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THE CATEGORY OF CONNECTED PARTIAL UNARY ALGEBRAS

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1. THE CLASSIFICAL FUNCTOR

1.0. Notation. We denote by Ord the class of all ordinals and by N the set of all finite ordinals. If $\alpha \in \text{Ord}$ then we put $W_\alpha = \{\beta \in \text{Ord}; \beta < \alpha\}$.

If A is a set we denote by $|A|$ the cardinal number of A. Let φ be a partial map from A into a set B. We put dom $\varphi = \{x \in A$; there exists $y \in B$ such that $(x, y) \in \varphi\}$. If dom $\varphi = A$ then we write $\varphi : A \to B$. Finally, if $C \subseteq A$ then we denote by $\varphi \mid C$ the restriction $\varphi \cap (C \times B)$ of φ.

Let $\mathscr{A}$ be a category. Then we denote by ob $\mathscr{A}$ the class of all objects of $\mathscr{A}$ and, for arbitrary $P, Q \in \mathscr{A}$, by $[P, Q]_{\mathscr{A}}$ the set of all morphisms from P to Q. In most cases we shall write shortly $\mathscr{A}$ instead of ob $\mathscr{A}$. Further, $\bigcup [P, Q]_{\mathscr{A}}$ means the class of all morphisms of $\mathscr{A}$. The sign $\cong$ means an isomorphism of categories and $\subseteq$ a full subcategory.

If $\mathscr{A}$ is a category such that, for each $P, Q \in \mathscr{A}$, $[P, Q]_{\mathscr{A}} \neq \emptyset$, $[Q, P]_{\mathscr{A}} \neq \emptyset$ implies $P = Q$ then $\mathscr{A}$ is called *antisymmetric*. If $\mathscr{A}$ is a category such that, for each $P, Q \in \in \mathscr{A}$, $|[P, Q]_{\mathscr{A}}| \leqq 1$ then $\mathscr{A}$ is called a *quasi-ordered class* or a *thin category*. An antisymmetric and thin category is called an *ordered class*. An ordered class $\mathscr{A}$ is called an *antichain* if $[P, Q]_{\mathscr{A}} = \emptyset$ for each $P, Q \in \mathscr{A}$, $P \neq Q$ and is called a *chain* if $[P, Q]_{\mathscr{A}} \neq \emptyset$ or $[Q, P]_{\mathscr{A}} \neq \emptyset$ for each $P, Q \in \mathscr{A}$.

In the paper [6], various arithmetic operations for categories are introduced. We want to use one of these operations specially only for thin categories. Thus, the definitions of composition of morphisms are evident.

The lexicographic sum $\sum^l_{G \in \mathscr{G}} \mathscr{A}_G$ of a system $\{\mathscr{A}_G; G \in \mathscr{G}\}$ of categories where $\mathscr{G}$ is an antisymmetric category is the class $\bigcup_{G \in \mathscr{G}} \bigcup_{P \in \mathscr{A}_G} (G, P)$ of objects and the class $\bigcup [(G_1, P), (G_2, Q)]_{\Sigma^l}$ of morphisms where

$$[(G_1, P), (G_2, Q)]_{\Sigma^l} = \begin{cases} [G_1, G_2]_{\mathscr{G}} \times [P, Q]_{\mathscr{A}_{G_1}} & \text{if } G_1 = G_2 \,, \\ [G_1, G_2]_{\mathscr{G}} & \text{if } G_1 \neq G_2 \,. \end{cases}$$

Further, if $\mathscr{G} = \{1, \ldots, n\}$ is a chain with natural order then we put $\sum\limits_{G \in \mathscr{G}}^{l} \mathscr{A}_G = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \ldots \oplus \mathscr{A}_n$.

If $\{\mathscr{A}, \mathscr{B}\}$ is a non-indexed system of two different categories then we suppose that $\{\mathscr{A}, \mathscr{B}\}$ is a chain with $[\mathscr{A}, \mathscr{B}]_{\{\mathscr{A}, \mathscr{B}\}} \neq \emptyset$ and that, for $\mathscr{A} \oplus \mathscr{B}$, ob $(\mathscr{A} \oplus \mathscr{B}) = \bigcup\limits_{P \in \mathscr{A}} (\mathscr{A}, P) \cup \bigcup\limits_{Q \in \mathscr{B}} (\mathscr{B}, Q)$.

Finally, we see that for the lexicographic sum $\mathscr{A} = \sum\limits_{G \in \mathscr{G}}^{l} \mathscr{A}_G$ with disjoint summands we can put $(G, P) = P$ for each $(G, P) \in \mathscr{A}$.

1.1. Definition. Let A be a non-empty set, f a partial map from A into A. Then the ordered pair $\boldsymbol{A} = (A, f)$ is called a *partial* unary algebra.

1.2. Definition. Let $\boldsymbol{A} = (A, f)$ be a partial unary algebra. Then we put $D\boldsymbol{A} = A - \operatorname{dom} f$. If $D\boldsymbol{A} = \emptyset$ then $\boldsymbol{A}$ is called a *complete* unary algebra.

