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A REALIZATION OF D-GROUPS

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T. NAKANO [6] introduced a ring-like system called an m-ring, which differs from the usual concept of rings by aditting a multivalued addition. In many cases, the results concerning m-rings can be applied to the theory of rings and lattice ordered groups.

The topic of this paper is an application of ideal-theoretic methods to the theory of m-rings and d-groups. The main result is a theorem about a realization of a d-group as a subdirect product of simply ordered d-groups. Since any l-group is a d-group, the theorem of Lorenzen (in commutative case) can be derived from our result.

Finally, in Section 4 we give a new proof of a conjecture about l-groups presented by W. KRULL. This proof is based on an approximation theorem for d-groups.

1. INTRODUCTION

In order to make this paper self-contained we repeat some basic facts about d-groups (see [6]).

By an *addition* in a set M, we mean a multivalued function assigning to every ordered pair of elements $(a, b) \in M^2$ a no-void subset $a \oplus b$ of M which satisfies the following axioms:

(i) $a \oplus b = b \oplus a$;

(ii) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, where $N \oplus K = \bigcup_{a \in N, b \in K} (a \oplus b)$, for any N, $K \subseteq M$;

(iii) $a \in b \oplus c$ implies $b \in a \oplus c$

for any $a, b, c \in M$.

An *m-ring* is a commutative semigroup $(M, \cdot)$ which admits an addition $\oplus$ and satisfies

(iv) $a(b \oplus c) = ab \oplus ac$.

In this paper all m-rings are required to obey the cancellation law and the existence of identity element. A *d-group* is a partially ordered group G which admits an addition $\oplus$ such that $(G, \cdot, \oplus)$ is an m-ring and satisfies

(v) $a, b \geqq c$ and $x \in a \oplus b$ imply $x \geqq c$.

Throughout this paper we denote by $U(X)$ the group of units of a semigroup X.

Let $(A, \cdot, \oplus)$ be an m-ring and let $Q(A)$ be the quotient group of the semigroup A. It can be easily verified that the addition $\oplus$ in A can be extended to the addition in $Q(A)$. Then the factor group $D(A) = Q(A)/U(A)$ can be partially ordered by division with respect to a semigroup A and becomes a d-group with respect to the addition

$$(a\, U(A))\,(b\, U(A))^{-1} \oplus' (c\, U(A))\,(d\, U(A))^{-1} =$$
$$= ((ad\, U(A) \oplus (cb\, U(A))\,(bd\, U(A))^{-1} .$$

$D(A)$ is then called a *d-group relative* to an m-ring A.

In what follows we assume that any m-ring contains an element 0 such that $0 \in a \oplus b$ if and only if $a = b$. The element 0 is then uniquely determined.

An m-ring R is called *local* provided that a sum of non-units does not contain a unit. This is equivalent to the assertion that the d-group $D(R)$ satisfies

(vi) $a > b$ implies $a \oplus b = \{b\}$.

A subset J of an m-ring A is called an m-ideal of A, if $a \oplus b \subseteq J$, $ar \in J$ for any $a, b \in J$, $r \in A$. An m-ideal J is called prime if J satisfies the condition

$$ab \in J \quad \text{implies} \quad a \in J \quad \text{or} \quad b \in J$$

for any $a, b \in A$.

It is easy to see that every maximal m-ideal is prime. Now for any m-ring A and a prime m-ideal P of A it is easy to see that

$$A_P = \{ab^{-1} : a \in A,\ b \in A - P\}$$

is a local m-ring with the maximal m-ideal

$$PA_P = \{ab^{-1} : a \in P,\ b \in A - P\} .$$

This m-ring is called an *m-ring of quotients* with respect to the multiplicative system $A - P$.

An m-ring R is called a *valuation m-ring* provided that the d-group $D(R)$ is simply ordered. Any valuation m-ring is local.

An element p of a d-group G is called *integral over* an m-subring I of G, if there exist elements $a_0, \ldots, a_n \in I$, $n \geqq 0$ such that

$$p^{n+1} \in a_n p^n \oplus \ldots \oplus a_0 .$$

An m-ring A is called *integrally closed* in a d-group G provided that every element of G integral over A is contained in A.

If H is a subgroup of a d-group G, then the factor group G/H becomes a d-group with respect to the addition

$$aH \oplus' bH = (aH \oplus bH)/H ,$$

where $\oplus$ is the addition in G, and the order relation

$$aH \geqq bH \Leftrightarrow a' \geqq b' \quad \text{for some} \quad a' \in aH , \quad b' \in bH$$

if and only if H is a *d-convex subgroup*, i.e. if it is a convex subgroup and $G_+H \oplus \oplus G_+H = G_+H$, where

$$G_+ = \{g \in G : g \geqq 1\} .$$

A d-convex subgroup H is called *prime* if the factor d-group G/H is local, i.e. $(G/H)_+$ is a local m-ring.

2. REALIZATION OF D-GROUPS

In this section we shall prove a theorem about the embedding of a d-group G into a direct product of simply ordered d-groups in such a way that a group G is a subditect product of these d-groups.

First, we shall prove several lemmas.

Lemma 1. *An m-ring A is a valuation m-ring if and only if there exist a simply ordered d-group G and a mapping $w : Q(A) \to G$ which satisfy the following conditions*:

(1) $w(xy) = w(x)\, w(y)$;
(2) $w(x \oplus y) = w(x) \oplus w(y)$;
(3) $A = w^{-1}(G_+)$

for any $x, y \in Q(A)$. A map w is then called an m-valuation associated with A.

Proof. Assume that A is a valuation m-ring and let $w : Q(A) \to D(A)$ be the canonical mapping. Then we have $w(x \oplus y) = w(x \oplus y)\; 1_{D(A)} = w(x \oplus y)\, w(U(A)) = (x\, U(A) \oplus y\, U(A))/U(A) = w(x) \oplus w(y)$. It can be easily verified that w satisfies the other conditions.

