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ON A CONJECTURE OF THE SEMIGROUP OF FULLY INDECOMPOSABLE RELATIONS

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The purpose of this note is to show that the conjecture in [4] does not hold in general. In order to avoid the multiplications of large matrices, we shall use some of the properties of directed graphs to show that, for each integer $n \geqq 5$, there exists a primitive binary relation ϱ on a set of n points such that none of the ϱ^i's is fully indecomposable for $i = 1, 2, \ldots, n$.

A binary relation on a finite set $\Omega = \{a_1, a_2, \ldots, a_n\}$ of n elements, $n > 1$, is a subset of $\Omega \times \Omega = \{(a_i, a_j);\ a_i, a_j \in \Omega\}$. Let $B = B(\Omega)$ be the set of all binary relations on Ω. (When there is no confusion, an element in B is also called a relation on Ω, or just a relation). Then B is a semigroup with the multiplication defined as follows: for ϱ and τ in B, $(a_i, a_j) \in \varrho\tau$ if there is a $a_k \in \Omega$ such that $(a_i, a_k) \in \varrho$ and $(a_k, a_j) \in \tau$. Let ω be the universal relation on Ω, i.e., $\omega = \Omega \times \Omega$, and $\Delta = \{(a_i, a_i);\ a_i \in \Omega\}$. Also, let M_n denote the set of all $n \times n$ matrices over the Boolean algebra of $\{0, 1\}$, then M_n is a semigroup under the ordinary matrix multiplication, and the map

$$\varrho \to M(\varrho) = (m_{i,j})$$

where

$$m_{i,j} = \begin{cases} 1 & \text{if } (a_i, a_j) \in \varrho\,, \quad \text{and} \\ 0 & \text{otherwise}\,, \end{cases}$$

is an isomorphism of B onto M_n. Let X_n be the set of all directed graphs on n vertices with allowable loops and simple directed edges. Each matrix in M_n can be considered as the adjacency matrix of a directed graph Y in X_n, and it determines Y uniquely up to isomorphism. Also, each graph in X_n with labelled vertices determines a unique matrix in M_n as its adjacency matrix. Hence, there is an one-to-one correspondence among B, M_n and X_n:

$$\varrho \to M(\varrho) \to Y(\varrho)\,.$$

Let $B_0 = B_0(\Omega)$ consist of all binary relations on Ω with $\mathrm{pr}_1(\varrho) = \mathrm{pr}_2(\varrho) = \Omega$ where

$$a_i\varrho = \{x \in \Omega;\ (a_i, x) \in \varrho\},\ \varrho a_i = \{y \in \Omega;\ (y, a_i) \in \varrho\},$$

$$\mathrm{pr}_1(\varrho) = \bigcup_{j=1}^{n} \varrho a_j \quad \text{and} \quad \mathrm{pr}_2(\varrho) = \bigcup_{j=1}^{n} a_i\varrho .$$

Clearly, B_0 is a subsemigroup of B. This means that, if $\varrho \in B_0$, then none of the columns and none of the rows in $M(\varrho)$ consist of all zeros, and every vertex in the graph $Y(\varrho) \in X_n$ is incident with at least one incoming edge, and at least one outgoing edge. (A loop is considered both as an incoming edge and as an outgoing edge). An element $\varrho \in B_0$ is said to be decomposable if there is a π belonging to the group Π of all permutation relations on Ω such that $M(\pi\varrho\pi^{-1})$ is of the form

