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Ladislav Bican

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THE SPLITTING LENGTH OF MIXED ABELIAN GROUPS OF RANK ONE

LADISLAV BICAN, Praha

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IRWIN, KHABBAZ and RAYNA [5] have studied the splitting properties of the tensor product of mixed abelian groups. They defined the splitting length of a mixed group G as the infimum of the set of all positive integers n such that the n-th tensor power G^n of G splits and they constructed a mixed group of rank one having the splitting length n for every positive integer n. The purpose of this paper is to characterize the mixed abelian groups of rank one having the splitting length n.

By the word "group" we shall always mean an additively written abelian group. As in [1], we use the notions "characteristic" and "type" in the broad meaning, i.e. we deal with these notions in mixed groups. The symbols $h_p^G(g)$, $\tau^G(g)$ and $\hat{\tau}^G(g)$ denote respectively the p-height, the characteristic and the type of the element g in the group G. π will denote the set of all primes. If T is a torsion group, then T_p is the p-primary component of T and similarly if $\pi' \subseteq \pi$ then $T_{\pi'}$ is defined by $T_{\pi'} = \sum_{p \in \pi'}{}_d\, T_p$. For a mixed group G with a torsion part T we denote by $\bar{G}$ the factor-group G/T and for $g \in G$, $\bar{g}$ is the element $g + T$ of $\bar{G}$. Other notation used will be essentially the same as in [3].

Let $\pi' \subseteq \pi$ and let G be a mixed group with $T_{\pi'} = 0$. If S is a subset of G then $\{S\}_{\pi'}^G$ denotes the π'-pure closure of S in G, the existence of which is easily seen. It was proved in [1] that a mixed group G of rank one splits if and only if it satisfies the following conditions (α), (β):

We say that a mixed group G with a torsion part T satisfies Condition (α) if to any $g \in G \dot{-} T$ there exists an integer m such that $\hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g})$.

Similarly, a mixed group G with a torsion part T satisfies Condition (β) if to any $g \in G \dot{-} T$ there exists an integer $m \neq 0$ such that for any prime p with $h_p^{\bar{G}}(\bar{g}) = \infty$, mg has a p-sequence (i.e. there exist elements $h_0^{(p)} = mg$, $h_1^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}, n = 0, 1, \ldots$).

Lemma 1. *Let p be a prime, G a mixed group and let $a_i \in G \dot{-} T$, $i = 0, 1, \ldots$ be such elements that $p^{r_i} a_i = p^{s_i - 1} a_{i-1}$, $i = 1, 2, \ldots$, $s_0 = 0$. If $\sum_{i=1}^{\infty} (r_i - s_i)$ has non-negative partial sums and $\sum_{i=1}^{\infty} (r_i - s_i) = \infty$ then a_0 has a p-sequence.*

Proof. The fact that $\liminf_{n\to\infty}\{\sum_{i=1}^{n}(r_i - s_i)\} = \infty$ enables us to define an increasing sequence $\{k_j\}_{j=0}^{\infty}$ of positive integers in the following way: Put $k_0 = 1$, $\gamma_0 = 0$ and if $k_0, k_1, \ldots, k_j$ are defined then let k_{j+1} be the greatest integer such that $\gamma_{j+1} = \sum_{i=1}^{k_{j+1}-1}(r_i - s_i) = \inf\{\sum_{i=1}^{n}(r_i - s_i),\ n \geqq k_j\} > \gamma_j$. For every $k_j \leqq m < k_{j+1}$, $j = 0, 1, \ldots$ we have $\sum_{i=k_j}^{m}(r_i - s_i) = \sum_{i=1}^{m}(r_i - s_i) - \sum_{i=1}^{k_j-1}(r_i - s_i) \geqq \gamma_{j+1} - \gamma_j > 0$ so that $\sum_{i=k_j}^{m}(r_i - s_i) + r_m - (\gamma_{j+1} - \gamma_j) \geqq s_m$. Now a p-sequence of a_0 can be defined as follows: $a_0 = p^{r_1}a_1, p^{r_1-1}a_1, \ldots, p^{r_1-\gamma_1}a_1 = p^{r_{k_1}}a_{k_1}, p^{r_{k_1}-1}a_{k_1}, \ldots, p^{r_{k_1}-(\gamma_2-\gamma_1)}a_{k_1} = p^{r_{k_2}}a_{k_2}, \ldots, p^{r_{k_j}}a_{k_j}, p^{r_{k_j}-1}a_{k_j}, \ldots, p^{r_{k_j}-(\gamma_{j+1}-\gamma_j)}a_{k_j} = p^{r_{k_{j+1}}}a_{k_{j+1}}, \ldots$.

Lemma 2. *Let p be a prime, G a mixed group and $a \in G \dot{-} T$ such that $h_p^G(p^l a) = \infty$. Then the element $p^l(a \otimes \ldots \otimes a)$ has a p-sequence in G^n for every $n \geqq 2$.*

Proof. If $l = 0$ then taking the elements $a_k \in G$ with $p^k a_k = a = p^{k-1}a_{k-1}$, $k = 1, 2, \ldots, a_0 = a$, we get $p^{k+n-1}(a_k \otimes \ldots \otimes a_k) = p^{k-1}(a_{k-1} \otimes \ldots \otimes a_{k-1})$, $k = 1, 2, \ldots$. If $l > 0$ then the elements $a_k \in G$ with $p^{(k+1)l}a_k = p^{kl}a_{k-1}$, $k = 1, 2, \ldots, a_0 = a$ have the property $p^{(k+1)l+(n+1)l}(a_k \otimes \ldots \otimes a_k) = p^{kl}(a_{k-1} \otimes \ldots \otimes a_{k-1})$, $k = 1, 2, \ldots$, and Lemma 1 completes the proof.

