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FUNCTIONAL APPROACH TO THE BRELOT-KELDYCH THEOREM

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In the classical theory of harmonic functions, some sets for which the Dirichlet problem is not solvable for any continuous function on their boundary are known. Nevertheless, we can assign to those functions something like a solution in a reasonable way. For the first time, it has been shown by M. V. KELDYCH in [6] that this "generalized solution" is in fact completely determined on the subspace of all "solvable functions". This result is in a certain sense surprising because the set of all "solvable functions" is in the uniform topology generally only a closed set of the first category, while the set of all "non-solvable functions" is of the second category.

Given a bounded open set $U \subset R^n$ and a continuous function f on the boundary U^* of U, then H_f^U stands for the generalized solution of the Dirichlet problem for f obtained by the Perron method. Let F be a linear and positive map associating with any continuous function f on U^* the harmonic function F_f on U satisfying $F_f = f$ if f is continuous on $\overline{U}$ and harmonic in U. The Keldych theorem states that $F_f = H_f^U$ for any continuous f on U^*.

An extension of the classical theorem of Keldych was given by M. BRELOT in [3] for a more general setting of the axiomatic theory including the domination axiom D, and in the framework of the general Bauer's axiomatic by the author in [10]. It was shown that the Brelot-Keldych theorem remains valid for any open set with a "negligible" set of all irregular points and this result is the best possible.

In the present paper we study certain topologies on the space of all continuous functions and we try to explain the functional-theoretical background of the Brelot-Keldych theorem. Also some new results will be derived. Let us remark at this point that an operator approach and methods of functional analysis were used in Landkof's book [8] to prove Keldych theorem in the classical case.

1. THE BRELOT — KELDYCH THEOREM

In what follows, $(X, \mathscr{H})$ denotes a strong harmonic space in the sense of Bauer's axiomatic [1], or more generally any P-harmonic space with countable base in the sense of CONSTANTINESCU and CORNEA [4].

If $U \subset X$ is an open relatively compact set, then ε_x^{CU} is the *balayaged measure* of the Dirac measure ε_x on the complement CU of U, and H_f^U stands for the *generalized solution* defined on the closure $\overline{U}$ of U by $H_f^U(x) = \varepsilon_x^{CU}(f)$ for any $f \in C(U^*)$ (= the space of all continuous functions on the boundary U^* of U). A point $z \in U^*$ is called a *regular boundary point* of U if $\varepsilon_z^{CU} = \varepsilon_z$. The set of all regular points will be denoted by U^*_{reg}, while $U^*_{ir} = U^* \setminus U^*_{\text{reg}}$ will be the set of all irregular points. A set U is termed *regular* if $U^*_{\text{reg}} = U^*$, i.e. if for any $f \in C(U^*)$ there exists a solution of the Dirichlet problem, which is non-negative if f is. We shall say that a set $M \subset U^*$ is *negligible* if M is of ε_x^{CU}-measure zero for every $x \in U$.

Let us denote by $S = S(\overline{U})$ the set of all continuous functions on $\overline{U}$ which are superharmonic in U. This min-stable convex cone determines a partial ordering on the set of all Radon measures on $\overline{U}$ and the question raised by E. G. Effros and J. L. Kazdan in [5] was whether the cone S is simplicial. This problem was affirmatively solved by J. Bliedtner and W. Hansen in [2]. Thus, for every $x \in \overline{U}$ there is a unique minimal measure δ_x^U, and it was shown in [2] that the following statements are equivalent:

(i) U^*_{ir} is negligible,

(ii) $\delta_x^U = \varepsilon_x^{CU}$ for any $x \in \overline{U}$,

(iii) $\varepsilon_x^{CU}(U^*_{ir}) = 0$ for every $x \in \overline{U}$

(see [2], Theorem 4.1).

It is almost obvious that the condition

(iv) $\delta_x^U = \varepsilon_x^{CU}$ for every $x \in U$

is also equivalent with (i). Indeed, if $x \in U$, and (iv) is assumed, then using the results from [2] we get $(\varepsilon_x^{CU})^{CU} = (\delta_x^U)^{CU} = \delta_x^U = \varepsilon_x^{CU}$. Hence, $\varepsilon_x^{CU}(U^*_{ir}) = 0$. (Compare also [10], Théorème 2.7.)