1.3. Definition. Let $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g)$ be partial unary algebras and $F : A \to B$ a map. Then F is called a *homomorphism* of $\boldsymbol{A}$ into $\boldsymbol{B}$ if $x \in \operatorname{dom} f$ implies $F(x) \in \operatorname{dom} g$ and $F(f(x)) = g(F(x))$ for each $x \in A$. We write $F : \boldsymbol{A} \to \boldsymbol{B}$.

1.4. Definition. Let $\boldsymbol{A} = (A, f)$ be a partial unary algebra.

(a) We put $f^0 = \mathrm{id}_A$. Suppose that we have defined a partial map f^{n-1} from A into A for $n \in N - \{0\}$. We denote by f^n the following partial map from A into A: if $x \in \operatorname{dom} f^{n-1}$ and $f^{n-1}(x) \in \operatorname{dom} f$, then we put $f^n(x) = f(f^{n-1}(x))$.

(b) Let $x \in A$ be arbitrary. Then we define $[x]_{\boldsymbol{A}} = \{y \in A;$ there is $n \in N$ with $x \in \operatorname{dom} f^n$ and $y = f^n(x)\}$.

(c) $\boldsymbol{A}$ is called a *connected* partial unary algebra (abbreviation a c-algebra) if, for any $x, y \in A$, $[x]_{\boldsymbol{A}} \cap [y]_{\boldsymbol{A}} \neq \emptyset$. (Compare [4], 1.5, 1.7, 1.9.)

1.5. Remark. Clearly, a partial unary algebra $\boldsymbol{A}$ is a c-algebra iff, for any $x, y \in A$, there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $y \in \operatorname{dom} f^n$ and $f^m(x) = f^n(y)$.

In our paper we want to study the category of all c-algebras. Throughout the paper, we denote this category by $\mathscr{U}^c$.

1.6. Problem. *Describe the category $\mathscr{U}^c$; i.e., find necessary and sufficient conditions for the existence of morphisms of $\mathscr{U}^c$ in the terminology of categories.*

In our considerations, we can apply most of the results of the paper [4] and some results of [3] and [5]. Therefore, let us recall the main notions and assertions of these papers first.

Let $\infty_1, \infty_2 \bar{\in} \mathrm{Ord}$ and suppose that $\alpha < \infty_1 < \infty_2$ for each $\alpha \in \mathrm{Ord}$.

1.7. Definition. Let $A = (A, f) \in \mathscr{U}^c$. Then we define

(a) the set $ZA = \{x \in A;$ there is $n \in N - \{0\}$ such that $f^n(x) = x\}$, $RA = |ZA|$; ZA is called the *cycle* and RA the *range* of A;

(b) the set $KA = \{x \in A - ZA;$ there is a sequence $(x_i)_{i \in N}$ such that $x_i \in \operatorname{dom} f$ for each $i \in N - \{0\}$, $x_0 = x$ and $f(x_{i+1}) = x_i$ for each $i \in N\}$; KA is called the *kernel* of A;

(c) the sets $A^{\infty_2} = ZA$, $A^{\infty_1} = KA$, $A^0 = \{x \in A; f^{-1}(x) = \emptyset\}$; if $\alpha \in \mathrm{Ord} - \{0\}$ is arbitrary and if the sets $A^\varkappa$ have been defined for all $\varkappa \in W_\alpha$ then we put $A^\alpha = \{x \in A - \bigcup_{\varkappa \in W_\alpha} A^\varkappa; f^{-1}(x) \subseteq \bigcup_{\varkappa \in W_\alpha} A^\varkappa\}$;

(d) $\vartheta A = \min\{\varkappa \in \mathrm{Ord}; A^\varkappa = \emptyset\}$;

(e) the map $SA : A \to \mathrm{Ord} \cup \{\infty_1, \infty_2\}$ by the condition $SA(x) = \varkappa$ for each $x \in A^\varkappa$, $\varkappa \in W_{\vartheta A} \cup \{\infty_1, \infty_2\}$; $SA(x)$ is called the *degree* of x.

(Compare [4], 2.4–2.8 (a), 2.13–2.16, 2.19, 2.20 and 2.23.)

1.8. Definition. We put

$\mathbf{0} - \mathscr{U}^c = \{A \in \mathscr{U}^c;\ KA = \emptyset,\ RA = 0\}$,
$\mathbf{1} - \mathscr{U}^c = \{A \in \mathscr{U}^c;\ KA \neq \emptyset,\ RA = 0\}$,
$\mathbf{2} - \mathscr{U}^c = \{A \in \mathscr{U}^c;\ RA \neq 0\}$.

1.9. Lemma. *$\mathscr{U}^c = \mathbf{0} - \mathscr{U}^c \cup \mathbf{1} - \mathscr{U}^c \cup \mathbf{2} - \mathscr{U}^c$ with disjoint summands.*

The assertion is evident.

(i) If $A \in \mathscr{U}^c$ then $|DA| \leqq 1$. (See [4], 2.1.)

1.10. Definition. Let $A \in \mathscr{U}^c$ be such that $DA \neq \emptyset$. Then we denote by dA the only point with the property $\{dA\} = DA$.