Definition 1. Let a be an element of a mixed group G and let p a prime. Define the p-height sequence of a in G as the double sequence $\{k_i, l_i\}_{i=0}^{\infty}$ of elements of $N \cup \{0, \infty\}$ inductively as follows: Put $k_1 = k_0 = l_0 = 0$ and $l_1 = h_p^G(a)$. If k_i, l_i are defined and either $h_p^G(p^{k_i}a) = l_i = \infty$, or $l_i < \infty$ and $h_p^G(p^{k_i+k}a) = l_i + k$ for all $k \in N$ then put $k_{i+1} = k_i$ and $l_{i+1} = l_i$. If $l_i < \infty$ and there are $k \in N$ with $h_p^G(p^{k_i+k}a) > l_i + k$ then let k_{i+1} be the smallest positive integer for which $h_p^G(p^{k_{i+1}}a) = l_{i+1} > l_i + k_{i+1} - k_i$.

Lemma 3. *Let G be a mixed group of rank one with a p-primary torsion part T and let $\bar{G}$ be p-divisible. Further, let $\{k_i, l_i\}_{i=0}^{\infty}$ be the p-height sequence of an element $a_0 \in G \dot{-} T$, $l_i \neq \infty$, $p^{l_i}a_i = p^{k_i}a_0$, $i = 1, 2, \ldots$. If $a_{i_1}, a_{i_2}, \ldots, a_{i_n} \in \{a_0, a_1, \ldots\}$, $j \geqq \max\{i_1, i_2, \ldots, i_n\}$ then $p^{l_j + \sum_{r=2}^{n} l_j - k_j + k_{i_r} - l_{i_r}}(a_j \otimes \ldots \otimes a_j) = p^{k_j - k_{i_1} + l_{i_1}}(a_{i_1} \otimes \ldots \otimes a_{i_n})$.*

Proof. We have $p^{l_j}a_j = p^{k_j}a_0 = p^{k_j - k_{i_r} + l_{i_r}}a_{i_r}$ and the assertion follows easily.

Lemma 4. *Let p be a prime and G a mixed group with a p-primary torsion part T. Further, let $a_0 \in G \dot{-} T$ be such that $h_p^{\bar{G}}(\bar{a}_0) = \infty$ and let $\{k_i, l_i\}_{i=0}^{\infty}$ be its p-height sequence with $l_i \neq \infty$, $i = 1, 2, \ldots$. If $p^{l_i}a_i = p^{k_i}a_0$ then there exists a subgroup $U = \sum_{i=2}^{\infty}{}_d\{t_i\}$ pure in T generated by the elements $p^{l_{i+1}-l_i-k_{i+1}+k_i}a_{i+1} - a_i$, $i = 1, 2, \ldots$.*

Proof. Put $U_1 = 0$, $t_1 = 0$ and proceed by induction. Suppose that we have constructed the elements $t_1, t_2, \ldots, t_i$ such that

(1) $$U_i = \{t_1\} \dotplus \{t_2\} \dotplus \ldots \dotplus \{t_i\}\,, \quad T = U_i \dotplus T'$$

and the elements

(2) $$t'_j = p^{l_j - l_{j-1} - k_j + k_{j-1}} a_j - a_{j-1}\,, \quad j = 1, 2, \ldots, i$$

satisfy

(3) $$|t_j| = |t'_j| = p^{l_{j-1} + k_j - k_{j-1}} \quad \text{and} \quad \{t'_1, t'_2, \ldots, t'_i\} = U_i\,.$$

First, by the definition of the p-height sequence we have

(4) $$l_i + k_{i+1} - k_i > l_i > l_{i-1} + k_i - k_{i-1}\,, \quad i = 1, 2, \ldots$$

and

(5) $$h_p^G(p^{j - l_j + k_j} a_0) = j\,, \quad k_i \leqq j - l_i + k_i < k_{i+1}\,.$$

Now for $t'_{i+1} = p^{l_{i+1} - l_i - k_{i+1} + k_i} a_{i+1} - a_i$ we have

(6) $$t'_{i+1} = x + t_{i+1}\,, \quad x \in U_i\,, \quad t_{i+1} \in T'\,.$$

If $p^j t_{i+1} = 0$ for some $j < l_i + k_{i+1} - k_i$ then we can suppose that $j \geqq l_i$ and we have $p^j t_{i+1} = p^j t'_{i+1} = 0$ by (3) and (4) and consequently $p^{l_{i+1} - l_i - k_{i+1} + k_i + j} a_{i+1} = p^j a_i = p^{j - l_i + k_i} a_0$ which contradicts (5). Thus (3) holds for $j = i + 1$.

Further, if $h_p^G(p^j t_{i+1}) = j + s$, $s > 0$ for some $j < l_i + k_{i+1} - k_i$ then we can clearly assume $j \geqq l_i$ and we have $p^{l_{i+1} - l_i - k_{i+1} + k_i + j} a_{i+1} - p^j t_{i+1} = p^j a_i = p^{j - l_i + k_i} a_0$, which contradicts (5). Thus $\{t_{i+1}\}$, being pure in T', is a direct summand of T' and the assertion follows easily.