$$\begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \tag{1}$$

where B and D are square matrices of sizes $s \times s$ and $(n - s) \times (n - s)$ respectively, and $1 \leqq s \leqq n - 1$. Otherwise it is called indecomposable. An element $\varrho \in B_0$ is said to be partly decomposable if there are two elements π_1 and π_2 in Π such that $M(\pi_1\varrho\pi_2)$ is of form (1). Otherwise it is called fully indecomposable. Let $I = I(\Omega)$, $F = F(\Omega)$ and $H = H(\Omega)$ be, respectively, the set of all indecomposable relations in B_0, the set of all fully indecomposable relations in B_0 and the set of all relations in B_0 each of which contains a permutation relation. H is called the Hall relations on Ω. F and H are semigroups, in fact, F is a two sided ideal of H (see Theorems 1.2 and 2.3 in [4]). We note that if a matrix contains an $s \times (n - s)$ zero submatrix for some s, $1 \leqq s \leqq n - 1$, then the matrix does not belong to F. A relation $\varrho \in B_0$ is said to be primitive if there is an integer $k = k(\varrho)$ such that $\varrho^k = \omega$. Clearly, a primitive relation is indecomposable. The set of all primitive relations in B_0 is denoted by $P = P(\Omega)$. As stated in [4], we have

$$B_0 \supset I \supset P \supset F .$$

A graph Y in X_n is said to be strongly connected if, for any two vertices in Y, there is a directed path in Y from one vertex to the other. If ϱ is decomposable, then the corresponding graph $Y(\varrho)$ is not strongly connected. If $\varrho \in P$, then the corresponding graph $Y(\varrho)$ is strongly connected. However, the converse does not hold, e.g., a directed n-cycle is strongly connected, but its corresponding binary relation does not belong to P. Wielandt, in [6], was the first to state that for any $\varrho \in P$, there is a least integer $k = k(\varrho)$, called the index of primitivity of ϱ, such that $\varrho^k = \omega$ and $k \leqq (n - 1)^2 + 1$. It was proved by many others, e.g., Holladay and Varga [3]. (Wielandt and others dealt with the $n \times n$ matrices with non-negative real entries, but as far as the primitivity and the index of primitivity concern, they are the same as the $n \times n$ matrices over the Boolean algebra of $\{0, 1\}$). As stated on pp. 162–163 in [4]: "To any $\varrho \in P$ there is a least integer $l_1 = l_1(\varrho) \geqq 1$ such that $\varrho^{l_1} \in H$, and a least integer $l_2 = l_2(\varrho) \geqq l_1$ such that $\varrho^{l_2} \in F$. The problem to find exact upper bounds for l_1 and l_2 (in terms of n) seems to be (at this writing) rather difficult. There are some reasons for the following

Conjecture. *For any $\varrho \in P$, we have $l_2 = l_2(\varrho) \leqq n''$.*

Here, we show that the conjecture does not hold in general. To find the exact upper bounds for l_1 and l_2 remains to be very difficult and unanswered. Let $\Omega = \{a_1, a_2, \ldots, a_5\}$ and $\varrho \in B_0 = B_0(\Omega)$ such that

$$M = M(\varrho) = \begin{bmatrix} 0&1&0&0&0 \\ 0&0&1&0&1 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \end{bmatrix}.$$

Then M^2, M^3, M^4, M^5, M^6 and M^{10} are respectively

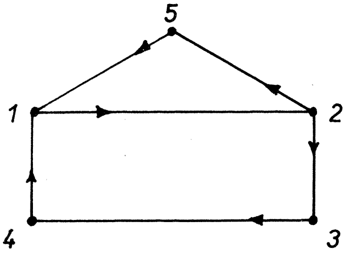
$$\begin{bmatrix} 0&0&1&0&1 \\ 1&0&0&1&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&1&0 \\ 1&1&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&1 \\ 0&0&1&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&0&0&0 \\ 0&1&1&0&1 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \\ 1&0&0&1&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&1&1&0&1 \\ 1&0&1&1&1 \\ 1&0&0&1&0 \\ 1&1&0&0&0 \\ 1&1&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&1&1&1 \\ 1&1&0&1&0 \\ 1&1&0&0&0 \\ 0&1&1&0&1 \\ 0&1&1&0&1 \end{bmatrix} \text{ and } \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix}.$$