The *Choquet boundary* $\text{Ch}_S U$ is defined as the set of all $x \in U^*$ such that $\delta_x^U = \varepsilon_x$. Of course, $\text{Ch}_S U \subset U^*_{\text{reg}}$. We know, in fact, that for any $x \in \overline{U} \setminus \text{Ch}_S U$, δ_x^U is exactly the balayaged measure of ε_x on the Choquet boundary. Comparing the definition of the regular points and the Choquet boundary with the above result, we obtain the following important lemma.

Lemma 1. *If U^*_{ir} is negligible, then* $\text{Ch}_S U = U^*_{\text{reg}}$.

Let us remark that if $H(\overline{U}) := S(\overline{U}) \cap -S(\overline{U})$ is an admissible subspace of $C(\overline{U})$ (i.e. it contains the constant functions and separates the points of $\overline{U}$), then $H(\overline{U})$ is also simplicial, $\text{Ch}_{H(\overline{U})} U = \text{Ch}_S U$, and for any $z \in \text{Ch}_S U$ there is a function $h \in H(\overline{U})$ such that $h(z) = 0$ and $h > 0$ otherwise on $\overline{U}$ ([2], Corollary 3.8).

Let $\mathscr{H}(U)$ be the set of all harmonic functions on U. A *Keldych operator* on U is a positive and linear mapping

$$L: C(U^*) \to \mathscr{H}(U)$$

such that $L(s \upharpoonright U^*) \leqq s$ on U for any $s \in S(\overline{U})$. (Here, and in the sequel, $f \upharpoonright A$ stands for the restriction of f on A.) Obviously, $L(h \upharpoonright U^*) = h$ on U for every Keldych operator and any $h \in H(\overline{U})$. If $H(\overline{U})$ is admissible and U^*_{ir} negligible, then the last condition turns out to be sufficient for a positive linear operator to be Keldych. Indeed, let L be such an operator. Then $H^U_w(x) = \delta^U_x(w) \leqq Lw(x) \leqq w(x)$ for any $x \in U$ and any $w \in W(\overline{U})$, where $W(\overline{U})$ is the set of all functions of the form $\min(h_1, \ldots, h_n)$, $h_i \in H(\overline{U})$. Hence, $\lim_{x \to z} Lw(x) = w(z)$ for any $z \in U^*_{\text{reg}}$. Since the set of all restrictions of $W(\overline{U}) - W(\overline{U})$ is uniformly dense in $C(U^*)$, we can conclude that $\lim_{x \to z} Lf(x) = f(z)$ for any $z \in U^*_{\text{reg}}$ and $f \in C(U^*)$. It follows that, for a given $s \in S(\overline{U})$, $\lim_{x \to z} (s(x) - Ls(x)) = 0$ for every $z \in U^*_{\text{reg}}$, and the minimum principle ([1], Satz 4.4.6) yields the statement.

Clearly, $f \mapsto H^U_f$ is a Keldych operator, and if we define $D^U_f(x) = \delta^U_x(f)$ for any $x \in \overline{U}$ and any $f \in C(U^*)$, then $f \mapsto D^U_f$ is also a Keldych operator. Another type of generalized solutions was introduced in [10]. The so-called *principal solution* L^U_f is again a Keldych operator. Possibly there are more Keldych operators on a set U. We shall say that U is a *Keldych set* if every Keldych operator on U coincides with the generalized Perron solution H^U_f. As a consequence of the results in [10] and Lemma 1 we obtain the following characterization of Keldych sets.

Proposition 2. *The following assertions are equivalent*:

(i) *U^*_{ir} is negligible,*

(v) *U is a Keldych set,*

(vi) *$D^U_f = H^U_f$ on U for any $f \in C(U^*)$,*

(vii) *$L^U_f = H^U_f$ on U for any $f \in C(U^*)$,*

(viii) *$(\varepsilon^{CU}_x)^{CU} = \varepsilon^{CU}_x$ for any $x \in U$,*

(ix) *$\varepsilon^{CU}_x(f) = \varepsilon^{CU}_x(F)$ on U for any $f \in C(U^*)$, where $F = \varepsilon^{CU}_x(f) \upharpoonright U^*$.*

