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LATTICE ENDOMORPHISMS OF 2^X

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I. INTRODUCTION

In providing a setting for this paper, one notes that much recent work has been concerned with the study of the semigroups of endomorphisms of algebraic structures. For example, we cite the work of CLIFFORD and MILLER [1] in which the union and symmetry preserving endomorphisms of the semigroup of binary relations on a set are characterized. In [4], MAXSON considers the lattice of all subsets of a set X and characterizes the lattice endomorphisms which preserve arbitrary unions and also fix the empty set. Further, SCHEIN [5] studies the semigroups of endomorphisms of several algebraic structures. Related also to the study of endomorphism semigroups is the recent work of FRIED and SICHLER [2] and GRATZER and SICHLER [3] in which the problem of representing an arbitrary monoid as a monoid of endomorphisms of a specified algebraic structure is considered.

In this paper, we consider the problem of characterizing all lattice endomorphisms of the lattice of subsets of a set X. In Section II, we find that information about these lattice endomorphisms can be obtained by restricting one's attention to those lattice endomorphisms which also fix the empty set, $\emptyset$. In Section III, we study the endomorphisms which fix $\emptyset$. We show that these endomorphisms have a decomposition into a complete part and a defective part. When the defective part has a finite image the endomorphisms are completely characterized. In the final section, we present some examples, one of which illustrates the construction given in our major theorem.

II. PRELIMINARIES

In this section, we introduce the terminology and notations to be used in the paper. We also present some general results which are of independent interest.

For any set X, let $2^X \equiv \langle 2^X, \cup, \cap, ', \emptyset, X \rangle$ denote the Boolean algebra of subsets of X. Let End 2^X denote the semigroup of lattice endomorphisms of 2^X, i.e.,

$$\text{End } 2^X = \{f : 2^X \to 2^X \mid f(A \cup B) = f(A) \cup f(B),\ f(A \cap B) = \\ = f(A) \cap f(B),\ A, B \in 2^X\},$$

and let $\mathrm{End}_\emptyset\, 2^X$ denote the subsemigroup of $\mathrm{End}\, 2^X$ consisting of those lattice endomorphisms of 2^X fixing $\emptyset$. It is well known that $\mathrm{End}_\emptyset\, 2^X$ is the semigroup of ring endomorphisms of 2^X when 2^X is considered as a ring in which the ring operations are symmetric difference and intersection.

Recall, for semigroups R and S, that R is a retract of S if there exist semigroup morphisms $f : R \to S$ and $g : S \to R$ such that $gf = 1_R$.

Consider now the set $2^X \times \mathrm{End}_\emptyset\, 2^X$. We define a product $\otimes$ on this set as follows:

$$(A, f) \otimes (B, g) = (A \cup f(B), fg)\,, \quad \text{for} \quad (A, f), (B, g) \quad \text{in} \quad 2^X \times \mathrm{End}_\emptyset\, 2^X\,.$$

It is easily verified that $2^X \times \mathrm{End}_\emptyset\, 2^X$, under the operation $\otimes$, is a semigroup with identity $(\emptyset, 1_X)$, and we denote this semigroup by $2^X \otimes \mathrm{End}_\emptyset\, 2^X$.

We are now ready for our first general result.

Theorem 1. $\mathrm{End}\, 2^X$ *is a retract of* $2^X \otimes \mathrm{End}_\emptyset\, 2^X$.

Proof. Choose f in $\mathrm{End}\, 2^X$ and let Z_f denote $f(\emptyset)$. Define $g_f : 2^X \to 2^X$ by $g_f(A) = f(A) - Z_f = f(A) \cap Z'_f$, $A \in 2^X$. It is easily verified that g_f is a lattice endomorphism of 2^X with $g_f(\emptyset) = \emptyset$ and so g_f is in $\mathrm{End}_\emptyset\, 2^X$. Since $Z_f \subseteq f(A)$, for every $A \subseteq X$, we also have $f(A) = g_f(A) \cup Z_f$.