Hence, $\varrho^{11} = \omega$ and ϱ is primitive. With some suitable permutations of rows and columns, we see none of ϱ^i for $i = 1, 2, \ldots, 5$ belonging to F, e.g., for M^4, we have

$$\begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \end{bmatrix} \begin{bmatrix} 1&1&0&0&0 \\ 0&1&1&0&1 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \\ 1&0&0&1&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \end{bmatrix} = \begin{bmatrix} 1&1&0&0&0 \\ 1&1&0&0&0 \\ 0&0&1&0&1 \\ 0&0&1&1&1 \\ 1&0&0&1&0 \end{bmatrix}$$

which is of form (1), and ϱ^4 is partly decomposable, i.e., $\varrho^4 \notin F$. The corresponding graph $Y(\varrho)$ is



In fact, we can prove the following

Theorem. *For each integer $n \geqq 5$, there exists a primitive binary relation ϱ on $\Omega = \{a_1, a_2, \ldots, a_n\}$ such that none of the ϱ^i is fully indecomposable for $i = 1, 2, \ldots, n$.*

In order to avoid the multiplication of large matrices in our proof, we shall use some of the elementary properties of directed graphs, namely, the following

Lemma 1. *Let $M = M(\varrho)$ be the adjacency matrix of the graph $Y = Y(\varrho)$. Then, in $M^t = (m_{i,j}^t)$, $m_{k,l}^t$ is 1 (is 0) if and only if there is at least one directed path (no directed path) of length t from the vertex k in Y to the vertex l in Y.*

Proof. It follows from the definition of adjacency matrix and the definition of matrix multiplication over the Boolean algebra of $\{0, 1\}$.

Lemma 2. *Let $\varrho \in B_0$ and $Y = Y(\varrho)$ be the corresponding graph. If every vertex of Y is on a k-cycle (not necessarily a simple k-cycle), then $\Delta \subseteq \varrho^k$.*

Proof. It follows from Lemma 1.

Lemma 3. *Let Z be a directed graph on n vertices such that each vertex of Z has a loop and Z contains a simple n-cycle, and let μ be its corresponding binary relation in $B_0 = B_0(\Omega)$. Then $\mu^{n-1} = \omega$ and μ is primitive.*

Proof. It is sufficient to assume $\mu = \Delta \cup \sigma$ where σ corresponds to the simple n-cycle in Z. Let $M = M(\Delta \cup \sigma)$, then $M = I + X$ where I is the identity matrix corresponding to Δ and $X = (x_{i,j})$ is the matrix corresponding to σ. Since X is the adjacency matrix of the simple n-cycle, we have, for any i_1 such that $1 \leqq i_1 \leqq n$,

$$x_{i_1,i_2} = x_{i_2,i_3} = x_{i_3,i_4} = \ldots = x_{i_{n-1},i_n} = x_{i_n,i_1} = 1$$

where $i_1, i_2, \ldots, i_n$ are pairwise distinct. By Lemma 1, in $X^k = (x_{i,j}^k)$, $x_{i_1,i_{k+1}}^k = 1$

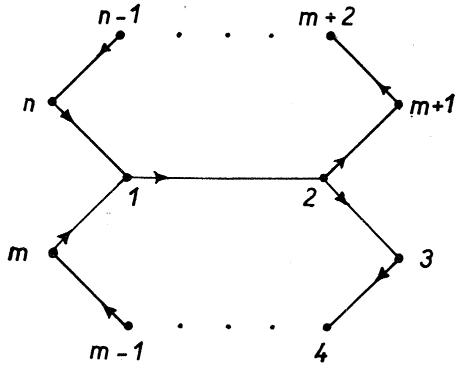
for $k = 1, 2, \ldots, n - 1$. Consequently,

$$M^{n-1} = (I + X)^{n-1} = I + X + X^2 + \ldots + X^{n-1}$$

consists of all one's. Hence, $\mu^{n-1} = \omega$ and μ is primitive.

Now the proof of our Theorem goes as follows:

Case 1. n is odd $\geqq 5$. Let $m = (n + 3)/2$. Construct a directed graph Y on n vertices with two directed cycles with length m and length $m - 1$ having one edge in common.