Lemma 5. *Let the hypotheses of Lemma 4 hold and let S be a basic subgroup of T containing U as a direct summand, $S = U \dotplus V$. If $H = \{S, a_0, a_2, \ldots\}^G_{\pi \div p}$ then $H \cap T = S$ and $H = V \dotplus \{a_0, a_1, \ldots\}^G_{\pi \div p}$.*

Proof. If $g \in \{a_0, a_1, \ldots\}^G_{\pi \div p} \cap T$ then for a suitable integer m with $(m, p) = 1$ it is $mg = \sum_{i=0}^{n} \lambda_i a_i$. Multiplying by p^{l_n} we get $\left(\sum_{i=0}^{n} \lambda_i p^{l_n - l_i + k_i}\right) a_0 \in T$ which yields $\sum_{i=0}^{n} \lambda_i p^{l_n - l_i + k_i} = 0$. However, by (4), for every $i = 0, 1, \ldots, n - 1$ we have $l_n - l_{i+1} - k_n + k_{i+1} < l_n - l_i - k_n + k_i$ and so $\lambda_n = p^{l_n - l_{n-1} - k_n + k_{n-1}} \lambda'_n$ and $mg = \lambda'_n(p^{l_n - l_{n-1} - k_n + k_{n+1}} a_n - a_{n-1}) + \sum_{i=0}^{n-1} \mu_i a_i \in U$ by induction, since the case $n = 0$ is trivial. Now the assertions follow without difficulties.

Lemma 6. *Let p be a prime and G a mixed group with a p-primary torsion part T. Further, let $a_0 \in G \dot{-} T$ be such that $h_p^{\bar{G}}(\bar{a}_0) = \infty$ and let $\{k_i, l_i\}_{i=0}^{\infty}$ be its p-height* *sequence with* $l_{m-1} < \infty$, $l_m = \infty$ *for some* $m \in N$. *If* $p^{l_i}a_i = p^{k_i}a_0$, $i = 1, 2, \ldots$ $\ldots, m-1$ *then there are elements* $a_m, a_{m+1}, \ldots$ *in* $G \dot{-} T$ *and a direct decomposition* $T = U \dotplus V$ *of* T *such that* $U = \sum_{i=2}^{m}{}_d \{t_i\} = \{p^{l_{m-1}+k_m-k_{m-1}}a_m - a_{m-1}, p^{l_{i+1}-l_i-k_{i+1}+k_i}a_{i+1} - a_i, i = 1, 2, \ldots, m-2\}$ *and* $\{p^{l_{m-1}+k_m-k_{m-1}}a_{m+i+1} - a_{m+i}, i = 0, 1, \ldots\} \subseteq V$.

Proof. Put $s = l_{m-1} + k_m - k_{m-1}$. Since $p^{k_m}a_0$ is of infinite p-height, we can choose elements a'_{m+i}, $i = 0, 1, \ldots$ such that $p^{(i+2)s}a'_{m+i} = p^{k_m}a_0$. Repeating the arguments of the proof of Lemma 4 one proves easily the existence of U generated by the elements t'_j from (2), $j = 2, \ldots, m-1$ and $t'_m = p^s a'_m - a_{m-1}$ such that $T = U \dotplus V$ and $p^s U = 0$. Now $p^s a'_{m+i+1} - a'_{m+i} = u_{m+i} + v_{m+i}$, $u_{m+i} \in U$, $v_{m+i} \in V$, $i = 0, 1, \ldots$. Setting

$$a_{m+i} = a'_{m+i} + u_{m+i} \tag{7}$$

we get $p^s a_{m+i+1} - a_{m+i} \in V$, $i = 0, 1, \ldots$ and $p^s a_m - a_{m-1} = t'_m$ owing to $p^s U = 0$.

Lemma 7. *Let* G *be of rank one and let the hypotheses of Lemma 6 be satisfied. If* $H = \{V, a_m, a_{m+1}, \ldots\}^G_{\pi \div p}$ *where* a_{m+i} *are the elements* (7) *then* $G = U \dotplus H$.

Proof. To prove $U \cap H = 0$, it clearly suffices to show that $mg = \sum_{i=1}^{l} \lambda_{m+i}a_{m+i} \in T$, $(m, P) = 1$ implies $g \in V$. But $p^{(l+2)s}mg = \left(\sum_{i=1}^{l} \lambda_{m+i}p^{k_m+(l-i)s}\right)a_0$ yields $\lambda_{m+1} = p^s\lambda'_{m+1}$, hence $mg = \lambda'_{m+1}(p^s a_{m+l} - a_{m+l-1}) + \sum_{i=1}^{l-1} \mu_i a_{m+i}$ and the induction can be applied.

Let $g \in G$ be arbitrary. Then $\varrho\bar{g} = \sigma\bar{a}_0$ for some integers ϱ, σ with $(\varrho, \sigma) = 1$. Suppose that $\varrho = p^k\varrho'$, $(\varrho', p) = 1$. There exist integers l, i such that $\bar{a}_0 = p^{k+l}\bar{a}_{m+i}$. Hence $\varrho' g = p^l\sigma a_{m+i} + t$ and consequently $\varrho' x = p^l\sigma a_{m+i}$ and $\varrho' y = a_{m+i}$ for some $x, y \in H$, since T is p-primary and $(\varrho', p^l\sigma) = 1$. Thus $g = p^l\sigma y + u + v \in U + H$, $u \in U$, $v \in V$.

