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PARAMETERS OF DISTRIBUTION OF $(n + 1)$-DIMENSIONAL MONOSYSTEMS IN THE EUCLIDEAN SPACE R^{2n+1}

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1. INTRODUCTION

A monosystem N is a manifold generated by a one-parameter family of linear spaces. If the dimension of N is $n + 1$, if $\mathbf{r}(s)$ (s always represents the arc length, while accents mean derivation to s) is a base curve and if $\mathbf{e}_1(s), \ldots, \mathbf{e}_n(s)$ constitute a base of the generating space $B(s)$ for all $s \in S$ (S is a domain in which the functions we consider are of sufficiently high class), then N can be represented by

$$\mathbf{R}(s, l_1, \ldots, l_n) = \mathbf{r}(s) + \sum_{i=1}^{n} l_i \, \mathbf{e}_i(s) , \quad s \in S , \quad l_i \in R \quad (i = 1, \ldots, n) .$$

Suppose that

$$\operatorname{rank} \left[\mathbf{r}'(s) \, \mathbf{e}_1(s) \ldots \mathbf{e}_n(s) \, \mathbf{e}'_1(s) \ldots \mathbf{e}'_n(s)\right] = 2n + 1 , \quad \forall s \in S .$$

This means that N is non-developable, or, in other words, that for every generating space $B(s)$, the mapping: point $\mapsto$ tangent space, $p \mapsto N_p$ is a non-singular projectivity.

There is just one central point in each generating space. The locus H_φ of the points p of a generating space $B(s)$, for which the tangent space N_p includes a constant angle φ $(0 < \varphi < \pi/2)$ with the tangent space N_a at the central point a of $B(s)$ is a central hyperquadric with standard equation

$$(1) \qquad \sum_{i=1}^{n} \frac{x_i^2}{d_i^2} = \operatorname{tg}^2 \varphi .$$

The (strict positive) half axes $d_1, \ldots, d_n$ of $H_{\pi/4}$ are the principal parameters of distribution of $B(s)$. These are the parameters of distribution of the axes of the hyperquadrics H_φ, which we call the principal axes of $B(s)$ (see [2.1.], [3], [5]). In fact we can attach a parameter of distribution to an arbitrary (finite) line of a generating space $B(s)$; first we define the central point a_R of a line R of $B(s)$ as follow:

a_R is the point of R, in which N_{a_R} is orthogonal with the tangent space N_{r_∞} at the infinite point r_∞ of R. To each finite line R of $B(s)$ belongs a strict positive parameter of distribution d, for which holds $a_R p = d \operatorname{tg} \theta$, where p is a variable point of R and where θ means the angle (N_{a_R}, N_p) $(0 \leqq \theta < \pi/2)$.

The central point a is also the central point for each line R of $B(s)$ passing through a. If the direction cosines of this line with respect to the principal axes of $B(s)$ are $\cos \theta_1, \ldots, \cos \theta_n$, one can easely prove (from (1)) that its parameter of distribution d is given by

$$\frac{1}{d^2} = \sum_{i=1}^{n} \frac{\cos^2 \theta_i}{d_i^2}.$$

In section 2 we determine the parameter of distribution of an arbitrary line using special submanifolds, for which also some deeper results are proved.

Finally we introduce in section 3 the notion of "dual parameter of distribution" and give in this connection a nice geometrical signification.

2. ORTHOGONAL SUBMANIFOLDS

Through each (finite) point of a generating space of N, there is passing just one orthogonal trajectory of N. The orthogonal trajectories through the points of a k-dimensional subspace $(1 \leqq k \leqq n)$ of the generating space $B(s_0)$ generate a $(k + 1)$-dimensional monosystem, which we call an orthogonal submanifold of N. Each (finite) line R of $B(s_0)$ determines in particular an orthogonal subsurface O_R of N.