If further, $h \in \mathrm{End}\, 2^X$ then $f\, h(A) = f(g_h(A) \cup Z_h) = f(g_h(A)) \cup f(Z_h) = g_f\, g_h(A) \cup g_f(Z_h) \cup Z_f$ and so $f\, h(\emptyset) = g_f\, g_h(\emptyset) \cup g_f(Z_h) \cup Z_f = g_f(\emptyset) \cup g_f(Z_h) \cup Z_f = g_f(Z_h) \cup Z_f$. Hence $Z_{fh} = g_f(Z_h) \cup Z_f$ and thus $g_{fh}(A) = g_f\, g_h(A)$. If we define $F : \mathrm{End}\, 2^X \to 2^X \otimes \mathrm{End}_\emptyset\, 2^X$ by $F : f \to (Z_f, g_f)$ then $(Z_f, g_f) \otimes (Z_h, g_h) = (Z_f \cup g_f(Z_h), g_f g_h)$ and so $F(fh) = F(f) \otimes f(h)$. This shows that F is a semigroup homomorphism.

Now consider the function $G : 2^X \otimes \mathrm{End}_\emptyset\, 2^X \to \mathrm{End}\, 2^X$ defined by $G(A, f) = \bar{f}$, where $\bar{f}(Y) = f(Y) \cup A$, for every $Y \subseteq X$. Clearly $\bar{f}$ is in $\mathrm{End}\, 2^X$. Also $G(A, f) \otimes G(B, g) = \bar{f}\bar{g}$ where $\bar{f}\,\bar{g}(Y) = \bar{f}(g(Y) \cup B) = \bar{f}(g(Y)) \cup \bar{f}(B) = f\, g(Y) \cup A \cup f(B) \cup A$ and $G((A, f) \otimes (B, g)) = G(A \cup f(B), fg) = \overline{fg}$, where $\overline{fg}(Y) = f\, g(Y) \cup f(B) \cup A$. Hence G is a semigroup homomorphism.

Since $G\, F(f) = G(Z_f, g_f) = \bar{f}$ and $\bar{f}(Y) = g(Y) \cup Z_f = f(Y)$ for every $Y \subseteq X$, $GF = 1_{\mathrm{End}\, 2^X}$. Thus $\mathrm{End}\, 2^X$ is a retract of $2^X \otimes \mathrm{End}_\emptyset\, 2^X$.

We note that $\mathrm{End}\, 2^X$ can never be isomorphic to $2^X \otimes \mathrm{End}_\emptyset\, 2^X$ when $X \neq \emptyset$. In fact $F(f) = (Z_f, g_f)$ implies $g_f(A) \cap Z_f = (f(A) - Z_f) \cap Z_f = \emptyset$. Therefore the image of F contains only those elements (B, e) for which $e(A) \cap B = \emptyset$, for all $A \subseteq X$.

From this theorem we see that $\mathrm{End}\, 2^X$ can be embedded in $2^X \otimes \mathrm{End}_\emptyset\, 2^X$. Consequently, to obtain more information concerning $\mathrm{End}\, 2^X$, we restrict our attention to lattice endomorphisms which fix $\emptyset$.

Recall that if $(L, \vee, \wedge)$ is a lattice then a lattice endomorphism of L is join complete if $f(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} f(a_i)$ whenever $\bigvee_{i\in I} a_i$ exists in L. In the next lemma we show that when the lattice is 2^X, a join complete lattice endomorphism has a simpler characterization.

Lemma 1. *If f is a lattice endomorphism of 2^X, then f is join complete if and only if, for every $A \subseteq X$, $f(A) = \bigcup_{x\in A} f(x)$.*

Proof. From the definition of join complete endomorphism we have $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$, for any index set I, and this clearly implies $f(A) = \bigcup_{x\in A} f(x)$. For the converse, consider a collection $\{A_i \mid A_i \subseteq X, i \in I\}$. Let $\bigcup_{i\in I} A_i = B \subseteq X$. Then $f(\bigcup_{i\in I} A_i) = f(B) = f(\bigcup_{x\in B} x) = \bigcup_{x\in B} f(x)$, by assumption. If x is in B, then x is in A_j, for some j in I and so $f(x) \subseteq f(A_j) \subseteq \bigcup_{i\in I} f(A_i)$. Therefore, $\bigcup_{x\in B} f(x) \subseteq \bigcup_{i\in I} f(A_i)$. As every lattice endomorphism preserves order we always have $\bigcup_{i\in I} f(A_i) \subseteq f(\bigcup_{i\in I} A_i)$ and thus f is join complete.