Conversely, assume that $w : Q(A) \to G$ has the desired properties. Then the mapping $f : g\, U(A) \mapsto w(g)$ defines an order isomorphism of $D(A)$ onto G. Therefore, $D(A)$ is simply ordered and A is a valuation m-ring.

Remark. If w is an m-valuation associated with a valuation m-ring A, then the following conditions hold:

(4) $x \in a \oplus b$ implies $w(x) \geqq \min \{w(a), w(b)\}$;

(5) $x \in a \oplus b$, $w(a) \neq w(b)$ imply $w(x) = \min \{w(a), w(b)\}$.

In fact, the condition (4) follows directly from the definition of a d-group. Now assume that $x \in a \oplus b$ and $w(a) < w(b)$. Since $a \in x \oplus b$, we have $w(x) = w(a)$ by (4).

Lemma 2. *Let G be a d-group and let I be an m-ideal of G_+. Then there exists a valuation m-ring R such that $I \subseteq M(R)$ and $G_+ \subseteq R \subset G$, where $M(X)$ is the maximal m-ideal of a local m-ring X.*

Proof. Put

$$\mathscr{A} = \{R' : R' \text{ is an m-ring, } IR' \neq R',\ G_+ \subseteq R' \subset G\}\ .$$

It is clear that $(\mathscr{A}, \subseteq)$ satisfies the conditions of Zorn's lemma; hence there exists a maximal element R of $\mathscr{A}$. If we suppose that R is not local, there exists a maximal m-ideal M of R such that $IR \subseteq M \neq R$ and $R \subset R_M$, where R_M is the m-ring of quotients with respect to the multiplicative system $R - M$. This contradicts the assumption on R. Hence R is a local m-ring.

Let $q \in G$ be the element which is not integral over R. By [7]; Lemma 3 there exists a local m-ring R_q which contains q^{-1} and R but not q and all the inverses of non-units of R. Especially, $IR_q \neq R_q$. Thus we have $R = R_q$. By [7]; Lemma 4 the integral closure R' of R is a valuation m-ring.

Suppose that $IR' = R'$. Thus we have $1 = ub$ for some $u \in I$, $b \in R'$. Hence

$$(1) \qquad w(u) > 1\ ,$$

$$b^{n+1} \in a_n b^n \oplus \ldots \oplus a_0\ ,$$

where the coefficients belong to R and w is the canonical mapping $G \to D(R)$. (Evidently, w satisfies the conditions (1), (2) of Lemma 1.) Let the number n be the smallest that satisfies the above relation. Then

$$1 = u^{n+1} b^{n+1} \in u^{n+1} a_n b^n \oplus \ldots \oplus u^{n+1} a_0\ ;$$

$$w(u^{n+1} a_i) \geqq w(u^{n+1}) > 1\ .$$

Thus

$$1 \in w(c_n)\, w(b)^n \oplus \ldots \oplus w(c_0)\ ,$$

where $w(c_i) > 1$. Since R is local, the rule 8); [7] implies

$$1 \in w(c_n)\, w(b)^n \oplus \ldots \oplus w(c_1)\, w(b)\ .$$

Multiplying by $w(b)^n$ on both sides we obtain

$$w(b)^n \in w(c_n)\, w(b)^{2n} \oplus \ldots \oplus w(c_1)\, w(b)^{n+1} .$$

Using (1) repeatedly, we have

$$w(b)^n \in w(d_n)\, w(b)^n \oplus \ldots \oplus w(d_0) ,$$

where $w(d_n b^n) > w(b^n)$. Again, we have

$$w(b)^n \in w(d_{n-1})\, w(b)^{n-1} \oplus \ldots \oplus w(d_0)$$

by 8); [7]. Thus there exist $p \in d_{n-1} b^{n-1} \oplus \ldots \oplus d_0$ and $j \in U(R)$ such that $b^n = pj$; hence we get

$$b^n \in (d_{n-1} j)\, b^{n-1} \oplus \ldots \oplus (jd_0) .$$

This contradicts the assumption on n. Therefore, $IR' \neq R'$ and we have $R' = R$. Especially, $I \subseteq IR \subseteq M(R)$.

Proposition 3. *Let G be a d-group and let P be a prime m-ideal of G_+. Then there exists a valuation m-ring R such that $G_+ \subseteq R \subset G$ and $M(R) \cap G_+ = P$.*

Proof. It is easy to see that $P(G_+)_P = \{pq^{-1} : p \in P, q \in G_+ - P\}$ is the maximal m-ideal of the m-ring of quotients $(G_+)_P$. By Lemma 2 there exists a valuation m-ring R such that $(G_+)_P \subseteq R$ and $P(G_+)_P \subseteq M(R)$. Thus $P(G_+)_P = M(R) \cap (G_+)_P$ and we have $P = P(G_+)_P \cap G_+ = M(R) \cap G_+$.

In what follows, G^* will denote the core of an ordered group G, i.e. $G^* = \{gh^{-1} : g, h \in G_+\}$.

Lemma 4. *Let G be a d-group and let H be a prime d-convex subgroup of G. Then there exists a valuation m-ring R such that $G_+ \subseteq R \subset G$ and $H^* = U(R)^*$.*

Proof. Setting

$$P = G_+ - (H \cap G_+) = G_+ - H_+$$

we shall prove that P is a prime m-ideal of G_+. In fact, assume that $x, y \in P$. Then $x, y \notin H$ and since H is prime, we have $x \oplus y \subseteq P$ by [6]; Lemma 6. Now assume that $x \in G_+$ and $y \in P$. Thus $xy \geqq y \geqq 1$ and since H is convex, we have $xy \in P$. The condition that P is a prime m-ideal can be verified easily. By Proposition 3 there exists a valuation m-ring R such that $G_+ \subseteq R \subset G$ and $M(R) \cap G_+ = P$. Hence $U(R)_+ = U(R) \cap G_+ = (R - M(R)) \cap G_+ = G_+ - P = H_+$. Therefore, we have $U(R)^* = H^*$.