Theorem. *The central point a_R and the parameter of distribution d of a line R of the generating space $B(s_0)$ are also the central point and the parameter of distribution of the generator R of the ruled surface O_R.*

Proof. Suppose that the base curve $\mathbf{r}(s)$ is the orthogonal trajectory through a point of R and that $\mathbf{e}_1(s), \ldots, \mathbf{e}_n(s)$ constitute a natural base system (this means that $\mathbf{e}_i\mathbf{e}_j = \delta_{ij}$ and $\mathbf{e}_i'\mathbf{e}_j = 0$, $(i, j = 1, \ldots, n)$, $\forall s \in S$). If $\cos \theta_1, \ldots, \cos \theta_n$ are direction cosines of R with respect to $\mathbf{e}_1(s_0), \ldots, \mathbf{e}_n(s_0)$ then O_R can be represented by

$$\mathbf{Z}(s, l) = \mathbf{r}(s) + l \sum_{i=1}^{n} \mathbf{e}_i(s) \cos \theta_i, \quad l \in R, \quad s \in S.$$

In any point $\mathbf{Z}(s_0, l)$ of R, the vector $\mathbf{r}'(s_0) + l_i \sum_{i=1}^{n} \mathbf{e}_i'(s_0) \cos \theta_i$ is orthogonal with R and orthogonal with $B(s_0)$. The angle of the tangent spaces of O_R at the points $p_1(s_0, l_1)$ and $p_2(s_0, l_2)$ of R, consequently is the same as the angle between the vectors $\mathbf{r}'(s_0) + l_1 \sum_{i=1}^{n} \mathbf{e}_i'(s_0) \cos \theta_i$ and $\mathbf{r}'(s_0) + l_2 \sum_{i=1}^{n} \mathbf{e}_i'(s_0) \cos \theta_i$ and thus also the same as the angle between the tangent spaces N_{p_1} and N_{p_2}, which completes the proof.

Corollary. *If d is the parameter of distribution of the line R, with central point a_R, of $B(s_0)$, if d' is the parameter of distribution of the line R', parallel with R, through the central point a of $B(s_0)$ and if $\varphi = \text{angle}\,(N_{a_R}, N_a)$, then*

$$d = d'/\cos\varphi\,.$$

Proof. Suppose that $\mathbf{e}_1(s), \ldots, \mathbf{e}_n(s)$ constitute a natural base system, that the vectors $\mathbf{e}_1(s_0), \ldots, \mathbf{e}_n(s_0)$ are parallel with the principal axes of $B(s_0)$ (this means $\mathbf{e}_i'(s_0)\,\mathbf{e}_j'(s_0) = 0 \;\; i \neq j \;\; (i, j = 1, \ldots, n)$) and that $\mathbf{r}(s)$ is the orthogonal trajectory through the central point a_R of R (this means if $\cos\theta_i$ $(i = 1, \ldots, n)$ are the direction cosines of R with respect to $\mathbf{e}_1(s_0), \ldots, \mathbf{e}_n(s_0)$, that $\mathbf{r}'(s_0)\sum_{i=1}^{n}\mathbf{e}_i'(s_0)\cos\theta_i = 0$).

Under these conditions, d will be given by

$$d^2 = 1/\sum_{i=1}^{n}\mathbf{e}_i'^2(s_0)\cos^2\theta_i\,,$$

while the principal parameters of distribution of $B(s_0)$ are given by

$$d_i^2 = \frac{1 - \displaystyle\sum_{j=1}^{n}\frac{(\mathbf{r}'(s_0)\,\mathbf{e}_j'(s_0))^2}{\mathbf{e}_j'^2(s_0)}}{\mathbf{e}_i'^2(s_0)} \qquad (i = 1, \ldots, n)\,.$$