Let ϱ be the binary relation in $B_0 = B_0(\Omega)$ corresponding to Y. We claim that ϱ is primitive:

We show $\Delta \subseteq \varrho^{n+2}$. Every vertex of Y is on the cycle of length $n + 2$.

$$1 \to 2 \to 3 \to \ldots \to m \to 1 \to 2 \to m + 1 \to m + 2 \to \ldots \to n \to 1 .$$

Hence, by Lemma 2, $\Delta \subseteq \varrho^{n+2}$.

Let Z be the directed graph on n vertices corresponding to ϱ^{n+2}. We show that Z contains a simple $(n + 2)$-cycle. Here the notation $1 \to {}^{\bullet}m$ means that, in Y, the vertex 1 reaches the vertex m by $(n + 2)$-length. Since $m = (n + 3)/2$ and since the two cycles in Y differ by 1 length, we have

$$1 \to {}^{\bullet}m \to {}^{\bullet}n \to {}^{\bullet}m - 1 \to {}^{\bullet}n - 1 \to {}^{\bullet}m - 2 \to \ldots \to {}^{\bullet}m + 1 \to {}^{\bullet}3 \to {}^{\bullet}2 \to {}^{\bullet}1 .$$

Since $\Delta \subseteq \varrho^{n+2}$, every vertex in Z has a loop. By Lemma 3, ϱ^{n+2} is primitive, and so is ϱ.

We claim that none of the ϱ^i's belongs to F for $i = 1, 2, \ldots, n$:

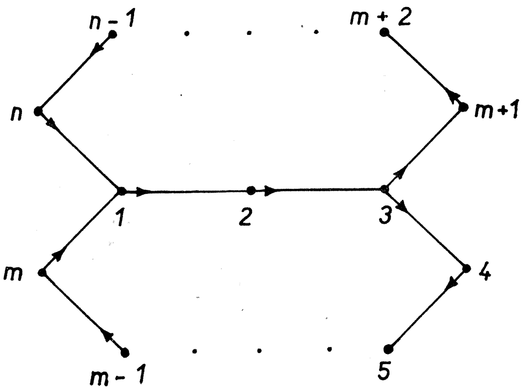
In the graph Y, we know that both $m - i$ and $n - i$ vertices reach 1 by $(i + 1)$-length for $i = 0, 1, 2, \ldots, m - 4$, i.e., in the matrix $M^{i+1} = (m_{k,l}^{i+1})$, we have $m_{(m-i),1}^{i+1} = m_{(n-i),1}^{i+1} = 1$, and the rest of $(m - i)$th row and $(n - i)$th row are zeros. Since M^{i+1} contains an $2 \times (n - 2)$ zero submatrix, we have $\varrho^{i+1} \notin F$ for $i = 0, 1, 2, \ldots, m - 4$.

Similarly, in M^{m-2}, $m^{m-2}_{4,2} = m^{m-2}_{(m+1),2} = 1$ and the rest of 4th row and $(m+1)$th row are zeros. Since M^{m-2} contains an $2 \times (n-2)$ zero submatrix, we have $\varrho^{m-2} \notin F$.

Similarly, for $l = 3, 4, \ldots, m+1$, in M^{m+l-4}, $m^{m+l-4}_{4,l} = m^{m+l-4}_{(m+1),l} = m^{m+l-4}_{4,(m+l-2)} = m^{m+l-4}_{(m+1),(m+l-2)} = 1$, and the rest of 4th row and $(m+1)$th row are zeros. Since M^{m+l-4} contains an $2 \times (n-2)$ zero submatrix, we have $\varrho^{m+l-4} \notin F$ for $l = 3, 4, \ldots, m+1$.

Hence, ϱ is primitive and none of ϱ^i belongs to F for $i = 1, 2, \ldots, n$.

Case 2. n is even > 5. Let $m = (n+4)/2$. Construct a directed graph U on n vertices with two directed cycles of length m and length $m-1$ having two edges in common.