Lemma 8. *Let* p *be a prime and* G *a mixed group of rank one with a* p*-primary torsion part* T *and* G p*-reduced. Let* $\{k_i, l_i\}_{i=0}^{\infty}$ *be the* p*-height sequence of an element* $a_0 \in G \dot{-} T$ *such that* $l_m - k_m = l = h_p^{\bar{G}}(\bar{a}_0) > l_{m-1} - k_{m-1}$ *for some integer* m. *If* $p^{l_i}a_i = p^{k_i}a_0$, $i = 1, 2, \ldots m$ *then* G *decomposes into* $G = U \dotplus V \dotplus \{a_m\}^G_{\pi \div p}$ *where* $U \dotplus V = T$ *and* $U = \sum_{i=2}^{m}{}_d \{t_i\} = \{p^{l_i - l_{i-1} - k_i + k_{i-1}}a_i - a_{i-1}, i = 2, \ldots, m\}$.

Proof. The decomposition $T = U \dotplus V$ can be proved by the methods used in the proof of Lemma 4. Further, $p^{l_m}a_m = p^{k_m}a_0$ yields $p^{l_m-k_m}\bar{a}_m = p^l\bar{a}_m = \bar{a}_0$ and $h_p^{\bar{G}}(\bar{a}_m) = 0$. So if $g \in G$ is arbitrary then $\varrho\bar{g} = \sigma\bar{a}_m$, $(\varrho, \sigma) = 1$, $(\varrho, p) = 1$ and $g \in T + \{a_m\}^G_{\pi \div p}$ similarly as in the proof of the preceding lemma. Now the assertion follows easily.

Lemma 9. *Let the hypotheses of Lemma* 4, (*Lemma* 6 *and Lemma* 8 *respectively*) *be satisfied and let* t'_i, $i = 2, 3, \dots$ ($i = 2, 3, \dots, m$, *respectively*) *be the elements* (2). Then $h_p^G(p^\alpha(t'_i \otimes \dots \otimes t'_i)) = h_p^G(p^\alpha(t_i \otimes \dots \otimes t_i)) = \alpha$ *for every* $\alpha < l_{i-1} + k_i - k_{i-1}$ (t_i *is the element given by* (6)).

Proof. It follows from the proof of Lemma 4 that the elements t'_i and $t'_i \otimes \dots \otimes t'_i$ are of the same order. Assuming that $p^{\alpha+s}x = p^\alpha(t'_i \otimes \dots \otimes t'_i)$, $s > 0$, $x \in U_i^n$ we obtain $0 \neq p^{l_{i-1}+k_i-k_{i-1}-1}(t'_i \otimes \dots \otimes t'_i) = p^{l_{i-1}+k_i-k_{i-1}-1+s}x = 0$, due to $p^{l_{i-1}+k_i-l_{i-1}}U_i = 0$. The rest is easy.

Lemma 10. *Under the hypotheses of the preceding lemma the element* $x_i = p^{n(l_i-l_{i-1}-k_i+k_{i-1})}(a_i \otimes \dots \otimes a_i) - (a_{i-1} \otimes \dots \otimes a_{i-1})$ *is of the order* $p^{l_{i-1}+k_i-k_{i-1}}$ ($i \leqq m$ *if the hypotheses of Lemmas* 6 *and* 8 *are assumed*).

Proof. If p^α is the order of x_i then $\alpha \leqq l_{i-1} + k_i - k_{i-1}$ by Lemma 3. Suppose that the strict inequality holds and put $\beta = l_i - l_{i-1} - k_i + k_{i-1} + \alpha < l_i$, $\beta > \alpha$. Then $p^\alpha(a_{i-1} \otimes \dots \otimes a_{i-1}) = p^\alpha((p^{\beta-\alpha}a_i - t'_i) \otimes \dots \otimes (p^{\beta-\alpha}a_i - t'_i)) = p^\beta u + (-1)^n p^\alpha(t'_i \otimes \dots \otimes t'_i)$, $u \in G^n$ and consequently $p^\alpha x_i = p^{n(\beta-\alpha)+\alpha} \times (a_i \otimes \dots \otimes a_i) - p^\beta u - (-1)^n p^\alpha(t'_i \otimes \dots \otimes t'_i) = 0$. Hence $h_p^G(p^\alpha(t'_i \otimes \dots \otimes t'_i)) > \alpha$ which contradicts Lemma 9.

Lemma 11. *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, $0 \to K \xrightarrow{\lambda} L \xrightarrow{\mu} M \to 0$ *be pure exact sequences with* A, K *p-primary and* C, M *p-divisible. Then*

(i) $0 \to A \otimes K \to B \otimes L \to C \otimes M \to 0$ *is exact,*

(ii) $0 \to A^n \to B^n \to C^n \to 0$ *is exact*

for every positive integer n.

