decompositions*

$$A_2 = \bigcup_{j \in A_1} A_{2j} , \quad B_2 = \bigcup_{j \in B_1} B_{2j}$$

such that at least two of the sets A_{2j} as well as at least two of the sets B_{2j} are nonvoid there exists an operator T satisfying the following postulates:

$$Tp(1, i) = \sum_{j \in A_{2i}} p(2, j) \quad \text{for} \quad i \in A_1,$$

$$Tp(1, i) = \sum_{j \in B_{2i}} p(2, j) \quad \text{for} \quad i \in B_1.$$

Such operators will be called operators of type 2.

3° *for each pair of decompositions*

$$A_2 = \bigcup_{j \in B_1} A_{2j}, \quad B_2 = \bigcup_{j \in A_1} B_{2j}$$

such that at least two of the sets A_{2j} as well as at least two of the sets B_{2j} are nonvoid there exists an operator T satisfying the following postulates:

$$Tp(1, i) = \sum_{j \in B_{2i}} p(2, j) \quad \textit{for} \quad i \in A_1,$$

$$Tp(1, i) = \sum_{j \in A_{2i}} p(2, j) \quad \textit{for} \quad i \in B_1.$$

Such operators will be called operators of type 3.

Then $W \in \mathrm{Hom}\,(E^{(1)}, E^{(2)})$ is an extreme element of $P(C_1, C_2)$ if and only if it is a positive multiple of an operator of one of the three types described above.

Proof. The fact that the postulates above define operators and that these operators are distinct in an immediate consequence of the preceding theorem since the matrices defining the operators are given in the canonical form.

We prove first that operators of type 1, 2 and 3 are all extreme. To this purpose, we shall prove the following proposition:

Let $B = (b_{ik})$ be an $n_1 + 1$ by $n_2 + 1$ matrix with nonnegative entries. Suppose that A is a matrix in canonical form such that $\psi(A)$ is one of the operators of the three types described in the theorem. Suppose that $B \leqq A$. Suppose that $B \in D(\psi)$, $Br_1 = \xi r_2$ where the column vectors r_1, r_2 are defined by $r_1 = (r_{1i})$, $r_{1i} = 1$ if $i \in A_1$, $r_{1i} = -1$ if $i \in B_1$, $r_2 = (r_{2i})$, $r_{2i} = 1$ if $i \in A_2$, $r_{2i} = -1$ if $i \in B_2$. Then $B = \omega A$ for some $\omega \geqq 0$.

Proof. Since $0 \leqq B \leqq A$, $b_{ik} = 0$ whenever $a_{ik} = 0$. Consider first operators of type 1. Thus there exist indices $i \in A_1$, $j \in B_1$ and $s \in N_2$ such that the only non-zero entries of the matrix $A = (a_{pq})$ are $a_{si} = a_{sj} = 1$. Since $Br_1 = \xi r_2$ and at most one row of B is non-zero, we have $\xi = 0$. Consequently, $b_{si} - b_{sj} = \xi r_{2j} = 0$ which implies $B = b_{si}A$.

Now let A be a canonical matrix of an operator of type 2 corresponding to the decompositions

$$A_2 = \bigcup_{j \in B_1} A_{2j}, \quad B_2 = \bigcup_{j \in B_1} B_{2j}.$$

Let $i \in N_2$ be a given index. It is easy to see that there exists exactly one index $k(i)$ such that $a_{i,k(i)} = 1$ and all remaining entries $a_{i,l}$ are equal to zero. Indeed, if $i \in A_2$ then $k(i) = t$ if $i \in A_{2t}$; if $i \in B_2$ then $k(i) = t$ if $i \in B_{2t}$. It follows from the relation $Br_1 = \xi r_2$ that $b_{i,k(i)} = \xi$ for all $i \in N_2$. Consequently, $B = \xi A$.

In the case of an operator of type 3 we obtain from $Br_1 = \xi r_2$ similarly $B = -\xi A$. The proof of the proposition is complete.

Now let T be an operator of one of the three types described in the theorem and let A be the corresponding matrix in the canonical form. Suppose that $T = T_1 + T_2$ where both T_1 and T_2 belong to $P(C_1, C_2)$. Let B_1 and B_2 be two nonnegative matrices defining T_1 and T_2 respectively. We observe first that the matrix A, being given in the canonical form, has at least two zero entries in each column, one with row index in A_2 and one with row index in B_2. It follows from Lemma (2,2) that $A = B_1 + B_2$. Consequently, $0 \leqq B_1 \leqq A$ so that $B_1 = \omega A$ according to the auxiliary proposition just proved. Hence $T_1 = \omega A$ which proves that A is extreme.