Proposition 2 is in fact a generalization of the classical theorem of Keldych. Let us mention briefly the ideas of the proof of this Brelot-Keldych theorem. In the classical case of harmonic functions, M. V. Keldych in 1941 constructed for any regular point z a non-negative function from $H(\overline{U})$ which vanishes exactly at z. This result together with the minimum principle for harmonic functions leads easily to the proof that any open bounded set is Keldych. The generalization to the axiomatic theory with axiom D was made by M. Brelot in 1960, the construction of the desirable function though with weakened properties being substantially simplified. Returning to the classical case, N. S. Landkof presented in 1965 an operator approach. Using a characterization of the annihilator of a set $H(\overline{U})$ in the space of all measures, the crucial step consists in the proof that $\lim_{x \to z} Lf(x) = f(z)$ if L is a Keldych operator,

$f \in C(U^*)$ and z is a regular point of U. This observation together with the minimum principle imply that any open set is Keldych. Taking account of the inequalities $Lw \leqq H_w^U \leqq w$ for any $w \in W(\overline{U})$ and the fact that the set of all differences of such functions is dense in $C(U^*)$, we get again that $\lim_{x \to z} Lf(x) = f(z)$ for any $z \in \mathrm{Ch}_S U$ as above. This yields the proof of the Brelot-Keldych theorem in the general axiomatic as is made in [10].

2. TOPOLOGIES ON $C(U^*)$

In what follows, let us observe the following notation. $U \subset X$ will be a fixed open relatively compact set, $C = C(U^*)$ will denote the set of all continuous functions on U^*, and $C_R \subset C$ will be the subspace of C consisting of the restrictions of the functions from $H(\overline{U})$ to U^*. Thus, C_R contains continuous functions for which the Dirichlet problem is always solvable, and the set U is regular if and only if $C_R = C$. The Banach subgroup theorem ([7], Theorem 10.5) gives the following characterization.

Proposition 3. *If we consider C with the supremum norm, then C_R is closed in C, and C_R is of the second category in C if and only if U is regular.*

Given any Keldych operator L on U, the mapping $L_x : f \mapsto Lf(x)$ is a positive Radon measure on U^* for any $x \in U$. Let us denote by $\mathfrak{M}$ the collection of all such measures. Thus, $\mu \in \mathfrak{M}$ if and only if there is a Keldych operator L on U and $x \in U$ such that $\mu = L_x$. Evidently, $\{\varepsilon_x^{CU}; x \in U\} \subset \mathfrak{M}$. Any element of $\mathfrak{M}$ is continuous on the Banach space C with the uniform norm, and on the subspace C_R coincides with the Perron solution ($L_x(f) = \varepsilon_x^{CU}(f)$ for any $f \in C_R$ and $x \in U$). Moreover, any L_x is representing measure for x with respect to $S(\overline{U})$, i.e. $L_x \in \mathscr{M}_x(S)$ in the notation of [2]. But if the set U is not regular, then C_R is only of the first category in C, while $C \setminus C_R$ is of the second category. Hence, the uniform topology on C is too strong and we shall seek other topologies on C which are weaker than the uniform one. One candidate is the weak-topology on C given by the duality of C and the space of Radon measures. However, this topology is still too strong. The set C_R is again weak-closed, because the weak closure of C_R and the uniform closure of C_R coincide. Nevertheless, if U^* is infinite, then C is of the first category in the weak topology, and thus C_R is so. The question whether there is a topology on C for which C_R is dense in C and all elements of $\mathfrak{M}$ are continuous can be trivially answered.

Theorem 4. *If there is a topology t on C such that C_R is t-dense in C and any element of $\mathfrak{M}$ is t-continuous, then U_{ir}^* is negligible.*

Proof. Assume $f \in C(U^*)$ and take a net $\{f_a\} \subset C_R$ such that $f_a \to f$ in t. Then $\mu(f_a) \to \mu(f)$ for any $\mu \in \mathfrak{M}$. In particular, $\varepsilon_x^{CU}(f_a) \to \varepsilon_x^{CU}(f)$ and $\delta_x^U(f_a) \to \delta_x^U(f)$. Since $\varepsilon_x^{CU}(f_a) = \delta_x^U(f_a)$, it follows that $D_f^U = H_f^U$ on U and we apply Proposition 2.

Subsequently, if U_{ir}^* is negligible, we shall show that there are some topologies on C with the properties mentioned in Theorem 4. And if there are such topologies, then the weak $w(C, \mathfrak{M})$-topology, which is the weakest topology making all elements of $\mathfrak{M}$ continuous, surely has also the desirable properties. However, in searching for such topologies we would like to find the strongest one possible.

