Ştefan Frunză

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A CHARACTERIZATION FOR THE SPECTRAL CAPACITY OF A FINITE SYSTEM OF OPERATORS

ŞTEFAN FRUNZĂ, Iaşi

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INTRODUCTION

The recent results of J. L. TAYLOR in the spectral theory for several operators ([10], [11]) allow to obtain an axiomatic theory of spectral decompositions for systems of commuting operators, very similar to that for one operator ([6]).

I have presented in ([9]) some results which seems to be essential for such a generalization. For example we have a uniqueness theorem for the spectral capacity and a characterization of it by means of a local spectrum. The uniqueness theorem may be proved directly ([12]), without to use a local spectrum, but this notion is very useful in problems of quasinilpotent equivalence and commutators. All systems of operators having an appropriate functional calculus ([1]) have also the spectral decompositions in our sense. Let us remark that such a system may have several functional calculi, no one of them valued in bicommutant of the system ([2]); however it has only one spectral capacity.

In this paper we obtain a new characterization of the spectral capacity, by extending to decomposable systems of commuting operators a result for one operator proved in a previous paper ([8]). This characterization has been suggested to me by a paper of E. BISHOP ([4]). Actually I have proved in ([8]) that for a $\mathcal{U}$-scalar operator, the two spectral manifolds of E. Bishop corresponding to any closed set of the complex plane are identical. This special spectral property remains valid also for any decomposable operator ([7]). In what follows we shall give a variant of this result for several operators.

PRELIMINARIES

For the convenience of the reader we shall summarize some definitions and results of general spectral theory in several variables which will be used here.

Consider a complex Banach space X and let $a = (a_1, \ldots, a_k)$ be a system (k-tuple) of mutually commuting linear continuous operators on X (we shall denote the operators by small letters as in ([10]) instead of capital letters as it is in use). The non-singularity of a on X in the sense of J. L. Taylor means the exactness of a certain complex defined conveniently in the terms of a and X; this may be either a chain complex of Koszul type or a cochain complex of exterior forms, the two complexes giving the same notion of non-singularity. The spectrum of a on X, denoted by $\mathrm{sp}(a, X)$, is the set of all $z \in \mathbf{C}^k$ such that the system $z - a = (z_1 - a_1, \ldots, z_k - a_k)$ is singular; the complement $\mathbf{C}^k \setminus \mathrm{sp}(a, X)$ of the spectrum will be called the resolvent set of a on X and will be denoted by $\mathrm{r}(a, X)$.

The cochain complex of exterior forms which defines the non-singularity is obtained as follows. Let $\sigma = (s_1, \ldots, s_k)$ be a system of indeterminates and let us denote by $\Lambda^p[\sigma, X]$, $p \in \mathbf{Z}$, the space of exterior forms of degree p in $s_1, \ldots, s_k$ having the coefficients in X (we put $\Lambda^p[\sigma, X] = 0$ for $p < 0$ or $p > k$ and $\Lambda^0[\sigma, X] = X$). Then the cochain complex is the object consisting of the sequence $(\Lambda^p[\sigma, X])_{p \in \mathbf{Z}}$ and of the left exterior multiplication by $a_1 s_1 + \ldots a_k s_k$ as coboundary operator. Therefore a is said to be non-singular if this complex is exact.