Denote by $\mathscr{T}$ the set of all operators of type 1, 2 or 3. We have seen that every operator T of $\mathscr{T}$ generates an extreme ray of the cone $P(C_1, C_2)$. To prove that every extreme ray is a multiple of some operator in $\mathscr{T}$, it suffices to prove that every element in $P(C_1, C_2)$ may be written as a convex combination of elements of $\mathscr{T}$.

First of all, we shall dispose of the following case.

(α) There exists a nonnegative matrix A such that $T = \psi(A)$ and $\lambda(A) = 0$. Then it is easy to see that

$$T = \sum_{s \in N_2} \sum_{i \in A_1} \sum_{j \in B_1} \frac{a_{is} a_{js}}{\sum_{k \in A_1} a_{ks}} T_{sij}$$

where a_{pq} are the elements of the matrix A so that the assertion holds.

We shall use induction with respect to the number $h(T)$ of positive entries in the canonical matrix A corresponding to T. If $h(T) = 0$, we have $T = 0 \in \operatorname{conv} \mathscr{T}$ trivially. Now let $h(T) > 0$ and suppose the assertion proved for all operators with smaller h.

Now we shall distinguish three cases.

1° $\mu(T) = 0$; in this case our assertion follows immediately from the preceding result (α).

2° $\mu(T) > 0$. Let A be the nonnegative matrix in canonical form such that $\psi(A) =$ $= T$ whence $\lambda(A) = \mu(T) > 0$.

We shall distinguish two subcases:

21° there exists a non-zero entry a_{si} in A such that either $(s, i) \in A_2 \times B_1$ or $(s, i) \in B_2 \times A_1$. Let first $(s, i) \in A_2 \times B_1$. It follows from the definition of $\lambda(A)$ that

$$\sum_{r \in A_1} a_{sr} > \lambda(A) > 0$$

so that $a_{sj} > 0$ for some $j \in A_1$. Since $h(T - \min(a_{si}, a_{sj}) T_{sij}) < h(T)$, we have, by the induction hypothesis, $T - \min(a_{si}, a_{sj}) T_{sij} \in \operatorname{conv} \mathscr{T}$. Consequently, $T \in \operatorname{conv} \mathscr{T}$ as well. If $(s, i) \in B_2 \times A_1$, it follows from

$$-\sum_{r \in B_1} a_{sr} < -\lambda(A)$$

that $a_{sj} > 0$ for some $j \in B_1$. Thus $h(T - \min(a_{si}, a_{sj}) T_{sji}) < h(T)$ and the induction hypothesis applies as well.

22° $\lambda(T) > 0$ and all non-zero entries a_{si} of A are in the union of the blocks $A_2 \times A_1$ and $B_2 \times B_1$.

221° First we dispose of the following simpler case: All non-zero entries of $B_2 \times B_1$ are concentrated in one column, with index k, say. According to 3° of Proposition (2,1) we have

$$a_{sk} = \lambda(A) \quad \text{for all} \quad s \in B_2 .$$

We shall denote by $\tilde{A}$ the matrix obtained from A by changing the k-th column of A as follows

$$\tilde{a}_{sk} = \lambda(A) \quad \text{for} \quad s \in A_2 , \qquad \tilde{a}_{sk} = 0 \quad \text{for} \quad s \in B_2 .$$

Since the difference $\tilde{A} - A$ has non-zero entries in the k-th column only and this column is a multiple of the relation r_2, the matrix A generates the same operator T. At the same time, $\tilde{A}$ is nonnegative and $\lambda(\tilde{A}) = 0$. It follows from (α) that $T \in \operatorname{conv} \mathscr{T}$.

222° It remains to deal with the case where at least two columns with indices in B_1 contain nonzero entries.

In order to use the induction hypothesis in this case it will be necessary to choose an operator of type 2.