The central points a_R and a have respective vector coordinates $\mathbf{r}(s_0)$ and

$$\mathbf{r}(s_0) - \sum_{i=1}^{n}\frac{\mathbf{r}'(s_0)\,\mathbf{e}_i'(s_0)}{\mathbf{e}_i'^2(s_0)}\,\mathbf{e}_i(s_0)\,,$$

and thus we get

$$\cos^2\varphi = \frac{\left(\left(\mathbf{r}'(s_0) - \displaystyle\sum_{i=1}^{n}\frac{\mathbf{r}'(s_0)\,\mathbf{e}_i'(s_0)}{\mathbf{e}_i'^2(s_0)}\,\mathbf{e}_i'(s_0)\right).\,\mathbf{r}'(s_0)\right)^2}{\left(\mathbf{r}'(s_0) - \displaystyle\sum_{i=1}^{n}\frac{\mathbf{r}'(s_0)\,\mathbf{e}_i'(s_0)}{\mathbf{e}_i'^2(s_0)}\,\mathbf{e}_i'(s_0)\right)^2} = 1 - \sum_{i=1}^{n}\frac{(\mathbf{r}'(s_0)\,\mathbf{e}_i'(s_0))^2}{\mathbf{e}_i'^2(s_0)}\,.$$

And

$$\frac{1}{d^2} = \sum_{i=1}^{n}\mathbf{e}_i'^2(s_0)\cos^2\theta_i = \left(\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}\right)\left(1 - \sum_{i=1}^{n}\frac{(\mathbf{r}'(s_0)\,\mathbf{e}_i'(s_0))^2}{\mathbf{e}_i'^2(s_0)}\right) = \frac{\cos^2\varphi}{d'^2}\,,$$

which has to be proved (for another method see [5]).

Theorem. *Consider the orthogonal submanifold Σ containing the k-dimensional subspace K of the generating space $B(s_0)$ $(1 \leq k \leq n)$. In each point of Σ and in each two-dimensional direction of Σ_p, Σ and N have the same sectional curvature.*

Proof. We first consider the case $k = 1$. We call "normal two-dimensional direction" in a point p of N, a two-dimensional direction of N_p, defined by a line R of the generating space B through p and the normal through p on B in N_p (that is the tangent to the orthogonal trajectory through p). Consider a (finite) line R of B, with central point a_R and parameter of distribution d. In [5] we proved that the sectional curvature K_σ of N in the normal two-dimensional direction determined by R in a variable point p of R, which lies at distance t of a_R, is given by

$$K_\sigma = -\frac{d^2}{(d^2 + t^2)^2}. \tag{2}$$

But d is the parameter of distribution and a_R is the central point of the generator R of the orthogonal submanifold of N containing R, and thus (2) gives also the expression of the Gauss curvature of this ruled surface in the point p of R. This proves our purpose for $k = 1$.

Next suppose that $k > 1$. The orthogonal subsurface determined by a line R of the subspace K of $B(s_0)$, is also an orthogonal submanifold of Σ and thus the theorem holds for any two-dimensional direction in the tangent space Σ_p in a point p of Σ, which is a normal two-dimensional direction of N_p. In [5], we found the following result: suppose that σ is an arbitrary two-dimensional direction of N_p. If $K_{\sigma 0}$ is the sectional curvature in the normal two-dimensional direction of N_p, passing through the line of intersection $\sigma \cap B$, and if δ_0 is the angle of σ and the normal through p on the generating space B in N_p, then

$$K_\sigma = K_{\sigma 0} \cos^2 \delta_0 .$$

And so, since $K_{\sigma 0}$ and δ_0 are the same for N and for Σ, the theorem is proved.

Theorem. *A $(k+1)$-dimensional orthogonal submanifold Σ of N is a total geodesic submanifold of N, iff any k-dimensional generating space $K(s)$ of Σ contains k principal axes of the corresponding generating space $B(s)$ of N.*

Proof. Suppose that the base curve $\mathbf{r}(s)$ of N is an orthogonal trajectory of N and of Σ and that for $s = s_0$ the vectors $\mathbf{e}_1(s_0), \ldots, \mathbf{e}_k(s_0)$ of the natural base system $\mathbf{e}_1(s), \ldots, \mathbf{e}_n(s)$ of N, are parallel with the generating space $K(s_0)$ of Σ in the generating space $B(s_0)$ of N. Then N and Σ can be represented by

$$\mathbf{R}(s, l_1, \ldots, l_n) = \mathbf{r}(s) + \sum_{i=1}^{n} l_i \, \mathbf{e}_i(s)$$

and

$$\mathbf{Z}(s, l_1, \ldots, l_k) = \mathbf{r}(s) + \sum_{j=1}^{k} l_j \, \mathbf{e}_j(s), \quad l_i \in R(i = 1, \ldots, n), \quad s \in S.$$