If U is an open set in $\mathbf{C}^k$ then $\mathscr{B}(U, X)$ will stand for the space of all X-valued functions defined on U which are ∞-times continuously differentiable with respect only to $\bar{z}_1, \ldots, \bar{z}_k$ in the sense of distributions; $\mathscr{U}(U, X)$ will stand for the space of all X-valued analytic functions on U endowed with the topology of compact convergence. We shall denote by α the operator on the exterior forms in $s_1, \ldots, s_k$ having the coefficients in $\mathscr{B}(U, X)$ (or $\mathscr{U}(U, X)$) defined by $[\alpha\psi](z) = [(z_1 - a_1) s_1 + \ldots$ $\ldots + (z_k - a_k) s_k] \wedge \psi(z)$, $z \in U$, and by $\alpha \oplus \bar{\partial}$ the operator on the exterior forms in $s_1, \ldots, s_k, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_k$ having the coefficients in $\mathscr{B}(U, X)$ defined by $[(\alpha \oplus \bar{\partial}) \psi] \cdot$ $\cdot (z) = [(z_1 - a_1) s_1 + \ldots + (z_k - a_k) s_k + (\partial/\partial \bar{z}_1) \mathrm{d}\bar{z}_1 + \ldots + (\partial/\partial \bar{z}_k) \mathrm{d}\bar{z}_k] \wedge \psi(z)$, $z \in U$. (the system $(s_1, \ldots, s_k, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_k)$ will be written $\sigma \cup \mathrm{d}\bar{z}$).

It is known that for any open set $U \subset \mathrm{r}(a, X)$ the complex consisting of the sequence $(\Lambda^p[\sigma, \mathscr{B}(U, X)])_{p \in \mathbf{Z}}$ and of α as coboundary operator, is exact ([11], Theorem 2.16). If we replace σ by $\sigma \cup \mathrm{d}\bar{z}$ and α by $\alpha \oplus \bar{\partial}$ then the complex obtained is exact so much the more ([11], Lemma 1.3). If we replace $\mathscr{B}(U, X)$ by $\mathscr{U}(U, X)$ then (for $k > 1$) the complex obtained is not necessarily exact but it is exact for any open polydisc $U \subset \mathrm{r}(a, X)$ ([11], Lemma 2.3).

We shall close this preliminaries by recalling the definition of the Cauchy-Weil integral. Consider an open neighbourhood U of the spectrum $\mathrm{sp}(a, X)$ and an analytic function $f \in \mathscr{U}(U, X)$. The Cauchy-Weil integral of f with respect to a is an element of X obtained as follows. If we regard the form $f s_1 \wedge \ldots \wedge s_k$ as an element of $\Lambda^k[\sigma \cup \mathrm{d}\bar{z}, \mathscr{B}(U, X)]$ then its cohomology class with respect to $\alpha \oplus \bar{\partial}$ contains a form χ with compact support ([11], § 3); it must keep the part of χ containing only $\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_k$, denoted by $\pi\chi$. Then the Cauchy-Weil integral of f, denoted by $\int_U R_{z-a} f(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_k$, is given by: $\int_U R_{z-a} f(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_k =$ $= \int_U (-1)^k \pi \chi(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_k$; it depends only on the cohomology class of χ and it is continuous as a function of $f \in \mathscr{U}(U, X)$. With the preceding results of exactness at hand it is easy to obtain directly such a form χ. Indeed, taking into account that $(\alpha \oplus \bar{\partial}) f s_1 \wedge \ldots \wedge s_k = o$ and denoting $V = U \setminus \mathrm{sp}(a, X)$, we deduce there exists a form $\varphi \in \Lambda^{k-1}[\sigma \cup d\bar{z}, \mathscr{B}(V, X)]$ such that $f s_1 \wedge \ldots \wedge s_k = (\alpha \oplus \bar{\partial}) \varphi$; now we have only to multiply the form φ by a suitable C^∞ scalar function h on U and to define $\chi = f s_1 \wedge \ldots s_k - (\alpha \oplus \bar{\partial}) h\varphi$. Now we can pass to our proper subject.