(ii) If $A = (A, f) \in \mathscr{U}^c$ with $DA \neq \emptyset$ then, for each $x \in A$, there is precisely one $n \in N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) = dA$. (See [4], 2.3.)

(iii) If $A = (A, f) \in \mathscr{U}^c$ then $DA \neq \emptyset$ iff the following conditions hold: $RA = 0$ and there is $x \in A$ such that $|[x]_A| < \aleph_0$. (See [4], 2.9.)

(iv) If $A = (A, f) \in \mathscr{U}^c$ then $(KA \cup ZA,\ f \mid KA \cup ZA)$, $(ZA, f \mid ZA)$ are subalgebras of A. (See [4], 2.10 and 2.15 (a).)

(v) If $A \in \mathscr{U}^c$ and $x \in A - (KA \cup ZA)$, $n \in N$ are such that $x \in \operatorname{dom} f^n$ then $SA(f^n(x)) \geqq SA(x) + n$. (See [4], 2.26 (a).)

1.11. Lemma. *Let $A \in \mathbf{1} - \mathscr{U}^c$ ($A \in \mathbf{2} - \mathscr{U}^c$) and $x \in A$. Then there is $n \in N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) \in KA$ ($f^n(x) \in ZA$ resp.).*

Proof. Indeed, the assertions follow from 1.8, 1.4 (b) and (iv). (Compare [3], 1.17 (a).)

1.12. Lemma. *Let $A \in \mathbf{0} - \mathscr{U}^c$. Then $DA \neq \emptyset$ iff ϑA is a successor ordinal.* (See [7].)

Proof. The condition is necessary by [4], 2.26 (c). Let, on the other hand, $A \in \mathbf{0} - \mathscr{U}^c$ and let ϑA be a successor ordinal. Then $A^{\vartheta A - 1} \neq \emptyset$ and, for $x \in A^{\vartheta A - 1}$, we have $x \in DA$ by (v) because $KA \cup ZA = \emptyset$. Thus, $DA \neq \emptyset$.

(vi) If $A \in \mathbf{0} - \mathscr{U}^c$ and $DA \neq \emptyset$ then $SA(dA) = \vartheta A - 1$. (See [4], 2.26 (c).)

(vii) If $A \in \mathbf{1} - \mathscr{U}^c$ and $DA \neq \emptyset$ then $SA(dA) = \infty_1$. (See [4], 2.26 (d).)

(viii) If $A \in \mathscr{U}^c$ is such that $DA = \emptyset$, λ, $\mu \in W_{\vartheta A}$, $\lambda < \mu$ and $x \in A^\mu$ then there are $x' \in A^\lambda$, $n \in N - \{0\}$ such that $f^n(x') = x$. (See [3], 1.16.)

(ix) If $A \in \mathscr{U}^c$ is such that $DA = \emptyset$, λ, $\mu \in W_{\vartheta A}$, $\lambda \leqq \mu$ then $|A^\mu| \leqq |A^\lambda|$. (See [3], 3.3.)

(x) Let $A \in \mathbf{0} - \mathscr{U}^c$ be such that $DA = \emptyset$. Then ϑA is cofinal with ω_0. Further, if $\lambda \in W_{\vartheta A}$ is such that $|A^\lambda| < \aleph_0$ then there is $\mu \in W_{\vartheta A}$ such that $|A^\mu| = 1$. (See [3], 3.5.)

1.13. Lemma. *Let $A \in \mathbf{0} - \mathscr{U}^c$, $DA = \emptyset$ and let $\lambda \in W_{\vartheta A}$ be such that $|A^\lambda| = 1$. If $x \in A^\lambda$ then $SA(f(x)) = \lambda + 1$.*

Proof. $SA(f(x)) \geqq \lambda + 1$ by (v). Let us assume $SA(f(x)) > \lambda + 1$. Thus $A^{\lambda+1} \neq \emptyset$ and $|A^{\lambda+1}| = 1$ by (ix) and if $SA(f(x)) = \mu$ then $|A^\mu| = 1$ by (v) and (ix). Since $\mu > \lambda + 1 > \lambda$, (viii) implies that for $y \in A^{\lambda+1}$ there are $m, n \in N - \{0\}$ such that $f^m(y) = f(x)$ and $f^n(x) = y$. Thus, $f(x) = f^m(y) = f^m(f^n(x)) = f^{m+n-1}(f(x))$ and we have $f(x) \in ZA$ which is a contradiction.

Clearly, the assertions (viii)–(x) and 1.13 can be proved without the condition $DA = \emptyset$.

1.14. Definition. Let $A = (A, f) \in \mathscr{U}^c$, $B = (B, g) \in \mathscr{U}^c$ be arbitrary. Then we put $H(A, B) = \{(x, x') \in A \times B;$ for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$ and $SA(f^n(x)) \leqq SB(g^n(x'))\}$. $(x, x') \in H(A, B)$ is said to form a *pair of h-elements* of A and B. (Compare [4], 3.4.)

1.15. Definition. Let $A, B \in \mathscr{U}^c$ be arbitrary. Then we put $(A, B) \in \mathrm{Ad}$ iff the following conditions hold:

(a) $RB \neq 0$ implies $RB \mid RA$*),

(b) $RB = 0$ implies $H(A, B) \neq \emptyset$.