For a ring endomorphism of 2^X, we only need to consider $f(X)$, when checking for join completeness of f.

Lemma 2. *If $f \in \mathrm{End}_\emptyset\, 2^X$, f is join complete if and only if $f(X) = \bigcup_{x\in X} f(x)$.*

Proof. The necessity of the condition is obvious. Suppose $f(X) = \bigcup_{x\in X} f(x)$. Then for every A contained in X,

$$f(A) = f(X \cap A) = f(A) \cap f(X) = f(A) \cap \Big[\bigcup_{x\in A} f(x) \cup \bigcup_{x\in X-A} f(x)\Big] =$$
$$= \Big\{f(A) \cap \Big[\bigcup_{x\in A} f(x)\Big]\Big\} \cup \Big\{f(A) \cap \Big[\bigcup_{x\in X-A} f(x)\Big]\Big\}.$$

Since x is in $X - A$ implies $f(A) \cap f(x) \subseteq f(A) \cap f(X - A) = f(\emptyset) = \emptyset$, we get $f(A) = f(A) \cap [\bigcup_{x\in A} f(x)] = \bigcup_{x\in A} f(x)$ as $\bigcup_{x\in A} f(x) \subseteq f(A)$. Hence f is join complete.

To illustrate the utility of the above lemma, we present the following example.

Consider a set X and $\{x_1, x_2\} \subseteq X$. Then the map f from 2^X into 2^X defined by

$$f(A) = \begin{cases} \emptyset & \text{if } x_1 \notin A \text{ and } x_2 \notin A, \\ x_1 & \text{if } x_1 \in A \text{ and } x_2 \notin A, \\ x_2 & \text{if } x_1 \notin A \text{ and } x_2 \in A, \\ \{x_1, x_2\} & \text{if } x_1 \text{ and } x_2 \in A \end{cases}$$

is a complete ring endomorphism of 2^X. In fact, by definition $f(\emptyset) = \emptyset$ and by checking cases $f(A \cup B) = f(A) \cup f(B)$ and $f(A \cap B) = f(A) \cap f(B)$. Since $f(X) = \{x_1, x_2\} = \bigcup_{x \in X} f(x)$ we see that f is a complete ring endomorphism.

III. MAIN RESULTS

The objective of this section is to obtain a representation theorem for the ring endomorphisms of 2^X. Our objective is reached through a sequence of lemmas culminating in our main result, theorem 2.

Lemma 3. *Let f be a ring endomorphism of 2^X and define $e_f : 2^X \to 2^X$ by $e_f(A) = f(A) - D_f(X)$ for all A contained in X, where $D_f(X) = f(X) - \bigcup_{x \in X} f(x)$. Then e_f is a complete ring endomorphism of 2^X.*

Proof. Since $e_f(A) = f(A) \cap [D_f(X)]'$ and since f is a ring endomorphism of 2^X, e_f is a ring endomorphism of 2^X. Also

$$e_f(X) = f(X) \cap (D_f(X))' = f(X) \cap [f(X) \cap (\bigcup_{x \in X} f(x))']' =$$
$$= f(X) \cap [(f(X))' \cup (\bigcup_{x \in X} f(x))] = \bigcup_{x \in X} f(x) .$$

Also for every x in X, $e_f(x) = f(x) \cap [(f(X))' \cup \bigcup_{x \in X} f(x)] = f(x)$. Hence $e_f(X) = \bigcup_{x \in X} e_f(x)$ and so by Lemma 2, e_f is complete.