DEFINITIONS AND SOME AUXILIARY RESULTS

We present firstly a result of general spectral theory which completes Theorem 2.1 from ([9])

Proposition 1. *Let U be an open polydisc containing* $\mathrm{sp}(a, X)$. *Then an element* $f s_1 \wedge \ldots \wedge s_k \in \Lambda^k[\sigma, \mathscr{U}(U, X)]$ *belongs to the range of the operator* $\alpha :$ $: \Lambda^{k-1}[\sigma, \mathscr{U}(U, X)] \to \Lambda^k[\sigma, \mathscr{U}(U, X)]$ *if and only if the Cauchy-Weil integral of f with respect to a is equal to zero.*

Proof. If $f s_1 \wedge \ldots \wedge s_k$ belongs to the range of the operator α then $f s_1 \wedge \ldots$ $\ldots \wedge s_k = \alpha\varphi'$ where $\varphi \in \Lambda^{k-1}[\sigma, \mathscr{U}(U, X)]$ and in the definition of the Cauchy-Weil integral (see Preliminaries) we may take $\chi = o$, whence $\int_U R_{z-a} f(z) \wedge dz_1 \wedge \ldots$ $\ldots dz_k = o$. Conversely suppose that $\int_U R_{z-a} f(z) \wedge dz_1 \wedge \ldots \wedge dz_k = o$. Since $U = U_1 \times \ldots \times U_k$ is a polydisc, by the projection property of the Taylor spectrum ([10], Lemma 3.1), we have $\mathrm{sp}(a_j, X) \subset U_j$, $1 \leqq j \leqq k$. Let Γ_j be a circumference in U_j containing inside $\mathrm{sp}(a_j, X)$, $1 \leqq j \leqq k$. Then we have $\int_U R_{z-a} f(z) \wedge dz_1 \wedge \ldots$ $\ldots \wedge dz_k = \int_{\Gamma_1} \ldots \int_{\Gamma_k} (z_1 - a_1)^{-1} \ldots (z_k - a_k)^{-1} f(z)\, dz_1 \ldots dz_k$. Let us write now the Cauchy integral formula for $f : f(w) = (1/(2\pi i)^k) \int_{\Gamma_1} \ldots \int_{\Gamma_k} (z_1 - w_1)^{-1} \ldots$ $\ldots (z_k - w_k)^{-1} f(z)\, dz_1 \ldots dz_k$, where w_j is inside of Γ_j $(1 \leqq j \leqq k)$. We have the following identity $f(z) = (z_1 - a_1) \ldots (z_k - a_k)(z_1 - a_1)^{-1} \ldots (z_k - a_k)^{-1} f(z) =$ $= [(z_1 - w_1) + (w_1 - a_1)] \ldots [(z_k - w_k) + (w_k - a_k)] (z_1 - a_1)^{-1} \ldots (z_k - a_k)^{-1}$. $. f(z)$. By performing in the last expression the product of the first k factors and passing to the integral we shall obtain terms containing at least a factor $z_j - a_j$, except the term containing $(z_1 - w_1) \ldots (z_k - w_k)$; but, by assumption, the integral corresponding to it is equal to zero. Since the integrals are analytic functions of w when w_j is inside of Γ_j $(1 \leqq j \leqq k)$ and are not modified by the dilatation of the circumferences Γ_j, we shall obtain actually an equality of the form $f(w) = (w_1 - a_1)$. $. g_1(w) + \ldots + (w_k - a_k) g_k(w)$, $w \in U$, where g_j are analytic functions on U, so that the proof is finished.

If we consider on $\Lambda^p[\sigma, \mathscr{U}(U, X)]$ the topology of convergence of the coefficients, we obtain the following

Corollary. *For any open polydisc U containing* $\mathrm{sp}(a, X)$, *the operator α of the preceding proposition has the closed range.*

Proof. Indeed, by Proposition 1, the range of α consists of all forms $f\, s_1 \wedge \ldots \wedge s_k$ having the Cauchy-Weil integral equal to zero. Since this integral is continuous as a function of f, the corollary is proved.

Before to state the following result it must give some definitions. We shall deal with systems of operators having a certain type of spectral decomposition, so-called decomposable systems of operators. For one operator there are two equivalent definitions for this type of decomposition; the one is in the terms of maximal spectral spaces ([5], Chapter 2), the other in the terms of spectral capacities ([3], [6]). The most natural seems to be the second definition which is very similar to that of the spectral operators in the sense of Dunford. The notion of spectral capacity is a broad generalization of that of spectral measure (compare for instance the property of additivity requested for the spectral measure to that for the spectral capacity).