A valuation m-ring R such that $G_+ \subseteq R \subset G$ for a d-group G is called *well centred* on G provided that $R = G_+ U(R)$.

Lemma 5. *A valuation m-ring R is well centred on G if and only if for any $\boldsymbol{q} \in D(R)_+$ there exists $q \in G_+$ such that $w(q) = \boldsymbol{q}$, where w is the m-valuation associated with R.*

Proof. Let R be well centred on G and assume that $\boldsymbol{q} \in D(R)_+$. Then $\boldsymbol{q} = r\,U(R)$ for some $r \in R$. Hence we have $r = uq$, where $u \in U(R)$, $q \in G_+$. Therefore, $\boldsymbol{q} = w(r) = w(q)$. The converse is trivial.

Lemma 6. *Let G be a d-group and let A be an m-ring such that $G_+ \subseteq A \subset G$. If $D(A)_+$ is integrally closed in $D(A)$, then A is integrally closed in G.*

Proof. Assume that x is an element of G such that

$$x^{n+1} \in a_n x^n \oplus \ldots \oplus a_0\,,$$

where the coefficients belong to A, and let $w : G \to D(A)$ be the canonical homomorphism. Then

$$w(x)^{n+1} \in w(a_n)\, w(x)^n \oplus \ldots \oplus w(a_0)$$

and we obtain $w(x) \in D(A)_+$. Therefore, there exist elements $g \in A$, $j \in U(A)$ such that $x = gj \in A\,U(A) \subseteq A$. Hence A is integrally closed in G.

Lemma 7. *Let G be a d-group and let A be an m-ring such that $G_+ \subseteq A \subset G$. Assume that a group $U(R)$ is directed for every valuation m-ring R, $G_+ \subseteq R \subset G$. Then the group $U(\boldsymbol{R})$ is directed for every valuation m-ring $\boldsymbol{R}$ such that $D(A)_+ \subseteq \boldsymbol{R} \subset D(A)$.*

Proof. Assume that $\boldsymbol{R}$ is a valuation m-ring such that $D(A)_+ \subseteq \boldsymbol{R} \subset D(A)$. Setting

$$R = \{g \in G : g\,U(A) \in \boldsymbol{R}\}$$

we shall show that R is a valuation m-ring in G containing G_+. In fact, it suffices to prove that R is closed under the addition. But we have $(a \oplus b)\,U(A) \subseteq \{c\,U(A) : c \in a' \oplus b',\ a' \in a\,U(A),\ b' \in U(A)\} = a\,U(A) \oplus b\,U(A)$. It is clear that $U(\boldsymbol{R}) = \{j\,U(A) : j \in U(R)\}$. Now assume that $\boldsymbol{a}, \boldsymbol{b} \in U(\boldsymbol{R})$; hence $\boldsymbol{a} = a\,U(A)$, $\boldsymbol{b} = b\,U(A)$ for some $a, b \in U(R)$. Since $U(R)$ is directed (in G), there exists an element $c \in U(R)$ such that $a, b \leqq c$. Since $G_+ \subseteq A$, we have $a^{-1}c, b^{-1}c \in A$. Therefore $\boldsymbol{a} \leqq c\,U(A)$, $\boldsymbol{b} \leqq c\,U(A)$ in $D(A)$ and $U(\boldsymbol{R})$ is directed.

An m-ring A is called a *Prüfer m-ring* provided that an m-ring of quotients A_P with respect to every prime m-ideal P of A is a valuation m-ring.

Theorem 8. *Let G be a directed d-group and suppose that $U(R)$ is directed for every valuation m-ring R such that $G_+ \subseteq R \subset G$. Then the following conditions are equivalent:*

(1) *A factor d-group G/H is simply ordered for every prime d-convex subgroup H of G.*

(2) *G_+ is a Prüfer m-ring.*

(3) *Every m-ring A such that $G_+ \subseteq A \subset G$ is integrally closed in G.*

(4) *Every valuation m-ring R such that $G_+ \subseteq R \subset G$ and whose group of units is a prime d-convex subgroup of G is well centred on G.*

(5) *Every valuation m-ring R such that $G_+ \subseteq R \subset G$ is well centred on G.*

Proof. (4) $\Rightarrow$ (1). We denote by $\mathfrak{M}$ the set of prime d-convex subgroups of G. Assume that $H \in \mathfrak{M}$. By Lemma 4 there exists a valuation m-ring R such that $G_+ \subseteq R \subset G$ and $U(R)^* = H^*$. Now, since $U(R)$ is directed, we have $U(R)^* = U(R)$ and by [6]; Lemma 6 we obtain that $U(R)$ is a prime d-convex subgroup of G. Hence R is well centred on G. Moreover, on the set $G/U(R) = \{g\, U(R) : g \in G\}$ we can define two order relations. First, $G/U(R)$ can be ordered as the d-group relative to R; second, $G/U(R)$ can be ordered as the factor d-group. Hence

$$x\, U(R) \leqq y\, U(R) \Leftrightarrow w(x) \leqq w(y)\,,$$

$$x\, U(R) \lesssim y\, U(R) \Leftrightarrow x' \leqq y' \quad \text{in} \quad G \quad \text{for some} \quad x' \in x\, U(R)\,, \quad y' \in y\, U(R)\,,$$

where w is the m-valuation associated with R. We shall prove that these order relations are identical. In fact, assume that $x\, U(R) \lesssim y\, U(R)$. This means that $xi \leqq yj$ in G for some $i, j \in U(R)$. Thus there exists $g \in G_+$ such that $x^{-1}y = gij^{-1} \in G_+\, U(R) \subseteq R$. Hence $w(x) \leqq w(y)$. Conversely, assume that $w(x) \leqq w(y)$. Then $y = xr$ for some $r \in R = G_+\, U(R)$; hence there exist $j \in U(R)$ and $g \geqq 1$ such that $r = gj$, $y = jgx \geqq jx$. Therefore $y\, U(R) \gtrsim x\, U(R)$.