Proof. By [4], Theorem 60.4 we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \otimes K & \xrightarrow{\alpha\otimes 1_K} & B \otimes K & \xrightarrow{\beta\otimes 1_K} & C \otimes K & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A\otimes\lambda} & & \downarrow{\scriptstyle 1_B\otimes\lambda} & & \downarrow{\scriptstyle 1_C\otimes\lambda} & & \\
0 & \longrightarrow & A \otimes L & \xrightarrow{\alpha\otimes 1_L} & B \otimes L & \xrightarrow{\beta\otimes 1_L} & C \otimes L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A\otimes\mu} & & \downarrow{\scriptstyle 1_B\otimes\mu} & & \downarrow{\scriptstyle 1_C\otimes\mu} & & \\
0 & \longrightarrow & A \otimes M & \xrightarrow{\alpha\otimes 1_M} & B \otimes M & \xrightarrow{\beta\otimes 1_M} & C \otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and (i) follows easily since $A \otimes M = C \otimes K = 0$. Further, (ii) follows from (i) by induction.

Lemma 12. *Let G be a mixed group of rank one with a p-primary torsion part T and $\bar{G}$ p-divisible. Further, let $\{k_i, l_i\}_{i=0}^{\infty}$ be the p-height sequence of an element $a_0 \in G \dot{-} T$ such that $l_i \neq 0$, $i = 1, 2, \dots$. If $p^{l_i} a_i = p^{k_i} a_0$, $i = 1, 2, \dots$, $H = \{S, a_0, a_1, \dots\}^G_{\pi \div p}$ is the group from Lemma 5 and $\beta : H \to G$ is the canonical embedding then $\beta^n : H^n \to G^n$ is an isomorphism for every $n \geqq 2$.*

Proof. Consider the commutative diagram with canonical maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \xrightarrow{\varrho} & H & \xrightarrow{\eta} & H/S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & T & \xrightarrow{\lambda} & G & \xrightarrow{\mu} & G/T & \longrightarrow & 0\,.
\end{array}$$

γ is monic by Lemma 5. Let $g \in G$ be arbitrary. Then $\varrho\bar{g} = \sigma\bar{a}_0$ for some integers ϱ, σ with $(\varrho, \sigma) = 1$. Suppose that $\varrho = p^k\varrho'$, $(\varrho', p) = 1$, $\bar{a}_0 = p^{k+l}\bar{a}_i$ (such l and i exist since $l_n - k_n > l_{n-1} - k_{n-1}$ by (4)). Therefore $\varrho'\bar{g} = p^l\sigma\bar{a}_i$ and $\varrho' g = p^l\sigma a_i + t$, $t \in T$. But T is p-primary and $(\varrho', p) = 1$, so that $t = \varrho' t'$. Thus $\varrho' x = p^l\sigma a_i$ and $\varrho' y = a_i$ for some $x, y \in G$, since $(J', p^l\sigma) = 1$. Then $y \in H$ and $g = p^l\sigma y + t'$. It follows now that γ is an isomorphism, for $\gamma(p^l\sigma y + S) = p^l\sigma y + T = g + T$. Especially, H/S is p-divisible. Lemma 11 now yields the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^n & \xrightarrow{\varrho^n} & H^n & \xrightarrow{\eta^n} & (H/S)^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha^n} & & \downarrow{\scriptstyle \beta^n} & & \downarrow{\scriptstyle \gamma^n} & & \\
0 & \longrightarrow & T^n & \xrightarrow{\lambda^n} & G^n & \xrightarrow{\mu^n} & (G/T)^n & \longrightarrow & 0\,,
\end{array}$$

where γ^n is an isomorphism. Moreover, α^n is an isomorphism again by Lemma 11, since S is a basic subgroup of T. Consequently, β^n is an isomorphism by "Five Lemma".

Lemma 13. *Let U_i, $i = 0, 1, \dots$ be p-reduced torsion-free groups of rank one of the same type $\bar{\tau}$, $U = \sum_{i=0}^{\infty}{}_d\, U_i$ and let $a_i \in U_i$, $i = 0, 1, \dots$ be such elements that $h_p^{U_i}(a_i) = 0$, $i = 1, 2, \dots$. If $V = \{p^{l_i} a_i - a_0,\ l_{i+1} \geqq l_i,\ i = 1, 2, \dots\}^G_{\pi \div p}$ then no non-zero element of U/V has a p-sequence.*

Proof. Throughout this proof let $\bar{u}$ denote the canonical image of u in U/V. Suppose first, that $\{\bar{h}_i\}_{i=0}^{\infty}$ is a p-sequence of $p^r\bar{a}_0$. It is easily seen that $m\bar{h}_1 = \sum_{i=0}^{l} \lambda_i^{(1)}\bar{a}_i$ for a suitable integer m with $(m, p) = 1$ and so

$$(8) \qquad p\sum_{i=0}^{l} \lambda_i^{(1)} a_i - mp^r a_0 = \sum_{i=1}^{l} \mu_i(p^{l_i} a_i - a_0), \quad \mu_k^{(1)} \neq 0, \quad \mu_j^{(1)} = 0, \quad j = k+1, \dots, 1.$$

We can clearly assume that $m\bar{h}_j = \sum_{i=0}^{l} \lambda_i^{(j)} a_i$, $j = 1, 2, \dots, l_k + 1$. Then $\sum_{i=0}^{l} (p\lambda_i^{(j+1)} - \lambda_i^{(j)}) a_i = \sum_{i=1}^{l} \mu_i^{(j+1)}(p^{l_i} a_i - a_0)$, $j = 1, 2, \dots, l_k$ and hence

$$(9) \qquad p\lambda_0^{(1)} - mp^r = -\sum_{i=1}^{l} \mu_i^{(1)},$$

$$p\lambda_i^{(1)} = p^{l_i}\mu_i^{(1)}, \quad i = 1, 2, \dots, l,$$

$$p\lambda_i^{(j+1)} - \lambda_i^{(j)} = p^{l_i}\mu_i^{(j+1)}, \quad i = 1, 2, \dots, l, \quad j = 1, 2, \dots, l_k.$$