In ([9]) we have given the definition of the decomposable systems of operators in the terms of spectral capacities. Let us denote by $\mathscr{F}(\mathbf{C}^k)$ the family of all closed subsets of $\mathbf{C}^k$ and by $\mathscr{S}(X)$ the family of all linear closed subspaces of X.

Definition 1. *A spectral capacity on* $\mathbf{C}^k$ is an application $\mathscr{E} : \mathscr{F}(\mathbf{C}^k) \to \mathscr{S}(X)$ satisfying the following conditions:

(i) $\mathscr{E}(\emptyset) = \{0\}$, $\mathscr{E}(\mathbf{C}^k) = X$.

(ii) $\mathscr{E}(\bigcap_{n=1}^{\infty} F_n) = \bigcap_{n=1}^{\infty} \mathscr{E}(F_n)$ for any sequence $(F_n)_{n\in N} \subset \mathscr{F}(\mathbf{C}^k)$.

(iii) For any open covering $\{U_j\}_{j=1}^m$ of $\mathbf{C}^k$, we have $X = \sum_{j=1}^{m} \mathscr{E}(\overline{U}_j)$ (i.e. any element $x \in X$ may be written, not necessarily in a unique manner, in the form $x = \sum_{j=1}^{m} x_j$, $x_j \in \mathscr{E}(\overline{U}_j)$).

Definition 2. A system $a = (a_1, \ldots, a_k)$ of mutually commuting operators is said to be *decomposable* if there exists a spectral capacity $\mathscr{E}$ on $\mathbf{C}^k$ such that:

(iv) $a_j \mathscr{E}(F) \subseteq \mathscr{E}(F)$, $1 \leqq j \leqq k$ and

(v) $\mathrm{sp}(a, \mathscr{E}(F)) \subseteq F$ for any set $F \in \mathscr{F}(\mathbf{C}^k)$.

Lemma 1. *If a is a decomposable system then for any set $F \in \mathscr{F}(\mathbf{C}^k)$, we have* $\mathrm{sp}(a, X/\mathscr{E}(F)) \subseteq \mathbf{C}^k \setminus F^\circ$.

Proof. For every $z^\circ \in F^\circ$ let us take an open set U such that F° and U cover $\mathbf{C}^k$ and $z^\circ \notin \overline{U}$. Then we have by (iii), $X = \mathscr{E}(F) + \mathscr{E}(\overline{U})$. By a well known isomorphism theorem we deduce $X/\mathscr{E}(F)$ is isomorphic to $\mathscr{E}(\overline{U})/\mathscr{E}(F) \cap \mathscr{E}(\overline{U}) = \mathscr{E}(\overline{U})/\mathscr{E}(F \cap \overline{U})$; on account of a result of Taylor on the spectral inclusions ([10], Lemma 1.2 or [11], Lemma 1.5), we have $\mathrm{sp}(a, \mathscr{E}(\overline{U})/\mathscr{E}(F \cap \overline{U})) \subseteq \overline{U}$ and therefore $z^\circ \notin \mathrm{sp}(a, X/\mathscr{E}(F))$, as desired.

In the following proposition we show that the decomposable systems of operators have a property similar to the property β of E. Bishop ([4]).