Lemma 4. *If f is a ring endomorphism of 2^X and D_f is the map defined by $D_f(A) = f(A) - \bigcup_{x \in A} f(x)$, then D_f is a ring endomorphism of 2^X, and the kernel of D_f contains all finite subsets of X.*

Proof. We first note that $f(A) \cap D_f(X) = f(A) \cap [f(X) - \bigcup_{x \in A} f(x)] = f(A) \cap f(X) \cap [\bigcup_{x \in X} f(x)]' = f(A) \cap [\bigcup_{x \in X} f(x) \cup \bigcup_{x \in X - A} f(x)]' = f(A) \cap [\bigcup_{x \in A} f(x)]' \cap [\bigcup_{x \in X - A} f(x)]'$. But $x \in X - A$ implies $f(x) \cap f(A) \subseteq f((X - A) \cap A) = f(\emptyset) = \emptyset$. Hence $f(A) \cap D_f(X) = f(A) \cap [\bigcup_{x \in X} f(x)]' = f(A) - [\bigcup_{x \in X} f(x)] = D_f(A)$ and so it is clear that D_f is a ring endomorphism of 2^X. Since $f(A) = \bigcup_{x \in X} f(x)$ for every finite set, the kernel of D_f contains all finite sets.

If f is a ring endomorphism of 2^X, we call the complete ring endomorphism e_f of Lemma 3 the *complete part of f* and the ring endomorphism D_f of Lemma 4 the *defective part of f*.

Lemma 5. *Let f be a ring endomorphism of 2^X. If e_f and D_f are the complete and defective parts of f, respectively, then $f(A) = e_f(A) \cup D_f(A)$ for every $A \subseteq X$.*

Proof. From the proof of Lemma 3, we get $e_f(x) = f(x)$ for every $x \in X$ and since e_f is complete, $e_f(A) = \bigcup_{x \in A} f(x)$. So $D_f(A) = f(A) - e_f(A)$. Also $e_f(A) = \bigcup_{x \in A} f(x) \subseteq f(A)$ and so $D_f(A) \cup e_f(A) = f(A)$.

If f and g are two ring endomorphism of 2^X, the map $f \cup g$ defined by $(f \cup g)(A) = f(A) \cup g(A)$ is not necessarily a ring endomorphism of 2^X. However, by repeated uses of the distributive laws, we obtain sufficient conditions for $f \cup g$ to be a ring endomorphism of 2^X.

Lemma 6. *If e and D are ring endomorphism of 2^X with $e(X) \cap D(X) = \emptyset$, then $f = e \cup D$ is also a ring endomorphism of 2^X.*

Generalizing the above lemma, we get $\bigcup_{i=1}^{n} f_i$ is a ring endomorphism of 2^X whenever each f_i is a ring endomorphism of 2^X and $(\bigcup_{i \neq j} f_i(X)) \cap f_j(X) = \emptyset$, for every j with $1 \leqq j \leqq n$. But then $f_i(X) \cap f_j(X) = \emptyset$ for every $i \neq j$. Conversely if $f_i(X) \cap f_j(X) = \emptyset$ for every $i \neq j$, then $(\bigcup_{i \neq j}^{n} f(X)) \cap f_j(X) = \emptyset$. Hence $\bigcup_{i=1}^{n} f_i$ is a ring endomorphism of 2^X if each f_i is a ring endomorphism of 2^X and $f_i(X) \cap f_j(X) = \emptyset$ whenever $i \neq j$.

We say a ring endomorphism f of 2^X is m-valued if the image of f contains exactly m elements. Since the image of f must be a Boolean ring, then as is well known, if this image of f is finite, it contains 2^n elements for some positive integer n. With these preliminaries, we get the next lemma.

Lemma 7. *Let D be a 2^n valued ring endomorphism of 2^X. Then $D = \bigcup_{i=1}^{n} D_i$, where each D_i, $1 \leqq i \leqq n$, is a two valued ring endomorphism of 2^X and $D_i(X) \cap D_j(X) = \emptyset$, whenever $i \neq j$. Also* ker $D_i \neq$ ker D_j *whenever $i \neq j$.*

Conversely, given a collection $\{D_i \mid 1 \leqq i \leqq n\}$ of two valued ring endomorphisms of 2^X, such that their non-zero values are pairwise disjoint, and their kernels are distinct, then $D = \bigcup_{i=1}^{n} D_i$ is a 2^n valued ring endomorphism of 2^X.