Now, since R is a valuation m-ring, the d-group $D(R) = (G/H^*, \leqq)$ is simply ordred. Hence we obtain that a factor d-group G/H^* is simply ordered. But, since $H^* \subseteq H$, the d-group G/H is simply ordered.

(1) $\Rightarrow$ (2). Let P be a prime m-ideal of G_+ and let H be the convex closure of a group generated by $G_+ - P$ in G. Since H is directed, it is a d-convex subgroup of G by [6]; Lemma 5. Now one may easily verify that $D((G_+)_P)$ is isomorphic to the factor d-group G/H. Since $(G_+)_P$ is local, H is prime. Hence G/H is simply ordered and we have that $(G_+)_P$ is a valuation m-ring. Therefore G_+ is a Prüfer m-ring

(2) $\Rightarrow$ (5). Suppose that G_+ is a Prüfer m-ring and let R be a valuation m-ring such that $G_+ \subseteq R \subset G$. Put

$$P = M(R) \cap G_+\,.$$

Then we have $(G_+)_P \subseteq R$ and if we assume that $x \in R - (G_+)_P$, we get $x^{-1} \in (G_+)_P \subsetneqq \subseteq R$. Hence x^{-1} is a unit in R, $x^{-1} \notin M(R)$. Thus

$$x^{-1} \in (G_+)_P - (M(R) \cap (G_+)_P),$$

so that

$$x = (x^{-1})^{-1} \in ((G_+)_P)_{[(G_+)_P - M(R) \cap (G_+)_P]} = (G_+)_P$$

and we obtain $(G_+)_P = R$. Finally, assume that $\boldsymbol{a} \in D(R)_+$ and let a be an element of G such that $a \in \boldsymbol{a}$. Hence $a = g_1 g_2^{-1}$ for some $g_1 \in G_+$, $g_2 \in G_+ - P \subseteq U(R)$. Thus $\boldsymbol{a} = w(g_1)$ and R is well centred on G by Lemma 5.

(5) $\Rightarrow$ (4). Trivial.

(2) $\Rightarrow$ (3). Suppose that A is an m-ring such that $G_+ \subseteq A \subset G$ and let M be a prime m-ideal of A. $P = M \cap G_+$ is a prime m-ideal of G_+ and $(G_+)_P \subseteq A_M$. Since G_+ is a Prüfer m-ring we obtain that A is a Prüfer m-ring.

Next, denote by $\boldsymbol{A}$ the integral part of $D(A)$. If $\boldsymbol{P}$ is a prime m-ideal of $\boldsymbol{A}$, one may easily verify that $U(\boldsymbol{A_P}) = U(A_P)/U(A) \subseteq D(A)$, where

$$P = \{a \in A : a\,U(A) \in \boldsymbol{P}\}$$

is a prime m-ideal of A. Now we get

$$D(\boldsymbol{A_P}) = D(A)/U(\boldsymbol{A_P}) = (G/U(A))/(U(A_P)/U(A)) \cong G/U(A_P) = D(A_P).$$

Thus we obtain that $\boldsymbol{A}$ is a Prüfer m-ring. By [7]; Theorem N we have

$$A \supseteq \bigcap \{AH : H \in \mathfrak{M}(A)\}$$

where $\mathfrak{M}(A)$ is the set of prime d-convex subgroups of $D(A)$. By Lemma 7 the implication (2) $\Rightarrow$ (5) can be applied to the d-group $D(A)$. Thus we obtain that every valuation m-ring $\boldsymbol{R}$ such that $\boldsymbol{A} \subseteq \boldsymbol{R} \subset D(A)$ is well centred on $D(A)$. Hence by Lemma 4, for any $H \in \mathfrak{M}(A)$ there exists a valuation m-ring $\boldsymbol{R}$ such that $\boldsymbol{R}$ is well centred on $D(A)$ and $U(\boldsymbol{R}) = U(\boldsymbol{R})^* = H^* \subseteq H$. We get

$$\boldsymbol{A} \supseteq \bigcap\{AH : H \in \mathfrak{M}(A)\} \supseteq \bigcap\{A\,U(\boldsymbol{R}) : \boldsymbol{R} \in \mathfrak{R}(A)\} = \\ = \bigcap\{\boldsymbol{R} : \boldsymbol{R} \in \mathfrak{R}(A)\} \supseteq \boldsymbol{A},$$

where $\mathfrak{R}(A)$ is the set of valuation m-rings such that $\boldsymbol{A} \subseteq \boldsymbol{R} \subset D(A)$ and for which $U(\boldsymbol{R})$ is a prime d-convex subgroup of $D(A)$. Hence

$$\boldsymbol{A} = \bigcap\{\boldsymbol{R} : \boldsymbol{R} \in \mathfrak{R}(A)\}$$

and by [7]; Main Theorem $\boldsymbol{A}$ is integrally closed in $D(A)$. Therefore A is integrally closed by Lemma 6.

(3) ⇒ (2). It suffices to prove that if G_+ is a local m-ring, it is a valuation m-ring. The proof of this part is substantially the same as that of [2a]; Proposition 13, but in order to make this paper self-contained we repeat it.

Let $x \in G - G_+$ and put

$$B = \bigcup (b_m x^{2m} \oplus \ldots \oplus b_0), \quad b_i \in G_+ .$$

Clearly, B is an m-ring and x is integral over B; hence $x \in B$. Let

$$x \in a_0 x^{2n} \oplus \ldots \oplus a_{n-1} x^2 \oplus a_n ; \quad a_i \in G_+ ,$$

where the number n is the smallest that satisfies the above relation. We have

$$a_0^{2n-2}(a_0 x) \in (a_0 x)^{2n} \oplus \ldots \oplus a_n a_0^{2n-1} .$$

This means that $a_0 x$ is integral over G_+ and by the assumption that G_+ is integrally closed we obtain that $a_0 \in G_+$.