Now for every $i = 1, 2, \dots, k$ we have $p^{l_i+1}\lambda_i^{(l_i+1)} = p^{l_i}(p^{l_i}\mu_i^{(l_i+1)} + \lambda_i^{(l_i)}) = p^{2l_i}\mu_i^{(l_i+1)} + p^{l_i-1}(p^{l_i}\mu_i^{(l_i)} + \lambda_i^{(l_i-1)}) = \dots = \sum_{j=0}^{l_i} p^{2l_i-j}\mu_i^{(l_i+1-j)}$, consequently $p \mid \mu_i^{(1)}$, which contradicts (9) and $(m, p) = 1$ in the case $r = 0$. If $r > 0$ then (8) yields $\bar{h}_1 = p^{r-1}\bar{a}_0$ and the assertion follows by induction.

The general case reduces to the above one, for U/V is of rank one and every two non-zero elements of such a group have a non-zero common multiple.

Lemma 14. *Let U_i, U and a_i be the same as in the preceding lemma, $V = \{p^l a_1 - a_0, p^{l_i} a_i - p^{l_1} a_1, l_{i+1} \geqq l_i, l \geqq l_1, i = 1, 2, \dots\}_{\pi \div p}^{U}$. Then no non-zero element of U/V has a p-sequence.*

Proof. Consider the endomorphism φ of U induced by $\varphi(a_1) = p^{h^U_p(a_0)} a_1 = a_1'$, $\varphi(a_0) = p^l a_1'$, $\varphi(a_i) = a_i$, $i = 2, 3, \dots$. It is easy to see that φ induces an epimorphism $\bar{\varphi} : U/V \to U'/V'$ where $U' = \operatorname{Im} \varphi$ and $V' = \{p^{l_i} a_i - p^{l_1} a_1'\}_{\pi \div p}^{U}$. Now it suffices to use Lemma 13.

Lemma 15. *Let U_i, U and a_i be the same as in Lemma 13, $V = \{p^{r_i} a_i - p^{s_{i-1}} a_{i-1}, i = 1, 2, \dots, s_0 = 0\}_{\pi \div p}^{U}$. Then $h_p^{U/V}(a_0 + V) = r = \max\{\sum_{i=1}^{n} (r_i - s_i) + r_{n+1}, n = 0, 1, \dots, k\}$ provided one of the following conditions holds:*

(i) $\sum_{i=1}^{n} (r_i - s_i) \geqq 0$ *for all* $n = 1, 2, \dots, k$ *and* $\sum_{i=1}^{k+1} (r_i - s_i) < 0$,

(ii) $\sum_{i=1}^{n} (r_i - s_i) \geqq 0$ *for all* $n = 1, 2, \dots$ *and* $r_i = s_i = s$, $i = k + 1, \dots$.

Proof. Considering the equality $p^{r+1}\sum_{i=0}^{n}\lambda_i a_i - q a_0 = \sum_{i=1}^{m}\mu_i(p^{r_i}a_i - p^{s_{i-1}}a_{i-1})$, $m > k+1$, $(q, p) = 1$, we get $p^{r+1}\lambda_0 - q = -\mu_1$, $p^{r+1}\lambda_i = \mu_i p^{r_i} - \mu_{i+1}p^{s_i}$, $i = 1, 2, \dots, m-1$. Now the assumption

$$(10) \qquad p^{1+\sum_{i=1}^{k}(r_i - s_i)} \mid \mu_{k+1}$$

leads to a contradiction, since then $p^{1+\sum_{i=1}^{k-1}(r_i-s_i)} \mid \mu_k, \dots, p^{1+r_1-s_1} \mid \mu_2, p \mid \mu_1$.

However, if (i) holds, then $\sum_{i=1}^{k+1}(r_i - s_i) = \sum_{i=1}^{k}(r_i - s_i) + r_{k+1} - s_{k+1} < 0$ yields

$$s_{k+1} - r_{k+1} > \sum_{i=1}^{k}(r_i - s_i)$$

and the equality $\mu_{k+1} = p^{(r-r_{k+1}+1)}\lambda_{k+1} - p^{(s_{k+1}-r_{k+1})}\mu_{k+2}$ implies (10). If (ii) is satisfied, $p^{r+1}\lambda_m = p^{r_m}\mu_m$ gives $p^{1+\sum_{i=1}^{m-1}(r_i-s_i)} \mid \mu_m$, hence $p^{1+\sum_{i=1}^{m-2}(r_i-s_i)} \mid \mu_{m-1}$. Continuing this process we obtain (10), again.

Lemma 16. *Let U_i, U, a_i, V be the same as in preceding lemma. If $\bar{a}_0 = a_0 + V$ has a p-sequence then the series $\sum_{i=1}^{\infty}(r_i - s_i)$ has all its partial sums non-negative and $\sum_{i=1}^{\infty}(r_i - s_i) = \infty$.*

Proof. The partial sums are non-negative by Lemma 15. Suppose that $\liminf_{n\to\infty} \cdot \{\sum_{i=1}^{n}(r_i - s_i)\} = \alpha < \infty$ and select an increasing sequence $\{k_j\}_{j=0}^{\infty}$ of integers such that $k_0 = 0$, $\sum_{i=1}^{k_j}(r_i - s_i) = \alpha$, $j = 1, 2, \dots$ and $\sum_{i=1}^{m}(r_i - s_i) \geqq \alpha$ for $m \geqq k_1$. Further, denote $\beta_j = \max\{\sum_{i=1}^{m-1}(r_i - s_i) + r_m, k_{j-1} < m \leqq k_j\}$, $j = 1, 2, \dots$.