B is said to be *admissible* for A.

(xi) If $A, B \in \mathscr{U}^c$ are such that $(A, B) \in \mathrm{Ad}$ then $RB = 0$ implies $RA = 0$. (See [5], 1.19 (a).)

(xii) If $A, B \in \mathscr{U}^c$ are arbitrary then $[A, B]_{\mathscr{U}^c} \neq \emptyset$ iff $(A, B) \in \mathrm{Ad}$. (With regard to 1.15 and (xi), see [4], 3.5, 3.11 and 3.14.)

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*) For arbitrary $m, n \in N - \{0\}$, $m \mid n$ means that m is a divisor of n.

(xiii) If $A, B \in \mathscr{U}^c$ are such that $(A, B) \in \mathrm{Ad}$ then $DB \neq \emptyset$ implies $DA \neq \emptyset$. (See [4], 3.6 (a).)

1.16. Definition. We put $\mathcal{O}^i = \{\alpha \in \mathrm{Ord} - \{0\};\ \alpha \text{ a successor ordinal}\}$, $\mathcal{O}^l = \{\alpha \in \mathrm{Ord};\ \alpha \text{ limit and cofinal with } \omega_0\}$. Further, we define the thin categories

(a) $\mathcal{O}$ such that $\mathrm{ob}\,\mathcal{O} = \mathcal{O}^i \cup \mathcal{O}^l$ and, for each $\alpha, \beta \in \mathcal{O}$,

$$[\alpha, \beta]_{\mathcal{O}} \neq \emptyset \quad \text{iff} \quad \alpha \leqq \beta \quad \text{and} \quad \alpha \in \mathcal{O}^l \quad \text{implies} \quad \beta \in \mathcal{O}^l\,;$$

(b) $\mathscr{K}$ such that $\mathrm{ob}\,\mathscr{K} = \{d, \bar{d}\}$ where $d, \bar{d}$ are elements such that $d, \bar{d} \in \mathrm{Ord} - \{0\}$, $d \neq \bar{d}$ and $[d, \bar{d}]_{\mathscr{K}} \neq \emptyset$, $[\bar{d}, d]_{\mathscr{K}} = \emptyset$ (a chain);

(c) $\mathcal{O}^*$ such that $\mathrm{ob}\,\mathcal{O}^* = \mathrm{ob}\,(\mathcal{O} \cup \mathscr{K})$ and, for any $a, b \in \mathcal{O}^*$,

$$[a, b]_{\mathcal{O}^*} \neq \emptyset \quad \text{iff} \quad a, b \in \mathcal{O} \quad \text{and} \quad [a, b]_{\mathcal{O}} \neq \emptyset \quad \text{or} \quad a, b \in \mathscr{K}$$
$$\text{and} \quad [a, b]_{\mathscr{K}} \neq \emptyset \quad \text{or} \quad a \in \mathcal{O}\,, \quad b \in \mathscr{K} \quad \text{and} \quad a \in \mathcal{O}^l \quad \text{implies} \quad b = \bar{d}\,;$$

(d) $\mathscr{N}$ such that $\mathrm{ob}\,\mathscr{N} = N - \{0\}$ and, for each $m, n \in \mathscr{N}$,

$$[m, n]_{\mathscr{N}} \neq \emptyset \quad \text{iff} \quad n \mid m\,;$$

(e) $\mathscr{C} = \mathcal{O}^* \oplus \mathscr{N}$.

1.17. Lemma. (a) *The categories $\mathcal{O}$, $\mathscr{K}$, $\mathcal{O}^*$, $\mathscr{N}$ and $\mathscr{C}$ are ordered classes.*
(b) *$\bar{d}$ is the greatest element in $\mathcal{O}^*$.*
The assertions are easy to prove.

1.18. Definition. Let $\mathscr{C}$ be defined as in 1.16. If $(\mathcal{O}^*, a), (\mathscr{N}, b) \in \mathscr{C}$, where $a \in \mathcal{O}^*$, $b \in \mathscr{N}$, are arbitrary then we put (for abbreviation) $a = (\mathcal{O}^*, a)$, $b = (\mathscr{N}, b)$. Further, for each $a, b \in \mathscr{C}$, we put

$$a \leqq b \quad \text{iff} \quad [a, b]_{\mathscr{C}} \neq \emptyset\,.$$

$\mathscr{C}$ is called the *classifical category* for the category $\mathscr{U}^c$.

1.19. Definition. Let $\mathscr{C}$ be the classifical category for $\mathscr{U}^c$. We define the functor $\chi : \mathscr{U}^c \to \mathscr{C}$ in this way: if $A \in \mathscr{U}^c$ then we put

$$\chi A = \begin{cases} RA & \text{if} \quad A \in \mathbf{2} - \mathscr{U}^c \\ \bar{d} & \text{if} \quad A \in \mathbf{1} - \mathscr{U}^c\,, \quad DA = \emptyset \\ d & \text{if} \quad A \in \mathbf{1} - \mathscr{U}^c\,, \quad DA \neq \emptyset\,. \\ \vartheta A & \text{if} \quad A \in \mathbf{0} - \mathscr{U}^c \end{cases}$$

Then χ is called the *classifical functor* for $\mathscr{U}^c$ and, for arbitrary $A \in \mathscr{U}^c$, χA is the *characterization* of A.