Now suppose that $n > 1$. Then

$$a_0^{2n-1} x \in a_0^n (a_0 x^2)^n \oplus \ldots \oplus a_0^{2n-2} a_{n-1} (x a_0)^2 \oplus a_n a_0^{2n-1} ,$$

so that

$$a_0^{n-1} x \in (a_0 x^2)^n \oplus \ldots \oplus a_0^{n-1} a_n .$$

Since $n > 1$, we have $a_0^{n-1} x \in G_+$. Hence

$$(a_0 x^2)^n \in a_1 (a_0 x^2)^{n-1} \oplus \ldots \oplus (a_0^{n-1} a_n \oplus a_0^{n-1} x)$$

and we obtain $a_0 x^2 \in G_+$. Now

$$x \in a_0 x^{2n} \oplus \ldots \oplus a_n = (a_0 x^2)\, x^{2n-2} \oplus \ldots \oplus a_n$$

and this contradicts the assumption on n. Therefore $n = 1$ and $x \in a_0 x^2 \oplus a_1$ for $a_0 x \in G_+$. Since $a_1 \in x(1 \oplus a_0 x)$ and $x \notin G_+$ we have $a_0 x = 1$ and we obtain $x^{-1} = a_0 (a_0 x)^{-1} \in G_+$. Therefore G is simply ordered and since $G = D(G_+)$ we obtain that G_+ is a valuation m-ring.

A set $\{G_i : i \in I\}$ is called a realization of a d-group G provided that G_i is a simply ordered d-group for any $i \in I$ and there exists an order isomorphism f of the d-group G into a group $\prod_{i \in I} G_i$ such that

$$f(a \oplus b) \subseteq f(a) \oplus' f(b) , \tag{2}$$

where $\oplus'$ is an addition defined in [6] and the group $f(G)$ is a subdirect product of the groups G_i. Note that for any directed d-group in which all d-convex subgroups are directed there exists an order isomorphism into a product of local d-groups which satisfies (2) ([6]; Theorem 6).

Theorem 9. *Let G be the same as in Theorem 8 and let all d-convex subgroups of G be directed. Then the conditions of Theorem 8 are equivalent to the condition*

(6) $$\{G/H : H \text{ is a prime d-convex subgroup of } G\} \text{ is a realization of } G .$$

Proof. (6) $\Rightarrow$ (1). Trivial.

(1) $\Rightarrow$ (6). By [6]; Theorem 6 there exists an order isomorphism $f : G \to \to \prod\{G/H : H$ is a prime d-convex subgroup of $G\}$ which satisfies (2) and such that $f(G)$ is a subdirect product of groups G/H. Since G/H is simply ordered for every prime d-convex subgroup H, the set $\{G/H\}$ is a realization of G.

In Section 3 we shall show that there exist a d-group G and a valuation m-ring R whose group of units $U(R)$ is not directed (in G), $G_+ \subseteq R \subset G$, and Theorem 8 is false for this d-group.

Proposition 10. *Let G be a d-group and let H be a directed d-convex subgroup of G. Then if G_+ is integrally closed in G, the m-ring $(G/H)_+$ is integrally closed in G/H.*

Proof. Assume that
$$p^{n+1}H \in a_n p^n H \oplus' \dots \oplus' a_0 H ,$$
where $a_i H \geqq H$ and $\oplus'$ is the addition in G/H. From the definition of the order relation in G/H it follows that there exist $h^{(i)} \in H$ $(i = 0, \dots, n)$ such that
$$a_i \geqq h^{(i)} ; \quad i = 0, \dots, n .$$
Further, from the definition of the addition $\oplus'$ it follows that there exist $b_0 \in a_0 H, \dots \dots, b_n \in a_n H$ such that

(3) $$p^{n+1} \in p^n b_n \oplus \dots \oplus b_0 ,$$

where $\oplus$ is the addition in G. Since H is directed, we can find an element $q \in H$ such that
$$q \geqq (h^{(i)})^{-1} \quad \text{for} \quad i = 0, \dots, n .$$
Then we have $a_i \geqq h^{(i)} \geqq q^{-1}$ for $i = 0, \dots, n$. Multiplying the relation (3) on both sides by q^{n+1}, we obtain

(4) $$(pq)^{n+1} \in (pq)^n (qa_n) h_n \oplus \dots \oplus (a_0 q) q^n h_0 ,$$

where $h_i = b_i a_i^{-1}$ $(0 \leqq i \leqq n)$. Again, since H is directed, we can find an element $h \in H$ such that
$$h \geqq 1 , \quad (q^i h_{n-1})^{-1} , \quad i = 0, \dots, n .$$

Multiplying the relation (4) on both sides by h^{n+1}, we get

$$(pqh)^{n+1} \in (pqh)^n d_n h h_n \oplus \ldots \oplus d_0 h^n q^n h_0 ,$$

where $d_i = a_i q \geqq 1$, $d_i h^{n-i+1} q^i h_{n-i} \geqq 1$. Since G_+ is integrally closed in G, we have $pqh \in G_+$. Thus we obtain $pH \geqq H$ and $(G/H)_+$ is integrally closed in G/H.

3. APPLICATIONS

In this section we shall give some applications of results from Section 2 to the theory of commutative integral domains and to the theory of abelian lattice ordered groups.

First, T. Nakano [6] showed that for any integral domain A the family $\bar{A} = \{\bar{x} = \{x, -x\} : x \in A\}$ is an m-ring with respect to the addition

$$\bar{x} \oplus \bar{y} = \{\overline{x + y}, \overline{x - y}\}$$

and the multiplication

$$\bar{x} . \bar{y} = \overline{xy} .$$

Analogically, it was proved that every abelian lattice ordered group G is a d-group with respect to the addition

$$a \oplus b = \{c \in G : a \wedge b = a \wedge c = b \wedge c\} .$$

It has proved useful on occasion to phrase a ring-theoretical or lattice-theoretical problem in terms of d-groups, first solve the problem there, and then pull back the solution if possibleto the original situation.

