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TOLERANCE RELATIONS ON PERIODIC COMMUTATIVE SEMIGROUPS

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A tolerance relation (or shortly tolerance) is a reflexive and symmetric binary relation on a set. The definition of a tolerance compatible with a given algebra was given in [1] and [2]. In these papers also some theorems concerning compatible tolerances on semigroups were proved. Among others, a theorem was proved stating that if a semigroup is a group, then each tolerance compatible with it is a congruence (even if we suppose that this tolerance is compatible only with the multiplication and do not suppose a priori that it is compatible with the operation of inversion).

Here we shall prove that for periodic commutative semigroups also the converse assertion is true.

Let S be a semigroup, let ξ be a tolerance on its set of elements. We say that ξ is compatible with S, if and only if for any four elements x_1, x_2, y_1, y_2 of S for which $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$ we have $(x_1x_2, y_1y_2) \in \xi$.

Theorem 1. *Let S be a periodic commutative semigroup with at least three elements. Then the following two assertions are equivalent:*
(1) S is a group.
(2) Each tolerance compatible with S is a congruence.

Proof. The implication $(1) \Rightarrow (2)$ was proved in [2]. We shall prove $(2) \Rightarrow (1)$. As is well-known, every periodic semigroup contains at least one idempotent. First suppose that S has at least three idempotents. Let p_1, p_2, p_3 be three pairwise distinct idempotents of S. Let $q = p_1p_2p_3$. As S is commutative, q is also an idempotent. It may be equal to some of the elements p_1, p_2, p_3; without loss of generality suppose $p_1 \neq q \neq p_2$. Let ξ be a tolerance on S consisting of the pairs (p_1, q), (q, p_1), (p_2, q), (q, p_2), (p_1p_2, q), (q, p_1p_2), (x, x), (p_1x, qx), (qx, p_1x), (p_2x, qx), (qx, p_2x), (p_1p_2x, qx), (qx, p_1p_2x) for all elements $x \in S$. The proof that ξ is compatible with S is left to the reader. Suppose that $(p_1, p_2) \in \xi$. As p_1, p_2, q are pairwise distinct, the pair (p_1, p_2) is distinct from the pairs (p_1, q), (q, p_1), (p_2, q), (q, p_2), (p_1p_2, q), (q, p_1p_2) and (x, x) for all $x \in S$. Suppose $(p_1, p_2) = (p_1x, qx)$ for some $x \in S$. This implies $p_2 = qx$. Multiplying by q we obtain $q = p_2q = q^2x = qx = p_2$, which is a contradiction. Analogously we prove that (p_1, p_2) is distinct from (qx, p_1x),

(p_2x, qx), (qx, p_2x), (p_1p_2x, qx), (qx, p_1p_2x). We have $(p_1, q) \in \xi$, $(q, p_2) \in \xi$, but $(p_1, p_2) \notin \xi$ and ξ is not transitive. Thus ξ is not a congruence.

Now suppose that S has exactly two idempotents p, q; without loss of generality suppose $pq = q$. To each element of a periodic semigroup there exists exactly one idempotent which is a power of this element. Let $S(p)$ (or $S(q)$) be the set of all elements of S with the property that some of their powers is p (or q respectively). We have $S(p) \cup S(q) = S$, $S(p) \cap S(q) = \emptyset$. Evidently $S(q)$ is an ideal of S. Suppose that $S(p)$ has at least two elements. Let ξ consist of all pairs (x, x), (x, y), (y, x), where $x \in S$, $y \in S(q)$. We shall prove the compatibility of ξ. Let $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \in \xi$ for some elements x_1, x_2, y_1, b_2 of S. If $x_1 = y_1$, $x_2 = y_2$, then $x_1x_2 = y_1y_2$ and $(x_1x_2, y_1y_2) \in \xi$. If this is not the case, then some of the four elements considered must be in $S(q)$. If some of the elements x_1, x_2 is in $S(q)$, then $x_1x_2 \in S(q)$, because $S(q)$ is an ideal of S, and $(x_1x_2, y_1y_2) \in \xi$. Analogously if some of the elements y_1, y_2 is in $S(q)$. Now let a be an element of $S(p)$ different from p. We have $(p, q) \in \xi$, $(q, a) \in \xi$ because $q \in S(q)$, but $(p, a) \notin \xi$ because $p \neq a$ and none of the elements p, a is in $S(q)$. Thus ξ is not a congruence. Now suppose that $S(q)$ has only one element; this means $S(p) = \{p\}$. As S has at least three elements, there exists at least one element $b \in S(q)$ different from q. Let ξ consist of the elements (p, p), (p, q), (q, p) and (y, z) for each $y \in S(q)$, $z \in S(q)$. It is easy to prove that ξ is compatible with S, by virtue of the fact that $S(q)$ is an ideal of S. Now we have $(p, q) \in \xi$, $(q, b) \in \xi$, but $(p, b) \notin \xi$, because $b \neq p$, $b \neq q$ and $p \notin S(q)$. The tolerance ξ is not a congruence.