Proof. Since D is a 2^n valued ring endomorphism, the image of D is isomorphic to $Z_2 \oplus \ldots \oplus Z_2$ (n copies). But $Z_2 \oplus \ldots \oplus Z_2$ is a n-dimensional vector space over Z_2 with $\{e_i \mid e_i = (0, \ldots, 1, 0, \ldots, 0)$, 1 in the i-th position$\}$ as a basis. Let F be the isomorphism between the image of D and $Z_2 \oplus \ldots \oplus Z_2$ and let $F(e_i) = A_i$, $i = 1, \ldots, n$. Then for $i \neq j$, $e_i e_j = 0$ implies $A_i \cap A_j = \emptyset$. Hence each element in the image of D can be expressed as the union of elements from the disjoint collection $\{A_i \mid i = 1, \ldots, n\}$. Thus if $A \in$ image of D then $A = (\alpha_1 \cap A_1) \cup (\alpha_2 \cap A_2) \cup \ldots \cup (\alpha_n \cap An)$ where $\alpha_i = A_i$ or $\emptyset$. Let $D_i = \pi_i D$, denote the i-th projection of D. Then $D_i(X) = A_i$ and D_i is a two valued ring endomorphism of 2^X. Also $D_i(X) \cap$

$\cap D_j(X) = A_i \cap A_j = \emptyset$. Since we can always find an element $(\alpha_1 \cap A_1) \cup \ldots \ldots \cup (\alpha_i \cap A_i) \cup \ldots \cup (\alpha_j \cap A_j) \cup \ldots \cup (\alpha_n \cap A_n)$ with $\alpha_i = \emptyset$ and $\alpha_j = A_j$ in 2^X, $\ker D_i \neq \ker D_j$ for $i \neq j$.

For the converse, since the nonzero values are pairwise disjoint, $D_i(X) \cap D_j(X) = \emptyset$, whenever $i \neq j$. Hence $D = \bigcup_{i=1}^{n} D_i$ is a ring endomorphism of 2^X. The fact that each D_i is two valued implies that the kernel of D_i, say, M_i is a maximal ideal of 2^X. Since we can always find an element in M_i, not in M_j whenever $i \neq j$, D is 2^n-valued.

In [4] Maxson obtains a representation theorem for complete ring endomorphisms by showing that there is an anti-isomorphism E between the semigroup $P\,T(X)$ of partial transformations on X and the semigroup $\mathrm{End}_{\emptyset,c}\, 2^X$ of complete ring endomorphisms of X. In fact for $\alpha \in P\,T(X)$, $E(\alpha) = f$ where $f(A) = \alpha^{-1}(A)$, $A \in 2^X$.

In the next theorem, we get an extension of this result by obtaining a ring endomorphism of 2^X with finite valued defective part.

Theorem 2. *Let ϱ be a partial transformation on X with domain of $\varrho \equiv \Delta(\varrho)$. Let D_i, $i = 1, 2, \ldots, m$ with $m \leqq |X - \Delta(\varrho)|$ be a collection of two valued ring endomorphisms of 2^X such that*

1) $\Delta(\varrho) \cap D_i(X) = \emptyset$, *for every i,*
2) $D_i(X) \cap D_j(X) = \emptyset$, *for $i \neq j$,*
3) $A \subseteq X$ *is finite implies* $D_i(A) = \emptyset$ *for every i.*

Then $g = E(\varrho) \cup (\bigcup_{i=1}^{m} D_i)$ is a ring endomorphism of 2^X, with finite valued defective part. Conversely, every ring endomorphism of 2^X with finite valued defective part can be found in this way.

Proof. We note that $\varrho \in P\,T(X)$ implies $E(\varrho)$ is a complete ring endomorphism of 2^X. By definition of E, $(E(\varrho))(X) = \varrho^{-1}(X) = \Delta(\varrho)$. From condition (2) and Lemma 7, $\bigcup_{i=1}^{m} D_i$ is a ring endomorphism of 2^X. By (1) and Lemma 7, we get $g = E(\varrho) \cup (\bigcup_{i=1}^{m} D_i)$ is a ring endomorphism of 2^X. We claim that the complete part of g, i.e. e_g is $E(\varrho)$ and the defective part of g, i.e. D_g is $\bigcup_{i=1}^{m} D_i$. Since $\bigcup_{i=1}^{m} D_i(x) = \emptyset$, for every x in X, $g(x) = (E(\varrho))(x)$ and so $D_g(X) = g(X) - \bigcup_{x \in X} (E(\varrho))(x) = (E(\varrho)(X)) \cup ((\bigcup_{i=1}^{m} D_i)(X)) - (E(\varrho))(X) = (\bigcup_{i=1}^{m} D_i)(X)$ as $E(\varrho)(X) \cap D_i(X) = \emptyset$ for all i. From the proof of Lemma 4, we have $D_g(A) = g(A) \cap D_g(X)$. But $g(A) \cap$