Proposition 11. (See [1]; Theorem 16.5.) *Let A be an integral domain with the quotient field K. If P is a prime ideal of A, then there exists a valuation ring R of K such that $M(R) \cap A = P$, where $M(R)$ is the maximal ideal of R.*

Proof. Let $\bar{A}$ denote the m-ring mentioned above. Put

$$\bar{P} = \{\bar{x} \in \bar{A} : x \in P\} .$$

One may easily verify that $\bar{P}$ is a prime m-ideal of $\bar{A}$. Now, by Proposition 3, there exists a valuation m-ring $\mathcal{R}$ such that $D(\bar{A})_+ \subseteq \mathcal{R} \subset D(\bar{A})$ and $M(\mathcal{R}) \cap D(\bar{A})_+ = w(\bar{P})$, where $w : \bar{K} \to \bar{K}/U(\bar{A}) = D(\bar{A})$ is the canonical homomorphism. We set

$$R = \{x \in K : w(\bar{x}) \in \mathcal{R}\} .$$

Now, one may easily verify that R is a valuation ring of K and $M(R) \cap A = P$.

Recall that a valuation ring R of the quotient field K of an integral domain A such that $A \subseteq R$ is called well centred on A provided that for any $\alpha \in \Gamma_+$ (the value group of R) there exists $a \in A$ such that $w(a) = \alpha$, where w is the valuation associated with R. M. GRIFFIN [2] proved that there exists an integral domain A such that every valuation ring of the quotient field of A is well centred on A, but A is not a Prüfer domain. This fact enables us to give an example of a d-group G such that $U(R)$ is not directed for a certain valuation m-ring R and for which Theorem 8 is false.

First, on the quotient field K of an integral domain A we can define a preorder relation

$$x \leqq_A y \Leftrightarrow x/_A y ,$$

where $/_A$ denotes the division relation with respect to A.

Next, the following proposition holds.

Proposition 12. *Let A be an integral domain with the quotient field K and let a group $U(R)$ be directed with respect to $\leqq_A$ for every valuation ring R of K containing A. Then the following conditions are equivalent:*

(1) *A is a Prüfer domain.*

(2) *Every valuation ring R of K such that $A \subseteq R$ is well centred on A.*

Proof. Let $\bar{A}$ be the same as in Proposition 11. We shall prove that $U(\mathscr{R})$ is directed for every valuation m-ring $\mathscr{R}$ such that $D(\bar{A})_+ \subseteq \mathscr{R} \subset D(\bar{A})$. In fact, set

$$R = \{x \in K : \bar{x}\, U(\bar{A}) \in \mathscr{R}\} .$$

Clearly, R is a valuation ring of K, $A \subseteq R$ and $\mathscr{R} = \bar{R}/U(\bar{A}) \subseteq D(\bar{A})$. Assume that $\bar{y}\, U(\bar{A}), \bar{x}\, U(\bar{A}) \in U(\mathscr{R})$. Evidently, $x, y \in U(R)$ and since $U(R)$ is directed, there exists $z \in U(R)$ such that $z \geqq_A x, y$. Therefore $\bar{x}\, U(\bar{A}), \bar{y}\, U(\bar{A}) \leqq \bar{z}\, U(\bar{A}) \in U(\mathscr{R})$.

Next we prove that $\mathscr{R}$ is well centred on $D(\bar{A})$. In fact, assume that $\alpha \in D(\mathscr{R})_+$. Thus we have

$$\alpha = (\bar{x}\, U(\bar{A}))\, U(\mathscr{R}) ,$$

where $\bar{x}\, U(\bar{A}) \in \mathscr{R}$. Let $w : K \to K/U(R)$ be the valuation associated with R. Since R is well centred on A, there exists $a \in A$ such that $w(a) = w(x)$. This means that $xa^{-1} \in U(R)$. Consequently, $\bar{x} \,.\, (\bar{a})^{-1}\, U(\bar{A}) \in U(\mathscr{R})$ and we have $(\bar{x}\, U(\bar{A}))\, U(\mathscr{R}) = (\bar{a}\, U(\bar{A}))\, U(\mathscr{R})$. Therefore $\mathscr{R}$ is well centred on $D(\bar{A})$.

Finally, $D(\bar{A})_+$ is a Prüfer m-ring by Theorem 8. Assume that B is an integral domain such that $A \subseteq B \subset K$. Put

$$\mathscr{B} = \{\bar{x}\, U(\bar{A}) : x \in B\} .$$

Evidently, $\mathscr{B}$ is an m-ring in $D(\bar{A})$, $D(\bar{A})_+ \subseteq \mathscr{B}$. By Theorem 8, $\mathscr{B}$ is integrally closed. Now assume that

$$x^{n+1} = b_n x^n + \ldots + b_0 ; \quad b_i \in B$$

for some $x \in K$. From the definition of addition in $\bar{A}$ we get

$$\bar{x}^{n+1} = \overline{b_n x^n + \ldots + b_0} \in \bar{b}_n \bar{x}^n \oplus \ldots \oplus \bar{b}_0 .$$

Hence

$$\bar{x}^{n+1} \, U(\bar{A}) \in \bar{b}_n \bar{x}^n \, U(\bar{A}) \oplus \ldots \oplus \bar{b}_0 \, U(\bar{A}) .$$

Since $\mathscr{B}$ is integrally closed, we obtain $\bar{x} \, U(\bar{A}) \in \mathscr{B}$. But, since $U(A) \subseteq U(B)$, we have $x \in B$. Therefore B is integrally closed and by [1]; Theorem 22.2 A is a Prüfer ring.

The converse is trivial.

Note that Proposition 12 can be proved directly without using the notion of d-group.

Now, if we assume that Theorem 8 holds for every d-group, we obtain from the proof of Proposition 12 that this one holds for every integral domain. However, this contradicts the result of Griffin.

Proposition 13 (Lorenzen). *Every abelian l-group has a realization.*

Proof. Let G be an abelian l-group. As it was mentioned above, G is a d-group with respect to the addition

$$a \oplus b = \{c \in G : a \wedge b = a \wedge c = b \wedge c\} .$$

Assume that R is a valuation m-ring such that $G_+ \subseteq R \subset G$. We shall prove that R is well centred on G and $U(R)$ is directed.