Now, take the group $K = \sum_{i=0}^{\infty}{}_d K_i$, $K_i \cong U_i$, the elements $b_i \in K_i$ such that $\bigvee_{m=k_j+1}^{k_{j+1}} \tau^U(a_m) \leqq \tau^K(b_{j+1})$, $j = 0, 1, \dots$ and set $L = \{p^{\beta_1}b_1 - b_0, p^{\beta_{j+1}-\alpha}b_{j+1} - p^{\beta_j-\alpha}b^j, j = 1, 2, \dots\}_{\pi\div p}^{K}$. The correspondences $a_0 \mapsto b_0$, $a_m \mapsto p^{\beta_j - r_m - \sum_{i=1}^{m-1}(r_i-s_i)} b_j$, $k_{j-1} < m \leqq k_j$ induce a homomorphism $\varphi : U \to K$. Now $\varphi(p^{r_{m+1}}a_{m+1} - p^{s_m}a_m) = 0$ for $k_{j-1} < m < k_j$ and $\varphi(p^{r_{k_j+1}}a_{k_j+1} - p^{s_{k_j}}a_{k_j}) = p^{\beta_{j+1}-\alpha}b_{j+1} - p^{\beta_j-\alpha}b_j \in L$, $j = 1, 2, \dots$, and $\varphi(p^{r_1}a_1 - a_0) = p^{\beta_1}b_1 - b_0$. Thus φ induces a homomorphism $\bar{\varphi} : U/V \to K/L$, $\bar{\varphi}(a_0 + V) = b_0 + L = \bar{b}_0$. If β_j are bounded then the sequence $\{k_j\}_{j=0}^{\infty}$ can be obviously chosen in such a way that $\beta_2 = \beta_3 = \dots$ and Lemma 15 yields a contradiction. If β_j are not bounded then it is easily seen that the sequence $\{k_j\}_{j=0}^{\infty}$ can be chosen so that $\beta_{j+1} > \beta_j$. However, $L = \{p^{\beta_1}b_1 - b_0, p^{\beta_{j+1}-\alpha}b_{j+1} - p^{\beta_j-\alpha}b_1, j = 1, 2, \dots\}_{\pi\div p}^{K}$ and Lemma 12 yields a contradiction.

Definition 2. Let p be a prime and n an integer, $n > 1$. We say that an element a of a mixed group G has the *(p, n)-property* if for its p-height sequence $\{k_i, l_i\}_{i=0}^{\infty}$ the sequence $\{(n-1)(l_i - k_i) - k_{i+1}\}_{i=0}^{\infty}$ has non-negative elements and $\lim_{i\to\infty} \{(n-1)(l_i - k_i) - k_{i+1}\} = nh_p^{\bar{G}}(\bar{a}) - \lim_{i\to\infty} l_i$, where we put $\infty - m = \infty$ for every $m \in N \cup \{0, \infty\}$.

Theorem. *The following properties are equivalent for a mixed group G of torsion-free rank one:*

(i) *G does not split and $n > 1$ is the smallest integer such that every element $a \in G \dot{-} T$ has a non-zero multiple ma which has the (p, n)-property for every prime p,*

(ii) *G does not split and $n > 1$ is the smallest integer such that $G \dot{-} T$ contains an element a which has the (p, n)-property for every prime p,*

(iii) *G has the splitting length $n > 1$.*

Proof. (i) implies (ii) trivially.

(ii) implies (iii). Suppose that $a \in G \dot{-} T$ has the (p, n)-property for every prime p. Let p be a prime and $\{k_i, l_i\}_{i=0}^{\infty}$ the p-height sequence of a. Put $a_0 = a$, $p^{l_i} a_i = p^{k_i} a_0$, $b_i = a_i \otimes \ldots \otimes a_i$, $r_i = l_i + (n-1)(l_i - l_{i-1} - k_i + k_{i-1})$, $s_i = l_i + k_{i+1} - k_i$, $i = 1, 2, \ldots$. Assume that $h_p^{\bar{G}}(\bar{a}) = l < \infty$. With respect to Definition 1 there is an integer t such that $k_t = k_{t+1} = \ldots$, $l_t = l_{t+1} = \ldots$. It follows from the proof of Lemma 3 that

$$(11) \qquad p^{r_i} b_i = p^{s_{i-1}} b_{i-1}, \quad i = 1, 2, \ldots, t.$$

Further,

$$(12) \qquad \sum_{i=1}^{j} (r_i - s_i) = (n-1)(l_j - k_j) - k_{j+1} \quad \text{and}$$

$$\sum_{i=1}^{t-1} (r_i - s_i) + r_t = n(l_t - k_t).$$

By hypothesis, $\sum_{i=1}^{j} (r_i - s_i) \geqq 0$ for all $j = 1, 2, \ldots, t-1$ and hence

$$(13) \qquad b_0 = p^{r_1} b_1 = p^{r_1 - s_1 + r_2} b_2 = \ldots = p^{\sum_{i=1}^{t-1} (r_i - s_i) + r} b_t$$

so that

$$(14) \qquad h_p^{G^n}(b_0) \geqq n(l_t - k_t).$$

On the other hand, $\lim_{i\to\infty} \{(n-1)(l_i - k_i) - k_{i+1}\} = (n-1)(l_t - k_t) - k_t = nl - l_t$, so that $l = l_t - k_t$. Thus $h_p^{G^n}(b_0) \geqq nl = h_p^{\bar{G}^n}(\bar{b}_0) \geqq h_p^{G^n}(b_0)$ by (14) and [4], § 60 Ex. 9(a).