Let τ be the binary relation in $B_0 = B_0(\Omega)$ corresponding to U. We claim that τ is primitive:

We show $\Delta \subseteq \tau^{n+3}$. Every vertex of U is on the cycle of length $n+3$.

$$1 \to 2 \to 3 \to \ldots \to m \to 1 \to 2 \to 3 \to m+1 \to m+2 \to \ldots \to n \to 1\,.$$

Hence, by Lemma 2, $\Delta \subseteq \tau^{n+3}$.

Let V be the directed graph on n vertices corresponding to τ^{n+3}. We show that V contains a simple $(n+3)$-cycle. Since $m = (n+4)/2$ and since the two cycles in U differ by one length, we have

$$1 \to {}^{\bullet}m \to {}^{\bullet}n \to {}^{\bullet}m-1 \to {}^{\bullet}n-1 \to {}^{\bullet}m-2 \to \ldots \to {}^{\bullet}m+1 \to$$
$$\to {}^{\bullet}4 \to {}^{\bullet}3 \to {}^{\bullet}2 \to {}^{\bullet}1$$

where $1 \to {}^{\bullet}m$ means, in U, the vertex 1 reaches the vertex m by $(n+3)$-length. Since $\Delta \subseteq \tau^{n+3}$, every vertex in V has a loop. By Lemma 3, τ^{n+3} is primitive, and so is τ.

In the graph U, we know that both $m - i$ and $n - i$ vertices reach 1 by $(i + 1)$-length for $i = 0, 1, 2, \ldots, m - 5$, i.e., in the matrix $M^{i+1} = (m_{k,l}^{i+1})$, we have $m_{(m-i),1}^{i+1} = m_{(n-i),1}^{i+1} = 1$, and the rest of $(m - i)$th row and $(n - i)$th row are zero. Since M^{i+1} contains an $2 \times (n - 2)$ zero submatrix, we have $\tau^{i+1} \notin F$ for $i = 0, 1, 2, \ldots, m - 5$.

Similarly, in M^{m-3} and M^{m-2}, $m_{5,2}^{m-3} = m_{(m+1),2}^{m-3} = 1$, and $m_{5,3}^{m-2} = m_{(m+1),3}^{m-2} = 1$, and the rest of 5th row and $(m + 1)$th row are zeros. Since each of M^{m-3} and M^{m-2} contains an $2 \times (n - 2)$ zero submatrix, we have $\tau^{m-3} \notin F$ and $\tau^{m-2} \notin F$.

Similarly, for $l = 4, 5, \ldots, m + 1$, in M^{m+l-5}, $m_{5,l}^{m+l-5} = m_{(m+1),l}^{m+l-5} = m_{5,(m+l-3)}^{m+l-5} = m_{(m+1),(m+l-3)}^{m+l-5} = 1$, and the rest of 5th row and $(m + 1)$th row are zeros. Since M^{m+l-5} contains an $2 \times (n - 2)$ zero submatrix, we have $\tau^{m+l-5} \notin F$ for $l = 4, 5, \ldots, m + 1$.

Hence, τ is primitive and none of τ^i belongs to F for $i = 1, 2, \ldots, n$.

Remark. It is well known [6, 5, 3, 2, 1] that if $\varrho \in P$ and $\Delta \subseteq \varrho$, then the index of primitivity is $\leqq n - 1$. Also, Proposition 3.2 in [4] states that if $\varrho \in P$ and $\Delta \subseteq \varrho$ then $\varrho \in F$. Consider $\varrho \in P$ and $\Delta \subseteq \varrho$. Since $\varrho \in P$, $Y(\varrho)$ is strongly connected and every vertex in $Y(\varrho)$ is on a cycle, not necessarily a simple cycle. Say, the smallest length of such a cycle in $Y(\varrho)$ is t, then $\varrho^t \in P$, $\Delta \subseteq \varrho^t$ and $\varrho^t \in F$. However, unfortunately, in general this t is not the least integer such that $\varrho^t \in F$. In the above example of the directed graph on 5 vertices, we have $t = 7$, but $\varrho^6 \in F$.

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