In fact, assume that $i, j \in U(R)$. Then we have $i \wedge j \in i \oplus j \subseteq R$. Since $(i \wedge j)^{-1} \geqq$ $\geqq (i^{-i} \wedge j^{-1}) \in R$, we get $i \wedge j, i \vee j \in R$. Especially, $i \wedge j, i \vee j \in U(R)$. Therefore $U(R)$ is directed.

Further, let $x \in R$. We set

$$x' = xj \wedge 1 ,$$

where j is a unit of R. Since $x' \in xj \oplus 1 \subseteq R$ and $(x')^{-1} \in G_+ \subseteq R$ we have $x' \in U(R)$. Moreover, since $xj \geqq x'$, there exists $g \in G_+$ such that $x = gj^{-1}x' \in G_+ \, U(R)$. Hence $R \subseteq G_+ \, U(R)$. The converse inclusion is trivial.

Since G is lattice ordered, one may easily verify that every d-convex subgroup of G is directed. Now, by Theorem 9, the d-group G has a realization. Hence G has a realization.

4. APPROXIMATION THEOREM

W. Krull [3] conjectured:

Let G be an abelian l-group and let $N_1, \ldots, N_k$ be prime l-ideals of G. Assume that a family $(a_1N_1, \ldots, a_kN_k) \in G/N_1 \times \ldots \times G/N_k$ satisfies the conditions

$$a_iN_iN_j = a_jN_iN_j \;; \quad i, j = 1, \ldots, k \,, \quad i \neq j \,.$$

Then there exists an element $a \in G$ such that

$$a_iN_i = aN_i \quad \text{for} \quad i = 1, \ldots, k \,.$$

The first proof of this conjecture was given by D. MÜLLER [4]. In this section we shall give an approximation theorem for d-groups. Since any l-group is a d-group, this theorem may serve a new proof of Krull's conjecture.

Proposition 14. *Let G be the same as in Theorem 8 and let G satisfy the equivalent conditions of this theorem. Assume that R_1, R_2 are valuation m-rings such that $G_+ \subseteq R_i \subset G$. Then*

$$R_1 \wedge R_2 = R_1R_2 \,, \quad U(R_1R_2) = U(R_1)\, U(R_2) \,,$$

where the set of valuation m-rings is ordered by the relation

$$R \leqq R' \Leftrightarrow R \supseteq R' \,.$$

Proof. First, to show that $R_1R_2 = R_1 \wedge R_2$ it suffices to prove that R_1R_2 is a valuation m-ring. Assume that $D(R_1R_2) = G/U$, where U is the group of units of R_1R_2. We denote by U_i the group of units of R_i. Now, since $U \supseteq U_1U_2 \supseteq U_i$, the canonical mapping of $D(R_i)$ onto $D(R_1R_2)$ satisfies the relation

$$xU_i \leqq yU_i \Rightarrow xU \leqq yU \,.$$

Hence R_1R_2 is a valuation m-ring.

Secondly, we shall prove that U_i is a prime d-convex subgroup of G. In fact, since U_i is directed, it is d-convex by [6]; Lemma 5. Moreover, since R_i is well centred on G, it can be verified analogous as in Theorem 8 that $(G/U_i, \precsim) = D(R_i)$, where $\precsim$ is the order relation on the factor d-group G/U_i. But since $D(R_i)$ is local, we have that U_i is prime.

Evidently, U_1U_2 is a d-convex subgroup of G. Now the canonical mapping of a factor d-group G/U_i onto a factor d-group G/U_1U_2 defines an order homomorphism. Hence G/U_1U_2 is simply ordered and we have that U_1U_2 is prime.

Finally, we shall prove that $D(R_1R_2) = (G/U_1U_2, \leqq)$, where $\leqq$ is the order relation on the factor d-group G/U_1U_2. In fact, let us assume that $xU_1U_2 \geqq U_1U_2$. Then there exists an element $j = j_1j_2 \in U_1U_2$ such that $x \geqq j$. Hence $xj_1^{-1} \in G_+U_2 \subseteq \subseteq R_2$ and we have $x \in R_1R_2$. Conversely, $xU_1U_2 \wedge U_1U_2 = (xU_1 \wedge U_1)\, U_2 = = U_1U_2$ for any $x \in R_1$. Assume that $w : G \to G/U_1U_2$ is the canonical homomorphism; then we have $R_1R_2 \subseteq w^{-1}((G/U_1U_2)_+)$. Therefore $R_1R_2 = = w^{-1}((G/U_1U_2)_+)$. Now, since U_1U_2 is a prime d-convex subgroup, by Lemma 4 there exists a valuation m-ring R such that $U(R) = U_1U_2$. From the fact that R is well centred we obtain that a factor d-group G/U_1U_2 is order isomorphic to the d-group $D(R)$. Therefore $R = R_1R_2$.

In what follows, we shall denote by $\mathfrak{N}$ the set of valuation m-rings R such that $G_+ \subseteq R \subset G$. If $R \in \mathfrak{N}$, $\boldsymbol{g} = g\,U(R) \in D(R)$, $\boldsymbol{g} \neq U(R)$, we set

$$\mathfrak{N}(\boldsymbol{g}) = \{R' \in \mathfrak{N} : g \notin U(R)\, U(R')\} .$$

An element $R \in \mathfrak{N}$ is called *weakly independent* provided that for any $R' \in \mathfrak{N}$ and $g \in U(R)\,U(R')$ there is an $a \in U(R)$ satisfying $a\,U(R') \geqq g\,U(R')$. It is easy to see that $R \in \mathfrak{N}$ is weakly independent if and only if for any $R' \in \mathfrak{N}$, $\boldsymbol{g} \in D(R')$ such that $R \notin \mathfrak{N}(\boldsymbol{g})$ there exists an element $a \in G$ such that $a \in U(R)$ and $a\,U(R') \geqq \boldsymbol{g}$.