1.20. Definition. Let $\mathscr{C}$ be the classifical category and χ the classifical functor for $\mathscr{U}^c$. If $a \in \mathscr{C}$ is arbitrary then we put $a - \mathscr{U}^c = \{A \in \mathscr{U}^c;\ \chi A = a\}$.

1.21. Lemma. $\mathscr{U}^c = \bigcup_{a \in \mathscr{C}} a - \mathscr{U}^c$, $\mathbf{0} - \mathscr{U}^c = \bigcup_{a \in \mathcal{O}} a - \mathscr{U}^c$, $\mathbf{1} - \mathscr{U}^c = \bigcup_{a \in \mathscr{K}} a - \mathscr{U}^c$, $\mathbf{2} - \mathscr{U}^c = \bigcup_{a \in \mathscr{N}} a - \mathscr{U}^c$ *all with disjoint summands.*

Proof. The assertion follows directly from definitions 1.8, 1.16, 1.19 and 1.20.

1.22. Definition. We put $\aleph - \mathscr{U}^c = \{\boldsymbol{A} = (A, f) \in \mathbf{0} - \mathscr{U}^c;\ \vartheta\boldsymbol{A} \in \mathcal{O}^I$ and $|A^\alpha| \geqq \aleph_0$ for ach $\alpha \in W_{\vartheta\boldsymbol{A}}\}$.

1.23. Lemma. *Let* $\boldsymbol{A} = (A, f)$, $\boldsymbol{B} = (B, g) \in \mathscr{U}^c$ *be such that* $(\boldsymbol{A}, \boldsymbol{B}) \in \mathrm{Ad}$. *Then* $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$.

Proof. (1) Let $\boldsymbol{B} \in \mathbf{2} - \mathscr{U}^c$. Then $\chi\boldsymbol{B} = R\boldsymbol{B} \in \mathscr{N}$. Further, if $\boldsymbol{A} \in \mathbf{0} - \mathscr{U}^c \cup \mathbf{1} - \mathscr{U}^c$ then $\chi\boldsymbol{A} \in \mathcal{O}^*$ and consequently $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$. If $\boldsymbol{A} \in \mathbf{2} - \mathscr{U}^c$ then $\chi\boldsymbol{A} = R\boldsymbol{A} \in \mathscr{N}$ and since $R\boldsymbol{B} \mid R\boldsymbol{A}$ by 1.15 we have $[\chi\boldsymbol{A}, \chi\boldsymbol{B}]_{\mathscr{N}} \neq \emptyset$ which means $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$.

(2) Let $\boldsymbol{B} \in \mathbf{1} - \mathscr{U}^c$. Then $\chi\boldsymbol{B} \in \mathscr{K}$. Further, if $\boldsymbol{A} \in \mathbf{2} - \mathscr{U}^c$ then $\chi\boldsymbol{A} = R\boldsymbol{A} \neq \emptyset$. Let $(x, x') \in A \times B$ be arbitrary. Then, by 1.11, there is $n \in N$ such that $x \in \operatorname{dom} f^n$, $f^n(x) \in Z\boldsymbol{A}$, i.e. $S\boldsymbol{A}(f^n(x)) = \infty_2$. However, for each $y \in [x']_{\boldsymbol{B}}$, we have $S\boldsymbol{B}(y) \in \mathrm{Ord} \cup \{\infty_1\}$ by 1.8. Thus $(x, x') \,\bar{\in}\, H(\boldsymbol{A}, \boldsymbol{B})$ which is a contradiction to $H(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$; this holds by 1.15 because $R\boldsymbol{B} = 0$.

It follows $\boldsymbol{A} \in \mathbf{0} - \mathscr{U}^c \cup \mathbf{1} - \mathscr{U}^c$. Thus, $\chi\boldsymbol{A} \in \mathcal{O}^*$. If $\chi\boldsymbol{B} = \bar{d}$ then $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$ by 1.17 (b). If $\chi\boldsymbol{B} = d$ then $D\boldsymbol{B} \neq \emptyset$ which implies $D\boldsymbol{A} \neq \emptyset$ by (xiii) because $(\boldsymbol{A}, \boldsymbol{B}) \in \mathrm{Ad}$. Thus, we obtain $\chi\boldsymbol{A} \in \mathcal{O}^i \cup \{d\}$ by 1.12 and 1.19 which implies $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$ by 1.16 (c).