The $n - k$ vectors $\partial \mathbf{R}/\partial l_r = \mathbf{e}_r(s)$ $(r = k+1, \ldots, n)$ constitute at each point $p(s, l_1, \ldots, l_k)$ of Σ an (orthonormal) basis of the $(n-k)$-dimensional space which is (total) orthogonal with Σ_p in N_p. Thus, Σ is total geodesic iff

$$\mathbf{r}''\mathbf{e}_r + \sum_{j=1}^{k} l_j \mathbf{e}_j''\mathbf{e}_r \equiv 0 \ \forall l_j \in R \quad (j = 1, \ldots, k) \quad \text{and} \quad \forall s \in S \quad (r = k + 1, \ldots, n).$$

These conditions become

$$(3) \qquad \mathbf{r}'\mathbf{e}_r' = 0 \quad \forall s \in S \quad (r = k + 1, \ldots, n)$$

and

$$(4) \qquad \mathbf{e}_j'\mathbf{e}_r' = 0 \quad \forall s \in S \quad (j = 1, \ldots, k;\ r = k + 1, \ldots, n).$$

The n coordinates of the central point a of a general generating space $B(s)$ of N are the solutions of the system of linear equations:

$$(5) \qquad \mathbf{r}'\mathbf{e}_i' + \sum_{t=1}^{n} l_t \mathbf{e}_t'\mathbf{e}_i' = 0 \quad (i = 1, \ldots, n).$$

From (3) and (4), we see that the $n - k$ last coordinates of each central point are zero, which means that the central point of a variable generating space $B(s)$ of N belongs to the corresponding generating space $K(s)$ of Σ. From the conditions (4), there follows that the vectors $\mathbf{e}_1(s), \ldots, \mathbf{e}_k(s)$ (resp. $\mathbf{e}_{k+1}(s), \ldots, \mathbf{e}_n(s)$) in each generating space $B(s)$ are parallel with the space generated by k principal axes of $B(s)$ (resp. $n - k$ princial axes of $B(s)$). This completes the proof.

Corollary. *The line of striction of a total geodesic orthogonal submanifold Σ of N is the same as the line of striction of N.*

Proof. From the proof of the last theorem, it follows that the line of striction of N is a curve of Σ. Moreover, the central point of N is determined by the system (5), while the central point of Σ is given by the same system, but with $t, i = 1, \ldots, k$.

Remarks. 1. The result of the last corollary remains true for any orthogonal submanifold of N which contains the line of striction of N.

2. Suppose that the line of striction of N is an orthogonal trajectory and that Σ is a $(k + 1)$-dimensional total geodesic orthogonal submanifold of N, then, in the proof of the last theorem, we can take the line of striction of N (and also of Σ) as the base curve. Under these conditions and, with the same notations, the orthogonal submanifold $\mathfrak{N}$, represented by

$$\mathbf{r}(s) + \sum_{r=k+1}^{n} l_r \mathbf{e}_r(s) \quad l_r \in R \quad (r = k + 1, \ldots, n), \quad s \in S,$$

is also total geodesic (because besides the conditions (3) and (4), there holds now: $\mathbf{r}'\mathbf{e}_i' = 0, \forall s \in S\ (i = 1, \ldots, k)$). In this case the line of striction of N (and of Σ and $\mathfrak{N}$) is a geodesic line of these three manifolds (cfr. theorem of Bonnet; see [5]).