Proposition 2. *Let U be an open polydisc in $\mathbf{C}^k$ and $(\psi_n) \subset \Lambda^{k-1}[\sigma, \mathscr{U}(U, X)]$ be a sequence such that for $n \to \infty$, $\alpha\psi_n \to o$, α being defined by the system $a = (a_1, \ldots, a_k)$. If a is decomposable then for every open relatively compact polydisc D, $\bar{D} \subset U$, there exists a sequence $(\varphi_n) \subset \Lambda^{k-1}[\sigma, \mathscr{U}(D, X)]$ such that $\alpha\psi_n = \alpha\varphi_n$ and for $n \to \infty$, $\varphi_n \to o$.*

Proof. Consider two open polydiscs D_1 and D_2 such that $D \subset \bar{D} \subset D_2 \subset \bar{D}_2 \subset D_1 \subset \bar{D}_1 \subset U$. Denoting $X_1 = \mathscr{E}(\bar{D}_1)$, $X_2 = \mathscr{E}(\mathbf{C}^k \setminus D_2)$ we have by (iii), $X = X_1 + X_2$, whence the following short sequence $0 \to X_1 \cap X_2 \to X_1 \oplus X_2 \to X \to 0$ is exact (the first application is given by $x \to x \oplus (-x)$ and the second by $x_1 \oplus x_2 \to x_1 + x_2$). Since U is a polydisc, on account of a result of exactness from ([10], Theorem 2.2), the sequence $0 \to \mathscr{U}(U, X_1 \cap X_2) \to \mathscr{U}(U, X_1) \oplus \mathscr{U}(U, X_2) \to (U, X) \to 0$ is also exact. By using the exactness in the last step (i.e. the surjectivity) we obtain there exist the sequences $(\psi'_n) \subset \Lambda^{k-1}[\sigma, \mathscr{U}(U, X_1)]$, $(\psi''_n) \subset \Lambda^{k-1}[\sigma, \mathscr{U}(U, X_2)]$ such that $\psi_n = \psi'_n + \psi''_n$. Consider now two other open polydiscs D'_1, D'_2 such that $\bar{D} \subset D'_2 \subset \bar{D}'_2 \subset D_2$ and $\bar{D}_1 \subset D'_1 \subset \bar{D}'_1 \subset U$ and let us denote $Y = X/\mathscr{E}(\bar{D}'_1 \setminus D'_2)$. By applying Lemma 1 we obtain $\mathrm{sp}(a, Y) \subset \bar{D}'_2 \cup (\mathbf{C}^k \setminus D'_1)$. Therefore $\mathrm{sp}(a, Y)$ has two separated parts and we may apply the variant given by J. L. Taylor for the theorem of idempotents; thus there exist two linear continuous projections p and q such that $p + q = 1$ and $\mathrm{sp}(a, pY) \subset \bar{D}'_2$, $\mathrm{sp}(a, qY) \subset \mathbf{C}^k \setminus D'_1$ ([11], Theorem 4.9). We shall prove that $p\, \widehat{\psi_n(z)} = \widehat{\psi'_n(z)}$ and $q\, \widehat{\psi_n(z)} = \widehat{\psi''_n(z)}$, $z \in U$, where $x \to \hat{x}$ stands for the quotient