(3) Finally, let $\boldsymbol{B} \in \mathbf{0} - \mathscr{U}^c$. Then $\chi\boldsymbol{B} = \vartheta\boldsymbol{B} \in \mathcal{O}$. Further, if $\boldsymbol{A} \in \mathbf{1} - \mathscr{U}^c \cup \mathbf{2} - \mathscr{U}^c$ then $\chi\boldsymbol{A} \in \mathscr{K} \cup \mathscr{N}$; thus, $K\boldsymbol{A} \cup Z\boldsymbol{A} \neq \emptyset$ by 1.8. Let $(x, x') \in A \times B$ be arbitrary. Then by 1.11, there is $n \in N$ such that $x \in \operatorname{dom} f^n$ and $f^n(x) \in K\boldsymbol{A} \cup Z\boldsymbol{A}$, i.e. $S\boldsymbol{A}(f^n(x)) \in \{\infty_1, \infty_2\}$. However, for each $y \in [x']_{\boldsymbol{B}}$, we have $S\boldsymbol{B}(y) \in \mathrm{Ord}$ by 1.8 and thus $(x, x') \,\bar{\in}\, H(\boldsymbol{A}, \boldsymbol{B})$ which is a contradiction to $H(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$ because $R\boldsymbol{B} = 0$.

It follows $\boldsymbol{A} \in \mathbf{0} - \mathscr{U}^c$. Thus $\chi\boldsymbol{A} = \vartheta\boldsymbol{A} \in \mathcal{O}$. Further, $W_{\vartheta\boldsymbol{A}} \subseteq W_{\vartheta\boldsymbol{B}}$.

Indeed, let $\alpha \in W_{\vartheta\boldsymbol{A}}$ be arbitrary. Since $R\boldsymbol{B} = 0$ we have $H(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$. Let $(x, x') \in H(\boldsymbol{A}, \boldsymbol{B})$ be arbitrary. Then for arbitrary $y \in A^\alpha$ there are $m, n \in N$ such that $x \in \operatorname{dom} f^m$, $y \in \operatorname{dom} f^n$ and $f^m(x) = f^n(y)$ by 1.5. Thus $\alpha = S\boldsymbol{A}(y) \leqq S\boldsymbol{A}(f^n(y)) = S\boldsymbol{A}(f^m(x)) \leqq S\boldsymbol{B}(g^m(x')) \in W_{\vartheta\boldsymbol{B}}$ by (v) and 1.14, which implies $\alpha \in W_{\vartheta\boldsymbol{B}}$.

It follows $\vartheta\boldsymbol{A} \leqq \vartheta\boldsymbol{B}$. Further, if $\vartheta\boldsymbol{A} \in \mathcal{O}^I$ then $D\boldsymbol{A} = 0$ by 1.12; thus $D\boldsymbol{B} = 0$ by (xiii) which implies $\vartheta\boldsymbol{B} \in \mathcal{O}^I$ by 1.12. We obtain $[\vartheta\boldsymbol{A}, \vartheta\boldsymbol{B}]_{\mathcal{O}} \neq \emptyset$ by 1.16 (a) and consequently $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$.

1.24. Definition. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathscr{U}^c$ be arbitrary. Then we put

$(\boldsymbol{A}, \boldsymbol{B}) \in \mathrm{Ad}^*$ iff the following conditions hold:

(a) $\chi\boldsymbol{A} \leqq \chi\boldsymbol{B}$,
(b) $\chi\boldsymbol{B} \in \mathcal{O}^I$, $\boldsymbol{A} \in \chi\boldsymbol{B} - \mathscr{U}^c \cap \aleph - \mathscr{U}^c$ implies $H(\boldsymbol{A}, \boldsymbol{B}) \neq \emptyset$.

1.25. Lemma. *Let $A, B \in \mathscr{U}^c$ be such that $(A, B) \in$ Ad. Then $(A, B) \in$ Ad*.*

Proof. The assertion follows directly from 1.15, 1.23 and 1.24 because, for $\chi B \in \mathcal{O}^l$, we have $RB = 0$ which implies $H(A, B) \neq \emptyset$.

1.26. Lemma. *Let $A = (A, f)$, $B = (B, g) \in \mathscr{U}^c$ be such that $(A, B) \in$ Ad*. Then $(A, B) \in$ Ad.*

Proof. (1) Let $RB \neq 0$. Then $\chi B = RB \in \mathscr{N}$. If $A \in \mathbf{0} - \mathscr{U}^c \cup \mathbf{1} - \mathscr{U}^c$ then $RA = 0$ and clearly $RB \mid RA$. If $A \in \mathbf{2} - \mathscr{U}^c$ then $\chi A = RA \in \mathscr{N}$ and since $\chi A \lesssim \chi B$, i.e. $[\chi A, \chi B]_{\mathscr{N}} \neq \emptyset$, we obtain $RB \mid RA$.

(2) Let $RB = 0$. Then $\chi B \in \mathcal{O}^*$ and thus $B \in \mathbf{0} - \mathscr{U}^c \cup \mathbf{1} - \mathscr{U}^c$. Since $\chi A \lesssim \chi B$ we have $\chi A \in \mathcal{O}^*$ and $A \in \mathbf{0} - \mathscr{U}^c \cup \mathbf{1} - \mathscr{U}^c$, too.

Now, let $B \in \mathbf{1} - \mathscr{U}^c$.