3. Example. Consider a curve $\mathbf{r}(s)$ $(s \in S)$ in R^5. The formules of Frenet are $(\mathbf{e}_1 = \mathrm{d}\mathbf{r}/\mathrm{d}s)$:

$$\frac{\mathrm{d}\mathbf{e}_1}{\mathrm{d}s} = \frac{\mathbf{e}_2}{\varrho_1}, \quad \frac{\mathrm{d}\mathbf{e}_k}{\mathrm{d}s} = -\frac{\mathbf{e}_{k-1}}{\varrho_{k-1}} + \frac{\mathbf{e}_{k+1}}{\varrho_k} \quad (k = 2, 3, 4), \quad \frac{\mathrm{d}\mathbf{e}_5}{\mathrm{d}s} = -\frac{\mathbf{e}_4}{\varrho_4}.$$

The manifold N represented by

$$\mathbf{R}(s, l_1, l_2) = \mathbf{r}(s) + l_1\,\mathbf{e}_2(s) + l_2\,\mathbf{e}_5(s), \quad l_1, l_2 \in R, \quad s \in S,$$

is non-developable iff $1/\varrho_2 \neq 0$ and $1/\varrho_4 \neq 0$ at each point of the curve $\mathbf{r}(s)$.

The submanifold represented by

$$\mathbf{Z}(s, l) = \mathbf{r}(s) + l\,\mathbf{e}_2(s) \quad l \in R, \quad s \in S,$$

is a total geodesic orthogonal submanifold of N.

3. DUAL PARAMETERS OF DISTRIBUTION

We require that from now on $n \geqq 2$. For each finite $(n - 2)$-dimensional subspace of each generating space, we can define a new parameter of distribution, which we call a dual parameter of distribution. Suppose that H is a $(n - 2)$-dimensional subspace of the generating space $B(s_0)$. In general, there are just two hyperplanes L_1 and L_2 of $B(s_0)$, which contain H, which are orthogonal, in such a manner that the $2n$-dimensional spaces T_{L_1} and T_{L_2}, generated by the tangent spaces of N at the points of L_1, resp. of L_2, are orthogonal too. (In $B(s_0)$, we obtain a model of an elliptic geometry by stating: distance pq = angle (N_p, N_q), for any two (finite or infinite) points p and q of $B(s_0)$. L_1 and L_2 are the hyperplanes of $B(s_0)$ which contain H and which are euclidean and elliptic orthogonal (see [5]))

Theorem. *Suppose that L is a variable hyperplane of $B(s_0)$, containing H, which forms with L_1 the angle θ $(0 \leqq \theta \leqq \pi/2)$, while θ' is the angle of the hyperplanes T_L and T_{L_1} of R^{2n+1} $(0 \leqq \theta' \leqq \pi/2)$. Then there exists a strict positive dual parameter of distribution δ, for which*

$$\operatorname{tg}\theta = \delta \operatorname{tg}\theta'.$$

(It is clear that $1/\delta$ is the dual parameter of distribution of H, calculated with respect to the hyperplane L_2.)

Proof. Suppose that the base curve $\mathbf{r}(s)$ is the orthogonal trajectory through the central point a of the generating space $B(s_0)$, while $\mathbf{e}_1(s), \ldots, \mathbf{e}_n(s)$ form a natural base system, in such a way that for $s = s_0$ the vectors $\mathbf{e}_1(s_0), \ldots, \mathbf{e}_n(s_0)$ have the same directions as the principal axes of $B(s_0)$ (this means $\mathbf{e}_i'\mathbf{e}_j' = 0$ $i \neq j$, $i, j = 1, \ldots, n$,

$s = s_0$). In the space $B(s_0)$, we choose an orthogonal coordinate system with origin a and base $\mathbf{e}_1(s_0), \ldots, \mathbf{e}_n(s_0)$. Consider the hyperquadric $\mathbf{\Gamma}$ of $B(s_0)$, given by the equation (d_i ($i = 1, \ldots, n$) are the principal parameters of distribution of $B(s_0)$)

$$\sum_{i=1}^{n} \frac{x_i^2}{d_i^2} = -1 .$$

It can be proved (see [5]) that the hyperplanes (of R^{2n+1}), generated by the tangent spaces of N at the points of two hyperplanes of $B(s_0)$, are orthogonal, iff the two hyperplanes of $B(s_0)$ are conjugated with respect to the hyperquadric $\mathbf{\Gamma}$ ($\mathbf{\Gamma}$ is the absolute hyperquadric of the earlier mentioned model of the elliptic geometry in $B(s_0)$).