Suppose we have found an operator T_0 of type 2 such that $\varepsilon A_0 \leqq A$ for some positive ε where A_0 is the canonical matrix of T_0. Then it is possible to find a positive ω such that $\omega A_0 \leqq A$ and, for $T_0 = \psi(A_0)$,

$$h(T - \omega T_0) < h(T) .$$

By induction hypothesis, we have then $T - \omega T_0 \in \operatorname{conv} \mathscr{T}$ so that $T \in \operatorname{conv} \mathscr{T}$ as well. This shows that it will be sufficient to find an operator T_0 with the properties mentioned above. In other words, we are to find two decompositions:

$$A_2 = \bigcup_{k \in A_1} A_{2k}$$

and

$$B_2 = \bigcup_{k \in B_1} B_{2k}$$

with the following properties:

1° each of these decompositions contains at least two nonvoid sets;

2° the following two implications hold:

$$\text{whenever} \quad s \in A_{2k} \quad \text{then} \quad a_{sk} \neq 0 \,;$$
$$\text{whenever} \quad s \in B_{2k} \quad \text{then} \quad a_{sk} \neq 0 \,.$$

The rest of this proof is devoted to the construction of these decompositions.

We shall define the decomposition

$$A_2 = \bigcup_{k \in A_1} A_{21}$$

as follows:

For each $k \in A_1$ denote by V_k the set

$$V_k = \{s \in A_2;\ a_{sk} \neq 0\}$$

and set

$$A_{2k} = V_k \setminus \bigcup \{V_t;\ t \in A_1,\ t < k\} \,.$$

Clearly the sets A_{2k} are disjoint. Since $\lambda(A) > 0$, each row of A contains at least one non-zero entry. Consequently, $\bigcup_{k \in A_1} A_{2k} = A_2$. We shall show now that at least two sets of decomposition $\{A_{2k}\}$ are nonvoid. To see that it suffices to observe that none of the sets V_k can fill the whole set A_2, the matrix A being in the canonical form. It is obvious from the construction that the decomposition possesses property 2°.

To define the decomposition

$$B_2 = \bigcup_{k \in B_1} B_{2k} \,,$$

we shall distinguish two cases:

Consider first the case that each row of A with row index in B_2 contains at most one non-zero entry.

We set then $B_{2k} = \{s \in B_2;\ a_{sk} \neq 0\}$. Since $\lambda(A) > 0$, each row contains at least, and thus exactly one non-zero entry. Consequently, the union of the B_{2k} is the whole of B_2 and the B_{2k} are disjoint. At the same time, at least two of them are nonvoid because the non-zero entries of A in $B_2 \times B_1$ are not all contained in one column.

It remains to deal with the case that some row, with row index $t \in B_2$ say, contains at least two non-zero entries, say $a_{ti} \neq 0$, $a_{tj} \neq 0$, $i, j \in B_1$. We define then for $k \in B_1$

$$\tilde{W}_k = \{s \in B_2;\ s \neq t,\ a_{sk} \neq 0\}$$

and set

$$\tilde{B}_{2k} = \tilde{W}_k \setminus \bigcup \{\tilde{W}_r;\ r \in B_1,\ r < k\} \,.$$

Now we set $B_{2k} = \tilde{B}_{2k}$ for all $k \in B_1$, different from both i and j. To define B_{2i} and B_{2j}, we shall distinguish two cases:

If $\tilde{B}_{2i} = B_2 \setminus \{t\}$, we set

$$B_{2i} = \tilde{B}_{2i} \,, \quad B_{2j} = \{t\} \,.$$

If $\tilde{B}_{2i} \neq B_2 - \{t\}$, hence is properly contained in $B_2 \setminus \{t\}$, we set

$$B_{2i} = \tilde{B}_{2i} \cup \{t\}\,, \quad B_{2j} = \tilde{B}_{2j}\,.$$

It follows from this construction that the properties 1° and 2° are satisfied in this case as well. The proof is complete.

3° $\mu(T) < 0$. In this case, the proof may be effected analogously to that of case 2°; one has only to use operators of type 3 instead of operators of type 2.

The attentive reader may have observed that one can proceed also as follows: The number $\mu(T)$ depends on the choice of one subset of the decomposition $N_1 = A_1 \cup B_1$ as A_1 and the other as B_1. If we interchange the notation, the number $\mu(T)$ will be changed to $-\mu(T)$ and the classes of operators of type 2 and 3 will be interchanged, too. Consequently, the validity of the assertion in the case $\mu(T) < 0$ follows from the validity of case 2°.