Before doing so we digress to a more general setting. Using the Choquet theory we derive a certain characterization of simplicial cones which permits one to introduce the above mentioned topology on C. (All the material in detail could be found in [2].) Let S be a min-stable admissible convex cone of continuous functions on a compact metrizable space X. The cone S determines a partial ordering on the space of positive Radon measures on X. Recall that S is called *simplicial* if for every $x \in X$ there is a unique minimal measure μ_x representing a point x. Further, denote by $B = B(X)$ the set of all *boundary measures* on X, i.e. the set of all measures μ on X such that $|\mu| (X \setminus \mathrm{Ch}_S X) = 0$. We note that a positive Radon measure is minimal if and only if it is a boundary measure. Let $H(S)$ be the space of all continuous *S-affine* functions on X. A (signed) Radon measure μ on X is a *$H(S)$-dependence* if $\mu h = 0$ for any $h \in H(S)$. Clearly, the set $H^{\perp}(X)$ of all $H(S)$-dependences on X is the annihilator of $H(S)$ in the duality of $C = C(X)$ and the space $M(X)$ of Radon measures on X. Let us remark that $B(X)$ and $H^{\perp}(X)$ are linear subspaces of $M(X)$.

Theorem 5. *The following assertions are equivalent*:

(i) *S is a simplicial cone.*

(ii) *$H(S)$ is a $w(C, B)$-dense subset of C.*

(iii) *X does not admit any non-zero boundary $H(S)$-dependence, i.e. $B(X) \cap H^{\perp}(X) = \{0\}$.*

(iv) *$H(S)$ separates the minimal measures on X.*

(v) *For any $x \in X$, $H(S)$ separates the minimal measures representing a point x.*

Proof. According to [7], Theorem 16.5, (ii) and (iii) are equivalent. Obviously, (iv) implies (v), and equivalence of (iii) and (iv) is trivial. Now assume that S is a simplicial cone and take any minimal measure μ. Using Proposition 1.1 of [2], we get $\mu(s) = \mu(\sup \{t \in -S;\ t \leqq s\})$ for any $s \in S$. Theorem 2.1 of [2] implies that the set of all t in the brackets is upper directed and, moreover,

$$\mu(s) = \sup \{\mu(t);\ t \in -S,\ t \leqq s\} \leqq \sup \{\mu(h);\ h \in H(S),\ h \leqq s\} \leqq \mu(s)\,.$$

Hence $\mu(s) = \sup \{\mu(h);\ h \in H(S),\ h \leqq s\}$ for any $s \in S$. By the density of $S - S$ in C one concludes that (i) implies (iv). The last implication (v) $\Rightarrow$ (i) is an immediate consequence of the definition of simliciality (compare also Proposition 2.4 of [2]).

Remark. There are many other characterizations of simplicial cones. One of them can be stated (in the case when $H(S)$ is uniformly closed and contains constants and separates the points of X) that S is simplicial if and only if $H(S)$ is a Lindenstrauss (or an L_1-predual) space. In particular, it follows that the Banach space $H(\bar{U})$ is, in any axiomatic of harmonic functions, a Lindenstrauss space.

Returning to our case of the cone of continuous superharmonic functions, taking into account the relation between C_R and $H(\overline{U})$ and regarding measures on $\overline{U}$ with support in U^* as measures on U^*, we can state that C_R is a $w(C, B(U^*))$-dense subset of C. Of course, $B(U^*)$ signifies the set of all Radon measures on U^* supported by the Choquet boundary of U. We know that U^*_{ir} is negligible if and only if $\mathfrak{M} \subset B(U^*)$.

Corollary 6. *If U^*_{ir} is negligible, then C_R is a $w(C, \mathfrak{M})$-dense subset of C.*

Proof. We have $\mathrm{Ch}_S U = U^*_{\mathrm{reg}}$ and $\mathfrak{M} \subset B(U^*)$.

The weak $w(C, \mathfrak{M})$-topology in the case of U^*_{ir} negligible can be described by the system of pseudo-norms $\{f \mapsto |\varepsilon_x^{CU}(f)|;\ x \in U\}$ and is the weakest topology on C such that $H_f^U(x)$ is a continuous function of f for every $x \in U$. The density of C_R in C for this topology was first established by M. Brelot in the general axiomatic theory with axiom D ([3], Théorème 3). Let us mention only that in this case all pseudo-norms $f \mapsto |\varepsilon_x^{CU}(f)|$ are equivalent for U connected.