natural map of X on Y. It will be enough for this to prove that $p\hat{x} = o$ for any $x \in X_2$ and $q\hat{x} = o$ for any $x \in X_1$. The two equalities follow directly by applying the definition of the Cauchy-Weil integral (see the preliminaries). Let x be an element of X_2; since $D_2 \subset \mathrm{r}(a, X_2)$ $(X_2 = \mathscr{E}(\mathbf{C}^k \setminus D_2))$ there exists a form $h \in \Lambda^{k-1}[\sigma \cup d\bar{z}, \mathscr{B}(D_2, X)]$ such that $x\, s_1 \wedge \ldots \wedge s_k = (\alpha \oplus \bar{\partial})\, h$, whence $p\, \hat{x}\, s_1 \wedge \ldots \wedge s_k = (\alpha \oplus \bar{\partial})\, p\hat{h}$, therefore, on account of the inclusion $\mathrm{sp}(a, pY) \subset D'_2$ we deduce $p\hat{x} = o$. Analogously one proves the other statement. By using now the equalities from above and the assumption of our proposition we deduce for $n \to \infty$, $\alpha\hat{\psi}'_n \to o$ and $\alpha\hat{\psi}''_n \to o$. Since D is an open polydisc contained in $\mathrm{r}(a, qY)$ and $\hat{\psi}''_n \in \Lambda^{k-1}[\sigma, \mathscr{U}(D, qY)]$, by the exactness result recalled in preliminaries, there exists the sequence $(\chi''_n) \subset \Lambda^{k-1}[\sigma, \mathscr{U}(D, qY)]$ such that for $n \to \infty$, $\chi''_n \to o$ on D and $\alpha\hat{\psi}''_n = \alpha\chi''_n$ on D. If we use for (ψ'_n) the Proposition 1, we obtain analogously that there exists a sequence $(\chi'_n) \subset \Lambda^{k-1}[\sigma, \mathscr{U}(U, pY)]$ with similar properties. Thus denoting $\chi'_n + \chi''_n$ by χ^*_n we have $\alpha\hat{\psi}_n = \alpha\chi^*_n$ and for $n \to \infty$, $\chi^*_n \to o$ on D. Let us use now the exactness of the sequence $0 \to \mathscr{U}(D, Y) \to \mathscr{U}(D, X) \to \mathscr{U}(D, X/Y) \to 0$; we obtain that $\chi^*_n = \hat{\chi}_n$ where for $n \to \infty$, $\chi_n \to o$ on D. The situation is now clear; we have $\alpha(\psi_n - \chi_n) \in \Lambda^k[\sigma, \mathscr{U}(D, \mathscr{E}(\bar{D}'_1 \setminus D'_2))]$ and for $n \to \infty$, $\alpha(\psi_n - \chi_n) \to o$; since D is disjoint from $\bar{D}'_1 \setminus D'_2$ we deduce $\alpha(\psi_n - \chi_n) = \alpha\eta_n$ where for $n \to \infty$, $\eta_n \to o$ on D. Denoting $\chi_n + \eta_n$ by φ_n we have $\alpha\psi_n = \alpha\varphi_n$, $\varphi_n \to o$ for $n \to \infty$ and thus the proof is finished.