(2,4) Theorem. *Denote by $r(T)$ the dimension of the range of the operator T. Then*

1° $r(T) = 1$ *for each operator of type* 1;

2° *if T is an operator of type 2 corresponding to the decompositions $\{A_{2j}\}$, $j \in A_1$ and $\{B_{2j}\}$, $j \in B_1$ then $r(T) = K_1 + K_2 - 1$ where K_1 is the number of non-empty sets A_{2j} and K_2 the number of non-empty sets B_{2j}.*

3° *if T is an operator of type 3 corresponding to the decompositions $\{A_{2j}\}$, $j \in B_1$ and $\{B_{2j}\}$, $j \in A_1$ then $r(T) = K_1 + K_2 - 1$ where K_1 is the number of non-empty sets A_{2j} and K_2 the number of non-empty sets B_{2j}.*

Proof. The assertion about the operator of type 1 is obvious. Let T be an operator of type 2 corresponding to the decompositions $\{A_{2j}\}$, $j \in A_1$ and $\{B_{2j}\}$, $j \in B_1$. Suppose $i \in A_1$ is such that $A_{2i} \neq \emptyset$. The vectors p_t with $t \in N_1 \setminus \{i\}$ form a basis of the space $E^{(1)}$. The dimension of the range of T is equal to the number of linearly independent vectors among the vectors Tp_t, $t \in N_1 \setminus \{i\}$. Denote by $\tilde{A}_1$ the set of all $j \in A_1$ such that $A_{2j} \neq \emptyset$; hence $i \in \tilde{A}_1$. Similarly denote by $\tilde{B}_1$ the set of all $j \in B_1$ such that $B_{2j} \neq \emptyset$. It is easy to see that the vectors Tp_j, $j \in \tilde{A}_1 \cup \tilde{B}_1 \setminus \{i\}$ are linearly independent and form a basis of the range of T. Thus $r(T) = K_1 + K_2 - 1$ as asserted.

The proof of 3° is analogous.

(2,5) Lemma. *Let C_1 and C_2 be two non-empty sets, let $1 \leqq r \leqq \min(|C_1|, |C_2|)$. Then there exists a decomposition $C_2 = \bigcup_{j \in C_1} C_{2j}$ with exactly r nonvoid components C_{2j}.*

Proof. May be left to the reader.

(2,6) Theorem. *Let C_1 and C_2 be two minimal cones in the spaces $E^{(1)}$ and $E^{(2)}$, respectively. Denote by n_1 and n_2, respectively the dimensions of C_1 and C_2. Then $P(C_1, C_2)$ is a polyhedral cone of dimension $n_1 n_2$. The set of all extreme operators of $P(C_1, C_2)$ is described in Theorem (2,3). Extreme operators have rank either* 1 *or greater than* 2. *If T is an extreme operator then T has rank* 1 *if and only if T is a positive multiple of an operator of type* 1. *Operators of type* 2 *have rank greater than* 2 *and each number r, $3 \leqq r \leqq \min(k_1, k_2) + \min(l_1, l_2)$ may be obtained as the rank of some extreme operator of type* 2. *Operators of type* 3 *have also rank greater than* 2 *and each number r, $3 \leqq r \leqq \min(k_1, l_2) + \min(k_2, l_1)$ may be obtained as the rank of some extreme operator of type* 3. *Here, $k_i = |A_i|$, $l_i = |B_i|$, $i = 1, 2$.*

Proof. We shall restrict ourselves to the proof that operators of type 2 realise all ranks between 3 and $d = \min(k_1, k_2) + \min(l_1, l_2) - 1$. Given r such that $3 \leqq r \leqq d$ there exist numbers r_1, r_2 such that $2 \leqq r_1 \leqq \min(k_1, k_2)$, $2 \leqq r_2 \leqq \min(l_1, l_2)$ $r = r_1 + r_2 - 1$. By Lemma (2,5) there exist decompositions $\{A_{2j}\}$, $j \in A_1$ of A_2 and $\{B_{2j}\}$, $j \in B_1$, of B_2 with exactly r_1 and r_2 nonvoid components. It follows then from Theorem (2,4) that the corresponding extreme operator has rank r.

The case of operators of type 3 is analogous.

In the papers [2], [3] it was proved that if C is an indecomposable cone then the identity is an extreme operator of $P(C, C)$. This leads naturally to the conjecture: if both C_1 and C_2 are indecomposable then some of the extreme operators of $P(C_1, C_2)$ has rank $\min(n_1, n_2)$ where n_1, n_2 are dimensions of C_1, C_2 respectively. Theorem (2,6) may be used to disprove this conjecture. Indeed, if we take $k_1 = 2$ and $k_2 = \frac{1}{2}n_2$ (for n_2 even), the maximal possible rank is $2 + \frac{1}{2}n_2 - 1 = \frac{1}{2}n_2$ which will be less than $\min(n_1, n_2)$ if $n_1 > n_2 > 2$.

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