If $\chi B = d$ then $DB \neq \emptyset$ which implies $SB(dB) = \infty_1$ by 1.19 and (vii). Further, since $\chi A \lesssim \chi B$ we have $\chi A \in \mathcal{O}^i \cup \{d\}$ by 1.16 (c). Hence $DA \neq \emptyset$ by (xiii) and $SA(dA) \in \text{Ord} \cup \{\infty_1\}$. Thus $(dA, dB) \in H(A, B)$ and we obtain $H(A, B) \neq \emptyset$.

Let $\chi B = \bar{d}$; then $DB = \emptyset$ and thus, for arbitrary $x' \in KB$, we have $x' \in \operatorname{dom} g^n$ for each $n \in N$ by 1.4 (a). Further, $\chi A \in \mathcal{O}^*$ and thus $A = \bigcup_{\alpha \in W_{\vartheta A}} A^\alpha \cup KA$. Let $x \in A$ be arbitrary. Then, for each $y \in [x]_A$, we have $SA(y) \in \text{Ord} \cup \{\infty_1\}$ and thus, for each $n \in N$, $x \in \operatorname{dom} f^n$ implies $x' \in \operatorname{dom} g^n$ nd $SA(f^n(x)) \leqq \infty_1 = SB(g^n(x'))$. Hence $(x, x') \in H(A, B)$ and we obtain $H(A, B) \neq \emptyset$.

Finally, let $B \in \mathbf{0} - \mathscr{U}^c$. Then $\chi B = \vartheta B \in \mathcal{O}$. Since $\chi A \lesssim \chi B$ we have $\chi A = \vartheta A \in \mathcal{O}$, $\vartheta A \leqq \vartheta B$ and conclude that $\vartheta B \in \mathcal{O}^i$ implies $\vartheta A \in \mathcal{O}^i$.

Let $\vartheta B \in \mathcal{O}^i$; then $DB \neq \emptyset$ by 1.12 and since $\vartheta A \in \mathcal{O}^i$ in this case, too, we have also $DA \neq \emptyset$ by 1.12. Further, $SA(dA) = \vartheta A - 1$ and $SB(dB) = \vartheta B - 1$ by (vi) and we obtain $(dA, dB) \in H(A, B)$. Thus $H(A, B) \neq \emptyset$.

Let $\vartheta B \in \mathcal{O}^l$. Then $DB = \emptyset$ by 1.12. Now, if $\vartheta A < \vartheta B$ we take $x' \in B$ such that $SB(x') \geqq \vartheta A$. Then $x' \in \operatorname{dom} g^n$ for each $n \in N$. Let us take $x \in A$ arbitrary; then, for each $n \in N$ such that $x \in \operatorname{dom} f^n$, we have $SA(f^n(x)) \leqq SB(g^n(x'))$. Thus $(x, x') \in H(A, B)$ and we obtain $H(A, B) \neq \emptyset$.

Further, let $\vartheta A = \vartheta B$. Thus $A \in \chi B - \mathscr{U}^c$. If $A \in \aleph - \mathscr{U}^c$ then $H(A, B) \neq \emptyset$ by 1.24. If $A \in \mathbf{0} - \mathscr{U}^c - \aleph - \mathscr{U}^c$ then there is $\lambda \in W_{\vartheta A}$ such that $|A^\lambda| < \aleph_0$. Further, there is $x \in A$ such that, for each $n \in N$, $x \in \operatorname{dom} f^n$ and $SA(f^n(x)) = SA(x) + n$ by (ix), (x) and 1.13, because $DA = 0$. Let $x' \in B$ be such that $SB(x') \geqq SA(x)$. If $n \in N$ is arbitrary, then $x' \in \operatorname{dom} g^n$ by 1.4 (a) and $SB(g^n(x')) \geqq SB(x') + n \geqq SA(x) + n = SA(f^n(x))$ by (v). Thus $H(A, B) \neq \emptyset$ because $(x, x') \in H(A, B)$.

1.27. Theorem. Ad = Ad*.

Proof. The assertion is a consequence of 1.25 and 1.26.

1.28. Theorem. *Let $A, B \in \mathscr{U}^c$ be arbitrary. Then there exists a homomorphism of A into B if and only if $\chi A \leqq \chi B$ and $\chi B \in \mathcal{O}^l$, $A \in \chi B - \mathscr{U}^c \cap \aleph - \mathscr{U}^c$ implies $H(A, B) \neq \emptyset$.*

Proof. The assertion follows from (xii) and 1.27.

1.29. Corollary. (a) *Let $A, B \in \mathscr{U}^c$ be such that $\chi B \in \mathcal{O}^l$ implies $\chi A \neq \chi B$. Then there exists a homomorphism of A into B if and only if $\chi A \leqq \chi B$.*

(b) *Let $A, B \in \mathscr{U}^c$ be such that $\chi B \in \mathcal{O}^l$ implies $A \bar{\in} \aleph - \mathscr{U}^c$. Then there exists a homomorphism of A into B if and only if $\chi A \leqq \chi B$.*

1.30. Corollary. *The functor $\chi : \mathscr{U}^c \to \mathscr{C}$ is covariant.*

Indeed, if $A, B \in \mathscr{U}^c$ are arbitrary then $[A, B]_{\mathscr{U}_c} \neq \emptyset$ implies $\chi A \leqq \chi B$ by 1.28, i.e. $[\chi A, \chi B]_{\mathscr{C}} \neq \emptyset$ and we put $\chi[A, B]_{\mathscr{U}_c} = [\chi A, \chi B]_{\mathscr{C}}$. Since $\mathscr{C}$ is a thin category χ is clearly covariant.