$\cap D_g(X) = g(A) \cap (\bigcup_{i=1}^{m} D_i(X))$, from above, and so $D_g(A) = (E(\varrho)(A) \cup \bigcup_{i=1}^{m} D_i(A)) \cap$ $\cap (\bigcup_{i=1}^{m} D_i(X)) = \bigcup_{i=1}^{m} D_i(A)$ as $D_i(A) \subseteq D_i(X)$, $D_i(X) \cap D_j(X) = \emptyset$ for $i \neq j$ and $(E(\varrho)) \cdot$ $\cdot (A) \cap D_i(X) = \emptyset$. Hence $D_g(A) = \bigcup_{i=1}^{m} D_i(A)$ or $D_g = \bigcup_{i=1}^{u} D_i$.

By Lemma 3, $e_g(A) = g(A) - D_g(X) = (E(\varrho))(A)$, and so $e_g = E(\varrho)$. Also the cardinality of the image of D_g is 2^m as each D_i is two valued.

For the converse let f be a ring endomorphism of 2^X with finite valued defective part. From Lemma 5, $f = e_f \cup D_f$ and since e_f is complete $e_f = E(\varrho)$ for some partial transformation ϱ of X. By hypothesis D_f is finite valued, say 2^m valued and so by Lemma 7, $D_f = \bigcup_{i=1}^{m} D_i$. Also $e_f(X) = (E(\varrho))(X) = \varrho^{-1}(X) = \Delta(\varrho)$ and from definitions of e_f and D_f, $\Delta(\varrho) \cap D_f(X) = \emptyset$. Hence $\Delta(\varrho) \cap \bigcup_{i=1}^{m} D_i(X) = \emptyset$ and thus, $\Delta(\varrho) \cap D_i(X) = \emptyset$ for every i. From Lemma 7, $D_i(X) \cap D_j(X) = \emptyset$, whenever $i \neq j$. Since D_f is the defective part of f, the kernel of D_f contains all finite sets, and since the kernel of $D_f = \bigcap_{i=1}^{m} \ker D_i$, $D_i(A) = \emptyset$ for every finite set A. Since $D_f(X) =$ $= f(X) - e_f(X) \subseteq X - e_f(X) = X - \Delta(\varrho)$ we get $|D_f(X)| \leqq |X - \Delta(\varrho)|$. Also $D_f(X) = D_1(X) \cup \ldots \cup D_m(X)$ implies $|D_f(X)| \geqq m$ and therefore $m \leqq |X - \Delta(\varrho)|$.

IV. EXAMPLES AND REMARKS

In this section, we present two examples. The first example illustrates the construction of an endomorphism f in $\mathrm{End}_\emptyset\, 2^X$ with finite valued defective part while the second example gives a ring endomorphism of 2^X in which the defective part is not finite valued.

Example A. Let X be the set of positive integers and let α be the partial transformation which maps all odd integers onto the integer 1. Then the map $e : 2^X \to 2^X$ defined by $e(A) = \alpha^{-1}(A)$, $A \subseteq X$ is a complete ring endomorphism.

Let M_1 and M_2 be two distinct maximal ideals containing all finite subsets of X and let Y be an infinite subset of the even integers not containing 2.