First we consider an arbitrary (finite) $(n-2)$-dimensional subspace H of $B(s_0)$, which does not contain the central point a of $B(s_0)$. Then H is the polar space with respect to the absolute hyperquadric $\mathbf{\Gamma}$ of a finite line R, which does not contain a and which thus can be represented by

$$x_i = l_i + t \cos \theta_i \quad t \in R \quad (i = 1, \ldots, n),$$

with $(l_1, \ldots, l_n) \neq (0, 0, \ldots, 0)$ and $\sum_{i=1}^{n} \cos^2 \theta_i = 1$. The equations of H are

$$\sum_{i=1}^{n} \frac{x_i \cos \theta_i}{d_i^2} = 0 , \quad \sum_{i=1}^{n} \frac{x_i l_i}{d_i^2} = -1 .$$

A variable hyperplane L through H has an equation of the form

$$\sum_{i=1}^{n} \frac{x_i}{d_i^2} (l_i + t \cos \theta_i) = -1 .$$

We now determine the ortogonal hyperplanes L_1 and L_2 through H for which T_{L_1} and T_{L_2} are also orthogonal. Thus we look for the common pair $L_1 \leftrightarrow L_2$ of the orthogonal involution of the hyperplanes of $B(s_0)$ through H and of the involution of the conjugate hyperplanes of $B(s_0)$ through H with respect to $\mathbf{\Gamma}$. By transition on the polar line R, we become on R a first involution, determined by the equation

$$\sum_{i=1}^{n} \frac{(l_i + t \cos \theta_i)(l_i + t' \cos \theta_i)}{d_i^4} = 0 ,$$

and a second involution, namely the involution of the conjugate points of R with respect to $\mathbf{\Gamma}$, with equation

$$\sum_{i=1}^{n} \frac{(l_i + t \cos \theta_i)(l_i + t' \cos \theta_i)}{d_i^2} = -1 .$$

The common conjugate pair of points of both involutions on R, correspond to the values t_1 and t_2 of the parameter, which are the solutions of the quadratic equation

$$t^2\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^4}-\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}\right)+$$

$$+\,t\left(\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^4}-\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}+\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^4}\right)+$$

$$+\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}+\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}-\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}=0\,.$$

Suppose that the point $l(l_1, \ldots, l_n)$ is one of the two points of R we are looking for, or in other words suppose that this quadratic equation has the solution $t = 0$, which means that

$$(6)\qquad \sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}+\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}-\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}=0\,.$$

Call L_1 the hyperplane corresponding with l. We find for the angle θ between the variable hyperplane L and L_1

$$\cos\theta=\frac{\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)\,l_i}{d_i^4}}{\sqrt{\left(\left(\displaystyle\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\right)^2\left(\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)^2}{d_i^4}\right)\right)}},$$

while the angle θ', between the hyperplanes T_L and T_{L_1}, is given by

$$\cos\theta'=\frac{\mathbf{r}'(s_0)+\displaystyle\sum_{i=1}^{n}l_i\,\mathbf{e}_i'(s_0)}{\sqrt{\left(1+\displaystyle\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\right)}}\cdot\frac{\mathbf{r}'(s_0)+\displaystyle\sum_{i=1}^{n}(l_i+t\cos\theta_i)\,\mathbf{e}_i'(s_0)}{\sqrt{\left(1+\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)^2}{d_i^2}\right)}}=$$

$$=\frac{1+\displaystyle\sum_{i=1}^{n}\frac{l_i(l_i+t\cos\theta_i)}{d_i^2}}{\sqrt{\left(\left(1+\displaystyle\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\right)\left(1+\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)^2}{d_i^2}\right)\right)}}\,.$$