2. THE CATEGORY $\mathscr{U}^c$

2.1. Definition. (a) Let $\mathscr{A}$ be a category. A thin category $\mathscr{A}(b)$ such that ob $\mathscr{A}(b) =$ = ob $\mathscr{A}$ and $[P, Q]_{\mathscr{A}(b)} \neq \emptyset$ iff $[P, Q]_{\mathscr{A}} \neq \emptyset$ for each $P, Q \in \mathscr{A}(b)$ is called a *basic* category for $\mathscr{A}$.

(b) A category $\mathscr{A}$ is called a category *with non-empty homs* if, for each $P, Q \in \mathscr{A}$, $[P, Q]_{\mathscr{A}} \neq \emptyset$.

A basic category $\mathscr{A}(b)$ for $\mathscr{A}$ is a thin category with the same objects and the same existence of morphisms. Thus, in our case, we can describe the category $\mathscr{U}^c$ by means of $\mathscr{U}^c(b)$.

2.2. Lemma. *Let $\mathscr{C}$ and $a - \mathscr{U}^c$ for arbitrary $a \in \mathscr{C}$ be defined as in* 1.18 *and* 1.20. *Then $\mathscr{U}^c(b) \cong \sum_{a \in \mathscr{C}}^l a - \mathscr{U}^c(b)$.*

Proof. Let us put, for arbitrary $A \in \mathscr{U}^c(b)$, $\varphi(A) = (a, A)$ if $A \in a - \mathscr{U}^c(b)$. Then $\varphi : \mathscr{U}^c(b) \to \sum_{a \in \mathscr{C}}^l a - \mathscr{U}^c(b)$ is clearly a bijection. Further, let $A, B \in \mathscr{U}^c(b)$ be arbitrary and let $A \in a - \mathscr{U}^c(b)$, $B \in a' - \mathscr{U}^c(b)$. Thus

$$[\varphi(A), \varphi(B)]_{\Sigma^l} = \begin{cases} [a, a']_{\mathscr{C}} \times [A, B]_{a - \mathscr{U}^c(b)} & \text{if} \quad a = a' \\ [a, a']_{\mathscr{C}} & \text{if} \quad a \neq a' \end{cases}$$

by the definition of $\sum^l$. Since $\chi A = a$, $\chi B = a'$ we obtain $[A, B]_{\mathscr{U}^c(b)} \neq \emptyset$ iff $[\varphi(A), \varphi(B)]_{\Sigma^l} \neq \emptyset$ by 1.29 (a). Thus φ is an isomorphism.

2.3. Corollary.

$$\mathscr{U}^c(b) \cong (\mathbf{0} - \mathscr{U}^c(b) \cup \mathbf{1} - \mathscr{U}^c(b)) \oplus \mathbf{2} - \mathscr{U}^c(b),$$

$$\mathbf{0} - \mathscr{U}^c(b) \cong \sum_{a\in\mathcal{O}}^{l} a - \mathscr{U}^c(b), \quad \mathbf{0} - \mathscr{U}^c(b) \cup \mathbf{1} - \mathscr{U}^c(b) \cong \sum_{a\in\mathcal{O}^*}^{l} a - \mathscr{U}^c(b),$$

$$\mathbf{1} - \mathscr{U}^c(b) \cong d - \mathscr{U}^c(b) \oplus \bar{d} - \mathscr{U}^c(b), \quad \mathbf{2} - \mathscr{U}^c(b) \cong \sum_{a\in\mathscr{N}}^{l} a - \mathscr{U}^c(b).$$

Indeed, the assertion follows from 2.2 and 1.21.

2.4. Lemma. (a) *If* $a \in \mathcal{O}^i \cup \mathscr{K} \cup \mathscr{N}$ *then* $a - \mathscr{U}^c$ *is a category with non-empty homs.*

(b) *If* $a \in \mathcal{O}^l$ *then* $a - \mathscr{U}^c - \aleph - \mathscr{U}^c$ *is a category with non-empty homs.*

Indeed, the assertions follow directly from 1.29 (b).

2.5. Main Theorem. $\mathscr{U}^c(b) \cong \sum_{a\in\mathscr{C}}^{l} a - \mathscr{U}^c(b)$ where $a - \mathscr{U}^c(b)$ *is with non-empty homs for each* $a \in \mathcal{O}^i \cup \mathscr{K} \cup \mathscr{N}$ *and* $a - \mathscr{U}^c(b) - \aleph - \mathscr{U}^c(b)$ *is with non-empty homs for each* $a \in \mathcal{O}^l$.

Theorem 2.5 gives a full simple description of the whole category $\mathscr{U}^c$ except the "internal" description of the subcategories $a - \mathscr{U}^c$ where a is a limit ordinal cofinal with ω_0.

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