Finally, let S have exactly one idempotent p. Suppose that there exists an element $a \in S$ such that $ax \neq a$ for each $x \in S$. Obviously $a \neq p$, because $p = p^2$. Let aS be the set of all elements ax, where $x \in S$. This is an ideal of S. We have $p \in aS$, because p must be a power of a. Suppose that aS contains at least two elements. Let ξ consist of the pairs (a, p), (p, a), (x, x), (y, z) for each $x \in S$, $y \in aS$, $z \in aS$. Let x_1, x_2, y_1, y_2 be elements of S, let $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$. If $x_1 = y_1$, $x_2 = y_2$, then $x_1x_2 = y_1y_2$ and $(x_1x_2, y_1y_2) \in \xi$. If $x_1 \neq y_1$, then either $x_1 = a$, or $x_1 \in aS$ and the same for y_1. As aS is an ideal, $x_1x_2 \in aS$, $y_1y_2 \in aS$ and $(x_1x_2, y_1y_2) \in \xi$. Analogously if $x_2 \neq y_2$. We have supposed that aS contains at least two elements. Thus let $b \in aS$, $b \neq p$. The pair (a, b) is not in ξ, because $a \neq p$, $b \neq p$, $a \neq b$ and $a \notin aS$. We have $(a, p) \in \xi$, $(p, b) \in \xi$, but $(a, b) \notin \xi$ and ξ is not transitive. Now suppose that aS has only one element; this means $aS = \{p\}$ and thus $ax = p$ for each $x \in S$. Then also $px = ax^2 = p$ and p is the zero element of S. Now let ξ consist of the pairs (x, x), (x, p), (p, x) for each $x \in S$. The compatibility of ξ is evident. If b, c are two different elements of $S - \{p\}$ (they exist, because S has at least three elements), then $(b, p) \in \xi$, $(p, c) \in \xi$, $(b, c) \notin \xi$ and ξ is not a congruence.

Now only one case remains: S has only one idempotent p and for each $a \in S$ there exists an element $x \in S$ such that $ax = a$. By induction we can prove $ax^n = a$ for each positive integer n. There must be a power of x which is an idempotent; as S has only one idempotent, there exists a positive integer m such that $x^m = p$. Then $ap = ax^m = a$. As a is an arbitrary element of S, the element p is the unit element of S. To each $a \in S$ there exists a positive integer n such that $a^n = p$; thus there exists an inverse element $a^{-1} = a^{n-1}$ such that $aa^{-1} = p$. We see that S is a group. The proof is complete.

We have excluded the case when S contains only one or two elements, because on every set with one or two elements each reflexive and symmetric binary relation is an equivalence.

This result is not true for non-commutative periodic semigroups, not even for those in which the product of two idempotents is again an idempotent.

Theorem 2. *Let n be an arbitrary cardinal number greater than one. Then there exists a non-commutative periodic semigroup S with $2n$ elements which is not a group and on which each compatible tolerance is a congruence.*

Proof. Let G be a periodic group of the order n. Let G_1, G_2 be two disjoint groups which are both isomorphic to G. Let e_1 (or e_2) be the unit element and $\bigcirc_1$ (or $\bigcirc_2$) the multiplication in G_1 (or G_2 respectively). Let φ be an isomorphic mapping of G_1 onto G_2. We shall construct a semigroup S. The support of S is the union of the supports of G_1 and G_2. The multiplication in S will be denoted by juxtaposition. Let $x \in S$, $y \in S$. If $x \in G_1$, $y \in G_1$, then $xy = x \bigcirc_1 y$. If $x \in G_2$ $y \in G_2$, then $xy = x \bigcirc_2 y$. If $x \in G_1$, $y \in G_2$, then $xy = \varphi(x) \bigcirc_2 y$. If $x \in G_2$, $y \in G_1$, then $xy = \varphi^{-1}(x) \bigcirc_1 y$. It is easy to see that $e_1x = xe_1 = e_2x = x$, $xe_2 = \varphi(x)$ for $x \in G_1$ and $e_2x = xe_2 = e_1x = x$, $xe_1 = \varphi^{-1}(x)$ for $x \in G_2$. The group G_1 is a left ideal of S, the group G_2 is a right ideal of S. Let ξ be a tolerance compatible with S, let ξ_1 (or ξ_2) be the restriction of ξ onto G_1 (or G_2); evidently ξ_1 (or ξ_2) is compatible with G_1 (or G_2 respectively). First suppose that $(x, y) \in \xi$ only if either $x \in G_1$, $y \in G_1$, or $x \in G_2$, $y \in G_2$. Here $\xi = \xi_1 \cup \xi_2$, $\xi_1 \subset G_1 \times G_1$, $\xi_2 \subset G_2 \times G_2$ and $G_1 \cap G_2 = \emptyset$. Since ξ_i $(i = 1, 2)$ is compatible with the group G_i, it is a transitive relation. Hence ξ is a congruence. Now suppose that there exist elements $x \in G_1$, $y \in G_2$ such that $(x, y) \in \xi$. Let m be the order of x in G_1 and n the order of y in G_2. We have $(x^{mn}, y^{mn}) \in \xi$. But $x^{mn} = (x^m)^n = e_1^n = e_1$, $y^{mn} = (y^n)^m = e_2^m = e_2$, thus $(e_1, e_2) \in \xi$ and from the symmetry $(e_2, e_1) \in \xi$. Given a pair a, b of elements of G_1 such that one of the pairs (a, b), $(a, \varphi(b))$, $(\varphi(a), b)$, $(\varphi(a), \varphi(b))$ is in ξ, then all others are also in ξ; this can be proved using the pairs (e_1, e_1), (e_1, e_2), (e_2, e_1), (e_2, e_2) which are all in ξ. Thus ξ consists of all the pairs (x, y), $(x, \varphi(y))$, $(\varphi(x), y)$, $(\varphi(x), \varphi(y))$, where $(x, y) \in \xi_1$. As ξ_1 is a congruence, ξ_2 is also a congruence. We have proved that each tolerance compatible with S is a congruence. Evidently the semigroup S is not a group.

References

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