So we find

$$\mathrm{tg}^2\,\theta=\frac{\left(\displaystyle\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\right)\left(\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)^2}{d_i^4}\right)-\left(\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)\,l_i}{d_i^4}\right)^2}{\left(\displaystyle\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)\,l_i}{d_i^4}\right)^2}$$

and

$$\operatorname{tg}^2\theta' = \frac{\left(1+\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\right)\left(1+\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)^2}{d_i^2}\right)-\left(1+\sum_{i=1}^{n}\frac{l_i(l_i+t\cos\theta_i)}{d_i^2}\right)^2}{\left(1+\sum_{i=1}^{n}\frac{l_i(l_i+t\cos\theta_i)}{d_i^2}\right)^2}.$$

After some calculations, the numerators of the expressions of $\operatorname{tg}^2\theta$ and $\operatorname{tg}^2\theta'$ become respectively

$$t^2\left(\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^4}-\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}\right)^2\right)$$

and

$$t^2\left(\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}+\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}-\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}\right)^2\right).$$

Moreover, we find using (6)

$$\left(\sum_{i=1}^{n}\frac{(l_i+t\cos\theta_i)\,l_i}{d_i^4}\right)\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}\right)=\left(1+\sum_{i=1}^{n}\frac{l_i(l_i+t\cos\theta_i)}{d_i^2}\right)\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}\right).$$

And so we obtain $\operatorname{tg}\theta = \delta\operatorname{tg}\theta'$,
with

$$\delta^2 = \frac{\left(\sum_{i=1}^{n}\frac{l_i^2}{d_i^4}\right)\left(\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^4}\right)-\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}\right)^2}{\left(\sum_{i=1}^{n}\frac{l_i^2}{d_i^2}\right)\left(\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}\right)-\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}\right)^2+\sum_{i=1}^{n}\frac{\cos^2\theta_i}{d_i^2}}\cdot\frac{\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^2}\right)^2}{\left(\sum_{i=1}^{n}\frac{l_i\cos\theta_i}{d_i^4}\right)^2}.$$

Next, suppose that H contains the central point a and that H is given by the two equations

(7) $$\sum_{i=1}^{n} b_i x_i = 0,$$

(8) $$\sum_{i=1}^{n} c_i x_i = 0.$$

Moreover we require that the hyperplanes L_1 and L_2 have respectively the equations (7) and (8), which means that

$$\sum_{i=1}^{n} b_i c_i = 0 \quad\text{and}\quad \sum_{i=1}^{n} b_i c_i d_i^2 = 0.$$

A variable hyperplane L through H has an equation of the form

$$\sum_{i=1}^{n}(b_i + tc_i)\,x_i = 0.$$

The angle θ between L and L_1 is given by

$$\cos\theta = \frac{\sum_{i=1}^{n}(b_i + tc_i)\, b_i}{\sqrt{((\sum_{i=1}^{n} b_i^2)(\sum_{i=1}^{n}(b_i + tc_i)^2))}} = \frac{\sum_{i=1}^{n} b_i^2}{\sqrt{((\sum_{i=1}^{n} b_i^2)^2 + t^2(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2))}},$$

while the cosinus of the angle θ' of T_L and T_{L_1} becomes

$$\cos\theta' = \frac{\sum_{i=1}^{n} b_i d_i^2\, \mathbf{e}_i'(s_0)}{\sqrt{(\sum_{i=1}^{n} b_i^2 d_i^2)}} \cdot \frac{\sum_{i=1}^{n}(b_i + tc_i)\, d_i^2\, \mathbf{e}_i'(s_0)}{\sqrt{(\sum_{i=1}^{n} b_i^2 d_i^2 + t^2 \sum_{i=1}^{n} c_i^2 d_i^2)}}$$

or

$$\cos\theta' = \frac{\sum_{i=1}^{n} b_i^2 d_i^2}{\sqrt{((\sum_{i=1}^{n} b_i^2 d_i^2)(\sum_{i=1}^{n} b_i^2 d_i^2 + t^2 \sum_{i=1}^{n} c_i^2 d_i^2))}}.$$