Theorem 1. *If a is a decomposable system then, for any open polydisc U, the operator $\alpha : \Lambda^{k-1}[\sigma, \mathscr{U}(U, X)] \to \Lambda^k[\sigma, \mathscr{U}(U, X)]$ has the closed range.*

Proof. Since $\Lambda^{k-1} = \Lambda^{k-1}[\sigma, \mathscr{U}(U, X)]$ is a Fréchet space and $\ker \alpha$ is a closed subspace, the quotient space $\Lambda^{k-1}/\ker \alpha$ is a Fréchet space. Let us denote $\psi \to \hat{\psi}$ the natural quotient map $\Lambda^{k-1} \to \Lambda^{k-1}/\ker \alpha$, and by $\hat{\alpha} : \Lambda^{k-1}/\ker \alpha \to \Lambda^k$, the operator coinduced by α on $\Lambda^{k-1}/\ker \alpha$, $\hat{\alpha}\hat{\psi} = \alpha\psi$. Obviously, the range of α is equal to the range of $\hat{\alpha}$ and it is enough to prove that $\hat{\alpha}\hat{\psi}_n \to o$ implies $\hat{\psi}_n \to o$. Indeed, if this last property is satisfied then $\hat{\alpha}$ has the continuous inverse and therefore is a linear topological isomorphism. Let $(\hat{\psi}_n) \subset \Lambda^{k-1}/\ker \alpha$ be a sequence such that $\hat{\alpha}\hat{\psi}_n \to o$ and let us prove that $\hat{\psi}_n \to o$. Let $K \subset U$ be an arbitrary compact set and D be an open relatively compact polydisc such that $K \subset D \subset \bar{D} \subset U$. Since $\alpha\psi_n = \hat{\alpha}\hat{\psi}_n \to o$, by applying to (ψ_n) the Proposition 2 it follows the existence of a sequence $(\varphi_n) \subset \subset \Lambda^{k-1}[\sigma, \mathscr{U}(D, X)]$ such that $\alpha\psi_n = \alpha\varphi_n$ and $\varphi_n \to o$ on D. Since a is decomposable, the sequence $\Lambda^{k-2}[\sigma, \mathscr{U}(D, X)] \xrightarrow{\alpha} \Lambda^{k-1}[\sigma, \mathscr{U}(D, X)] \xrightarrow{\alpha} \Lambda^k[\sigma, \mathscr{U}(D, X)]$ is exact ([9], Theorem 3.1); consequently, taking into account that $\alpha(\psi_n - \varphi_n) = o$ on D, there exists a sequence $(\chi_n) \subset \Lambda^{k-2}[\sigma, \mathscr{U}(D, X)]$ such that $\psi_n = \varphi_n + \alpha\chi_n$. By using, for any n, the Taylor expansion of χ_n, we may write $\chi_n = \chi_n' + \chi_n''$ where the coefficients of χ_n' are polynomials and for $n \to \infty$, $\chi_n'' \to o$ uniformly on K. Thus for any n, $\psi_n - \alpha\chi_n'' = \varphi_n + \alpha\chi_n''$ on D, the left hand being defined on U. Therefore we have obtained the sequence $(\psi_n - \alpha\chi_n') \subset \Lambda^{k-1}[\sigma, \mathscr{U}(U, X)]$ satisfying the following properties: $\alpha\psi_n = \alpha(\psi_n - \alpha\chi_n')$, $\psi_n - \alpha\chi_n' \to o$ uniformly on K. This fact means just that $\hat{\psi}_n \to o$ in the seminorm of $\Lambda^{k-1}/\ker \alpha$ corresponding to the compact set K; this compact set being chosen arbitrarily, we have $\hat{\psi}_n \to o$ in $\Lambda^{k-1}/\ker \alpha$ and the proof is finished.

The last theorem contains the characterization of the spectral capacity announced in the title. To state it let us make a convention of language.

We say that an element $x \in X$ may be uniformly approximated locally on an open set $G \subset \mathbf{C}^k$ by the functions of the form $\sum_{i=1}^{k} (z_i - a_i) f_i(z)$, if for any point $z^\circ \in G$ there exists an open relatively compact polydisc D, $z^\circ \in D \subset \bar{D} \subset G$ satisfying the following condition: for every $\varepsilon > o$ there exist n X-valued analytic functions $f_{1,\varepsilon}, \ldots, f_{k,\varepsilon} : D \to X$ such that $\|\sum_{i=1}^{k} (z_i - a_i) f_{i,\varepsilon}(z) - x\| < \varepsilon$.

Theorem 2. *Let $a = (a_1, \ldots, a_k)$ be a decomposable system of operators and $\mathscr{E}$ be its spectral capacity. Then for any closed set $F \subset \mathbf{C}^k$, $\mathscr{E}(F)$ consists of all elements $x \in X$ which may be uniformly approximated locally on $\mathbf{C}^k \setminus F$ by the functions of the form $\sum_{i=1}^{k} (z_i - a_i) f_i(z)$.*

Proof. By ([9], Theorem 3.2) $\mathscr{E}(F)$ consists of all elements $x \in X$ which may be represented locally on $\mathbf{C}^k \setminus F$ in the form $x = \sum_{i=1}^{k} (z_i - a_i) f_i(z)$, where f_i are analytic X-valued functions, $1 \leqq i \leqq k$. Therefore $\mathscr{E}(F)$ is the set of all elements $x \in X$ such that $x\, s_1 \wedge \ldots \wedge s_k$ belongs locally on $\mathbf{C}^k \setminus F$ to the range of α. By the preceding theorem this set is equal to the set described in the enunciation, as desired.

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Author's address: University of Iaşi, Mathematics Faculty, Iaşi, Socialist Republic of Romania.