The space $C(U^*)$ with the weak $w(C, B(U^*))$-topology is a locally convex linear topological space, and it is Hausdorff one if and only if U^* is the Šilov boundary of S. Indeed, it is clear that the $w(C, B(U^*))$-topology is Hausdorff if and only if the set $B(U^*)$ distinguishes continuous functions on U^*. If $\mathrm{Ch}_S U$ is dense in U^*, and $f \in C(U^*)$ is not identically zero, then there is $z \in \mathrm{Ch}_S U$ such that $f(z) \neq 0$. Hence, for $\varepsilon_z \in B(U^*)$ we have $\varepsilon_z(f) \neq 0$. On the other hand, if $\mathrm{Ch}_S U$ is not dense in U^*, then there is a point z and an $f \in C(U^*)$ satisfying $0 \leqq f \leqq 1$ on U^*, $f(z) = 1$, $f = 0$ on $\overline{\mathrm{Ch}_S U}$. Thus, for any $\mu \in B(U^*)$ we have $\mu f = 0$.

We know that the set U^*_{ir} is negligible if and only if $\mathfrak{M} \subset B(U^*)$. One implication is a direct consequence of the Brelot-Keldych theorem, and the other one follows from the observation that in the case of U^*_{ir} non-negligible there is $x \in U$ such that

$$\varepsilon_x^{CU}(U^* \setminus \mathrm{Ch}_S U) \geqq \varepsilon_x^{CU}(U^*_{ir}) > 0 .$$

In what follows, we shall derive the inclusion $\mathfrak{M} \subset B(U^*)$ whenever U^*_{ir} is negligible without using the Brelot-Keldych theorem explicitly.

We denote $\bar{A}_f(x) = \inf\{s(x);\ s \geqq f \text{ on } U^*,\ s \in S(\overline{U})\}$, $\underline{A}_f(x) = -\bar{A}_{-f}(x)$ for any $f \in C(U^*)$ and any $x \in U$. From the minimum principle it follows easily that $\underline{A}_f \leqq \bar{A}_f$ on $\overline{U}$. The set in the brackets of all superharmonic functions constitutes a saturated Perron family on U, so that $\underline{A}_f$ and $\bar{A}_f$ are harmonic functions on U. If $s \in S(\overline{U})$, $s \geqq f$, then $s \geqq H_s^U \geqq H_f^U$ on U. Hence, $\underline{A}_f \leqq H_f^U \leqq \bar{A}_f$ on U. The same inequalities hold for D_f^U instead of H_f^U.

Theorem 7. *The following assertions are equivalent:*

(i) *U^*_{ir} is negligible;*

(x) *$\underline{A}_f = \bar{A}_f$ on U for any $f \in C(U^*)$.*

Proof. *If $\underline{A}_f = \bar{A}_f$ on U, then $D_f^U = H_f^U$ on U whenever $f \in C(U^*)$* and U^*_{ir} is negligible in view of Proposition 2. Assume that U^*_{ir} is negligible. The function $\bar{A}_f$

is upper semicontinuous on $\overline{U}$ and harmonic in U. For any $x \in U^*_{\text{reg}} = \text{Ch}_S U$ we have $\overline{A}_f(x) = f(x)$, the equality ensueing from the characterization of the points in the Choquet boundary. Summarizing, we get $\limsup_{y \to x, y \in U} \overline{A}_f(y) \leqq \overline{A}_f(x) = f(x)$ for any $x \in$ $\in U^*_{\text{reg}}$. Hence, at any regular point x we get

$$\limsup_{y \to x} (\overline{A}_f(y) - H^U_f(y)) \leqq \limsup_{y \to x} \overline{A}_f(y) - \lim_{y \to x} H^U_f(y) \leqq f(x) - f(x) = 0$$

and the minimum principle ([1], Satz 4.4.6) results in $\overline{A}_f \leqq H^U_f$ on U. Thus, $\underline{A}_f =$ $= H^U_f = \overline{A}_f$ on U.