From this, we find

$$\operatorname{tg}^2\theta = t^2 \frac{\sum_{i=1}^{n} c_i^2}{\sum_{i=1}^{n} b_i^2} \quad \text{and} \quad \operatorname{tg}^2\theta' = t^2 \frac{\sum_{i=1}^{n} c_i^2 d_i^2}{\sum_{i=1}^{n} b_i^2 d_i^2},$$

and so

$$\operatorname{tg}\theta = \delta \operatorname{tg}\theta',$$

with

(9) $$\delta = \sqrt{\left(\frac{\sum_{i=1}^{n} b_i^2 d_i^2}{\sum_{i=1}^{n} c_i^2 d_i^2} \cdot \frac{\sum_{i=1}^{n} c_i^2}{\sum_{i=1}^{n} b_i^2}\right)},$$ which has to be proved.

Next we define dual principal parameters of distribution in the following way: consider the $\frac{1}{2}n(n-1)$ $(n-2)$-dimensional subspaces through the central point a of the generating space $B(s_0)$, which are generated by $(n-2)$ principal axes of $B(s_0)$. These $(n-2)$-dimensional subspaces have, with respect to the coordinate system used in the proof of the last theorem, the equations

$$x_i = x_j = 0 \quad i \neq j \quad (i, j = 1, \ldots, n),$$

while the corresponding orthogonal hyperplanes, for which the corresponding hyperplanes of R^{2n+1} are also orthogonal, are given by $x_i = 0$ and $x_j = 0$. A dual parameter of distribution of such a $(n-2)$-dimensional subspace is called a dual principal parameter of distribution, and calculated with respect to $x_i = 0$, we note it δ_{ij} (thus, there are two dual principal parameter of distribution δ_{ij} and δ_{ji} both belonging to the subspace $x_i = x_j = 0$, and $\delta_{ij}\delta_{ji} = 1$).

Theorem. *Suppose that θ_i $(i = 1, \ldots, n)$ are the (principal) angles between the generating space $B(s_0)$ and a variable generating space $B(s)$. If δ_{ij} are the dual principal parameters of distribution of $B(s_0)$, then*

$$\delta_{ij} = \left|\frac{\mathrm{d}\theta_j}{\mathrm{d}\theta_i}\right|_{s=s_0} \quad i \neq j \quad (i, j = 1, \ldots, n).$$

Proof. Note p the shortest distance between $B(s_0)$ and $B(s)$. If d_i $(i = 1, \ldots, n)$ are the principal parameters of distribution of $B(s_0)$, then one can proof that (see [2.1.])

$$d_i = \left|\frac{\mathrm{d}p}{\mathrm{d}\theta_i}\right|_{s=s_0} \quad (i = 1, \ldots, n).$$

From (9), we get

$$\delta_{ij} = \frac{d_i}{d_j},$$

and thus

$$\delta_{ij} = \frac{\left|\frac{\mathrm{d}p}{\mathrm{d}\theta_i}\right|_{s=s_0}}{\left|\frac{\mathrm{d}p}{\mathrm{d}\theta_j}\right|_{s=s_0}} = \left|\frac{\mathrm{d}\theta_j}{\mathrm{d}\theta_i}\right|_{s=s_0},$$

which completes the proof.

Remarks. 1. The dual principal parameters of distribution are not all independent; for instance

$$\delta_{ij}\delta_{jk} = \delta_{ik} \quad i \neq j \neq k \neq i \quad (i, j, k = 1, \ldots, n).$$

2. The results of sections 1 and 2 remain true for $(n + 1)$-dimensional monosystems N in R^k with $k \geqq 2n + 1$. This holds also for the theorems of section 3 if we change some details: for instance, if L is any hyperplane of a generating space B, then T_L is now a hyperplane of the $(2n + 1)$-dimensional space generated by the tangent spaces of N at the points of B. Moreover, two general n-dimensional spaces in R^k $(k \geqq 2n + 1)$ generate a $(2n + 1)$-dimensional space and in this space we calculate now the shortest distance p and the principal angles θ_i.

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