We say that a family $(g_1, \ldots, g_n) \in G^n$ is *compatible* with respect to $(R_1, \ldots, R_n) \in \in \mathfrak{N}^n$ provided that for any $1 \leqq i, j \leqq n$, $i \neq j$ it holds

$$g_iU_iU_j = g_jU_iU_j ,$$

where $U_i = U(R_i)$. Finally, we say that G satisfies the *approximation theorem* provided that for any family $(g_1, \ldots, g_n) \in G^n$ compatible with respect to a family $(R_1, \ldots, R_n) \in \mathfrak{N}^n$ there exists $a \in G$ such that

$$g_iU_i = aU_i , \quad i = 1, \ldots, n .$$

The proof of the following proposition is quite the same as that of [2]; Proposition 5. Nonetheless, we repeat it in order to make this paper self-contained.

Proposition 15. *Let G be the same as in Theorem* 8. *Then the following conditions are equivalent*:

(1) *G satisfies the approximation theorem.*

(2) *Every valuation m-ring of $\mathfrak{N}$ is weakly independent.*

Proof. (2) $\Rightarrow$ (1). The proof is by induction on n. For $n = 1$ the approximation theorem evidently holds. Now assume that a family $(g_1, \ldots, g_n) \in G^n$ is compatible with respect to $(R_1, \ldots, R_n) \in \mathfrak{N}^n$. We may assume that if $i \neq j$ then $R_i \nsubseteq R_j$. Indeed, if $R_i \subseteq R_j$ then by induction there exists $a \in G$ such that $aU_k = g_kU_k$ for each k, $1 \leqq k \leqq n$, $k \neq j$. Since (g_i, g_j) is compatible with respect to (R_i, R_j), we have $aU_j = g_iU_j = g_jU_j$ and the induction is complete.

Now there exists $b_1 \in G_+$ such that $b_1 \in U_1$ and $b_1 \notin U_i$ for $i = 2, \ldots, n$. In fact, since $U_1 \nsubseteq U_i$, there exist $b_i \in (U_1 - U_i) \cap G_+$ for each i, $2 \leqq i \leqq n$. We set $b_1 = b_2 \ldots b_n$.

By the induction hypothesis there exists $a_1 \in G$ such that $a_1 U_i = g_i U_i$ for $i = 1, \ldots, n - 1$. We may assume that $a_1 U_1 = g_1 U_1$, $a_1 U_i \geqq g_i U_i$ for $i = 2, \ldots, n$. Indeed, if $a_1 U_n < g_n U_n$ we have $g_n a_1^{-1} U_n \neq U_n$. Since (g_n, g_1) and (g_1, a_1) are compatible with respect to (R_n, R_1) we obtain that (g_n, a_1) is compatible. Now, since $R_1 \notin \mathfrak{N}(g_n a_1^{-1})$, there exists $a_1' \in G_+$ such that $a_1' \in U_1$, $a_1' U_n \geqq g_n a_1^{-1} U_n$ in virtue of the weak independence; letting $a_2 = a_1 a_1'$ we have

$$a_2 U_1 = g_1 U_1 \,,$$
$$a_2 U_i \geqq a_1 U_i = g_i U_i \quad \text{for} \quad i = 2, \ldots, n - 1 \,,$$
$$a_2 U_n \geqq a_1 g_n a_1^{-1} U_n = g_n U_n \,.$$

Now $a_1 b_1 U_1 = a_1 U_1 = g_1 U_1$, $a_1 b_1 U_i > a_1 U_i \geqq g_i U_i$ for $i = 2, \ldots, n$. Similarly for each valuation m-ring R_i we may find $a_i b_i \in G$ such that $a_i b_i U_i = g_i U_i$, $a_i b_i U_k > g_k U_k$ for $k \neq i$. Hence $a U_i = \min \{a_j b_j U_j : 1 \leqq j \leqq n\} = g_i U_i$ for any $a \in a_1 b_1 \oplus \ldots \oplus a_n b_n$.

(1) $\Rightarrow$ (2). Trivial.

Now using the method used in Section 3 we shall give an approximation theorem for lattice ordered groups.

Lemma 16. *Let G be an abelian l-group. Then every valuation m-ring R such that $G_+ \subseteq R \subset G$ is weakly independent.*

Proof. Let R be a valuation m-ring such that $G_+ \subseteq R \subset G$. Let $R' \in \mathfrak{N}$ and $\boldsymbol{g} = g\, U(R') \in D(R')$ be such that $g \in U(R)\, U(R')$. Hence $g = ij$ for some $i \in U(R')$, $j \in U(R)$. Now we set

$$a = j \vee 1 \,.$$

Then $a \in G_+ \subseteq R$ and $a^{-1} = (j \vee 1)^{-1} = j^{-1} \wedge 1 \in j^{-1} \oplus 1 \subseteq R$. Thus we have $a \in U(R)$. Moreover, there exists an element $g' \in G_+$ such that $a g^{-1} i = g'$. Hence we obtain $a\, U(R') \geqq g i^{-1}\, U(R') = g\, U(R')$.

Theorem 17. *Krull's conjecture is true.*

Proof. Let G be an abelian l-group and let $N_1, \ldots, N_k$ be prime l-ideals of G. Assume that $a_1, \ldots, a_k \in G$ are such that $a_i N_i N_j = a_j N_i N_j$. G satisfies the approximation theorem by Lemma 16 and Proposition 15 and by [6]; § 8 any prime l-ideal is a prime d-convex subgroup. Hence, by Lemma 4, for any N_i, $1 \leqq i \leqq k$, there exists a valuation m-ring R_i such that $U(R_i) = N_i$. Consequently, the family $(a_1, \ldots, a_k)$ is compatible with respect to the family $(R_1, \ldots, R_k)$. Therefore there exists $a \in G$ such that $a N_i = a_i N_i$ for $i = 1, \ldots, k$.

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