Corollary 8. *If U^*_{ir} is negligible, then for any $f \in C(U^*)$ and for any $x \in U$*

$$H^U_f(x) = \inf \{s(x);\ f \leqq s \text{ on } U^*,\ s \in S(\overline{U})\}\ .$$

Corollary 9. *If U^*_{ir} is negligible, then for any $f \in C(U^*)$ and for any $x \in U$*

$$H^U_f(x) = \inf \{s(x);\ s \geqq f \text{ on } U^*_{\text{reg}},\ s \in S(\overline{U})\}\ .$$

Proof. Obviously, $H^U_f(x) \geqq \inf \{s(x);\ s \geqq f \text{ on } U^*_{\text{reg}},\ s \in S(\overline{U})\}$ according to the preceding Corollary 8. If $s, -t \in S(\overline{U})$, $s \geqq f$ on U^*_{reg}, $t \leqq f$ on U^*, then $t \leqq s$ on $\overline{U}$ in virtue of the minimum principle again. Hence $\underline{A}_f \leqq \inf \{s \in S(\overline{U});\ s \geqq f \text{ on } U^*_{\text{reg}}\}$ and the proof is complete.

Given any positive Radon measure μ on U^*, a set $M \subset U^*$ is of outer μ-measure zero if and only if for any $\varepsilon > 0$ there is a sequence $\{f_n\} \subset C(U^*)$ with the properties $0 \leqq f_1 \leqq f_2 \leqq \ldots$ on U^*, $\mu(f_n) < \varepsilon$ for any n, $\sup_n \{f_n\} \geqq 1$ on M. This yields the next lemma.

Lemma 10. *The following assertions are equivalent:*

(i) *U^*_{ir} is negligible;*

(xi) *given any $x \in U$ and any $\varepsilon > 0$, there are sequences $\{f_n\} \subset C(U^*)$ and $\{s_n\} \subset S(\overline{U})$ such that*

$$0 \leqq f_1 \leqq f_2 \leqq \ldots \text{ on } U^*\ ,\quad f_n \leqq s_n \text{ on } U^*\ ,\quad s_n(x) < \varepsilon$$

for any n, $\sup_n \{f_n(y)\} \geqq 1$ for every $y \in U^*_{ir}$.

Proof. If the condition (xi) is fulfilled, then U^*_{ir} is negligible in view of the above remark and the inequality $H^U_s \leqq s$ for $s \in S(\overline{U})$. Let U^*_{ir} be negligible. We fix $x \in U$ and $\varepsilon > 0$. There are $\{f_n\} \subset C(U^*)$ with the properties mentioned above and we may apply Corollary 8.

Now, it is probably time to explain the Brelot-Keldych theorem. *If $L = \{L_x\}$ is any Keldych operator on a set U, and U^*_{ir} is negligible, then any measure $L_x \in \mathfrak{M}$ is supported by the set U^*_{ir}* according to the last lemma. (Of course, for every $\varepsilon > 0$,

$x \in U$ we can find the sequences $\{f_n\}$ and $\{s_n\}$ as above. Then $L_x(f_n) \leqq L_x(s_n) \leqq \leqq s_n(x) < \varepsilon$.) *Hence* $L_x \in B(U^*)$, *and* L_x *is continuous in the* $w(C, B(U^*))$-*topology. Finally, on the set* C_R *which is* $w(C, B(U^*))$-*dense in* $C(U^*)$, L_x *and* ε_x^{CU} *coincide.*

3. POINT KELDYCH FUNCTIONALS

Any Keldych operator L on a set U satisfies the inequalities $D_s^U \leqq Ls \leqq H_s^U$ on U whenever $s \in S(\bar{U})$. The first inequality is clear, the second one is a consequence of the fact that Ls is a lower function to s on U. Hence, by Proposition 2, $Ls = H_s^U$ if U_{ir}^* is negligible. Nevertheless, the validity of the Brelot-Keldych theorem can be accounted for probably in the best way by Corollary 9.

The requirement of Lf being a harmonic function on the whole set U whenever $f \in C(U^*)$ has been important in all variants of proofs of the Brelot-Keldych theorem. The question arises if this condition can be omitted and an analogous theorem stated for the point Keldych functionals, a suitable definition of which seems to be as follows: For $x \in U$, a (weak) *point Keldych functional at* x is every monotone functional P on the set $C(U^*)$ such that $Ps \leqq s(x)$ for any $s \in S(\bar{U})$ and $Pt \geqq t(x)$ for any $t \in -S(\bar{U})$. On account of Corollary 8, the proof of the following theorem is straightforward.

Theorem 11. *Let* U_{ir}^* *be a negligible set, and let* P *be a point Keldych functional at* $x \in U$. *Then* $P = \varepsilon_x^{CU}$.

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