Define $D_1, D_2 : 2^X \to 2^X$ by

$$D_1(A) = \begin{cases} \emptyset & \text{if } A \in M_1 \\ Y & \text{if } A \notin M_1 \end{cases}$$

and

$$D_2(A) = \begin{cases} \emptyset & \text{if } A \in M_2 \\ \{2\} & \text{if } A \notin M_2 . \end{cases}$$

Since $D_1(X) \cap D_2(X) = \emptyset$, $D = D_1 \cup D_2$ is a four valued ring endomorphism of 2^X. Since $e(X) = \alpha^{-1}(X) = \{n \mid n \text{ is an odd integer}\}$, $e(X) \cap D_1(X) = e(X) \cap$ $\cap D_2(X) = \emptyset$. Hence $f = e \cup D_1 \cup D_2$ is a ring endomorphism of 2^X with four valued defective part. A similar procedure can be used to construct a ring endomorphism of 2^X with 2^n valued defective part for any positive integer n.

In the next example, we use the following theorem of Sikorski.

Theorem 3 [6]. *Let V be a subalgebra of a Boolean algebra U. Every homomorphism h_0 of V into a complete algebra W can be extended to a Boolean algebra homomorphism h of U into W (and hence a ring homomorphism).*

Example B. Let U be 2^X where X is some infinite set containing the integers, and let W be 2^Y, Y any infinite subset of X. Let $V_0 = F(X)$, the finite cofinite Boolean algebra on X. Let $V_1 = (F(X) \cup (3))$, the subalgebra of 2^X generated by $F(X)$ and the multiples of 3. Let $V_n = (V_{n-1} \cup (2n + 1))$. All of the above subalgebras are distinct. Define $h_0 : V_0 \to 2^Y$ by

$$h_0(A) = \begin{cases} \emptyset & \text{if } A \text{ is finite} \\ Y & \text{if } A \text{ is cofinite}, \end{cases}$$

and $h_n : V_n \to 2^Y$ by

$$h_n((2n + 1)) = \{y_n\}, \quad y_n \in Y - \{y_1, y_2, \ldots, y_{n-1}\}, \quad \text{and} \quad h_n|_{V_{n-1}} = h_{n-1}.$$

From the proof of Sikorski's theorem all of these maps are Boolean ring endomorphisms extending h_0. Note that the cardinality of the image of h_n is greater than that of h_{n-1}.

Let $Z = \bigcup_{n=1}^{\infty} V_n$. Clearly Z is a subalgebra of 2^X. Define $g : Z \to 2^Y$ by $g(A) = h_n(A)$ for $A \in V_n$. The map g is well defined, for if $A \in V_n$ and V_m, $V_n \subseteq V_m$, say, then h_m restricted to V_n is h_n and so $h_n(A) = h_m(A)$. Further g is easily seen to be a Boolean algebra homomorphism. Also the image of g contains $\{y_n \mid n \text{ is any positive integer}\}$ and thus is infinite. Using Sikorski's theorem again, there exists an extension $\bar{g}$ of g from 2^X into 2^Y. This $\bar{g}$ is the desired ring endomorphism of 2^X. First we note that the image of $\bar{g}$ is infinite and $\bar{g}$ restricted to Z takes all finite sets to $\emptyset$. Hence $e_{\bar{g}}$ the complete part of $\bar{g}$ is the zero map and $\bar{g} = D_{\bar{g}}$. Thus the defective part of $\bar{g}$ is infinite valued.

The above example suggests another way of looking at the defective part D_f of a ring endomorphism f of 2^X. Since D_f maps all finite sets onto $\emptyset$, the restriction of D_f to $F(X)$ is a two valued map. In fact, if A is any cofinite set and $D_f(X) = Y$, then $D_f(A) = D_f(X - A') = D_f(X) - D_f(A') = D_f(X)$. Conversely starting with a two valued ring homomorphism f of $F(X)$ into 2^X, which takes all finite sets to $\emptyset$, we obtain an extension $\bar{f}$ of f and $\bar{f}$ is a ring endomorphism of 2^X. Further $\bar{f}(x) = \emptyset$ for every $x \in X$ implies $e_{\bar{f}}$ is the zero map and $\bar{f} = D_{\bar{f}}$. Hence we get the following result.

Theorem 4. *D_f is the defective part of a ring endomorphism of 2^X if and only if D_f restricted to $F(X)$ is a two valued ring homomorphism determined by the maximal ideal $\{A \mid A \subseteq X$ is finite$\}$.*

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