Now we are going to show that b_0 has a p-sequence for every prime p with $h_p^{\bar{G}}(\bar{a}) = \infty$. First, let $l_t = \infty$ for some t. Then $p^{l_{t-1}+k_t-k_{t-1}}a_{t-1} = p^{k_t}a_0$ is of infinite p-height and $p^{l_{t-1}+k_t-k_{t-1}}b_{t-1}$ has a p-sequence by Lemma 6. However, $b_0 = p^{n(l_{t-1}-k_{t-1})}b_{t-1}$ by (13) and (12) and $(n-1)(l_{t-1}-k_{t-1}) - k_t \geqq 0$ yields $l_{t-1} + k_t - k_{t-1} \leqq n(l_{t-1} - k_{t-1})$, which shows that b_0 has a p-sequence.

Finally, if $l_t < \infty$ for all $t \in N$ then Lemma 11 yields the relations (11) between b_t, $t = 0, 1, \ldots$ and the (p, n)-property of a together with (12) and Lemma 5 imply the existence of a p-sequence of b_0.

Thus G^n satisfies Conditions (α), (β) and G^n splits by [1], Theorem 2.

To complete the proof of Theorem it suffices to show that if G^n splits then every element $a \in G \dot{-} T$ has a non-zero multiple having the (p, n)-property for every prime p.

(iii) implies (i). Assume that G^n splits and let $a' \in G \dot{-} T$ be arbitrary. Due to [1], Lemmas 1–4 and Theorem 2 a' has a non-zero multiple $a = ma'$ such that for the element $g = a \otimes \ldots \otimes a$, $\tau^{G^n}(g) = \tau^{\bar{G}^n}(\bar{g})$ and g has a p-sequence for every prime p such that $\bar{G}$ is p-divisible.

Let p be a prime. If $G \xrightarrow{\beta} G/T_{\pi \div p} \to 0$ is the canonical projection then it follows from [4], Corollary 60.3 that $\operatorname{Ker} \beta^n \subseteq T(G^n)$ and we can suppose that T is p-primary.

Let $\bar{G}$ be p-divisible, let $\{k_i, l_i\}_{i=0}^{\infty}$ be the p-height sequence of a and assume that $l_i < \infty$ for all $i = 1, 2, \ldots$, $p^{l_i}a_i = p^{k_i}a_0 = a$. It follwos from Lemmas 10 and 5 that we can suppose that $G = \{a_0, a_1, \ldots\}_{\pi \div p}^{G}$. Using Lemmas 3, 10 and factorizing G^n by $\{p^{\sum_{r=1}^{n}(l_j - k_j + k_{i_r} - l_{i_r})}(a_j \otimes \ldots \otimes a_j) - (a_{i_1} \otimes \ldots \otimes a_{i_n}),\ i_1, \ldots, i_n,\ j \in N,\ j \geqq \max\{i_1, \ldots, i_n\} > \min\{i_1, \ldots, i_n\}\}$ we obviously obtain a group isomorphic to U/V where U, V are of the form from Lemma 15 with $r_i = l_i + (n-1)(l_i - l_{i-1} - k_i + k_{i-1})$, $s_i = l_i + k_{i+1} - k_i$. Now (12) and Lemma 16 show that a has the (p, n)-property.

Let $\bar{G}$ be p-divisible, let $\{k_i, l_i\}_{i=0}^{\infty}$ the p-height sequence of a such that $l_{m-1} < \infty$ and $l_m = \infty$. By Lemma 7, G decomposes into $G = U \dot{+} H$. Multiplying $t_i' = p^{l_i - l_{i-1} - k_i + k_{i-1}}a_i - a_{i-1}$, $i = 2, \ldots, m-1$, by $p^{l_{i-1}-k_{i-1}}$ and $t_m' = p^s a_m - a_{m-1}$, $s = l_{m-1} + k_m - k_{m-1}$, by $p^{l_{m-1}-k_{m-1}}$ we get $a = p^{2s-k_m}a_m - t$, where $t = \sum_{i=2}^{m} p^{l_{i-1}-k_{i-1}}t_i'$. Moreover, it follows from the proof of Lemma 6 (see (6) in the proof of Lemma 4) that $t_i' = t_i + \sum_{j=2}^{i-1} \lambda_j^{(i)} t_j$ and consequently $t = \sum_{i=2}^{m} p^{l_{i-1}-k_{i-1}}(1 + \sum_{j=i+1}^{m} p^{l_{j-1}-k_{j-1}}\lambda_i^{(j)})\, t_i = \sum_{i=2}^{m} p^{l_{i-1}-k_{i-1}}\alpha_i t_i$, $(\alpha_i, p) = 1$. Now $g = a \otimes \ldots \otimes a = (p^{2s-km}a_m \otimes \ldots \otimes p^{2s-km}a_m) + \ldots + (-1)^n (t \otimes \ldots \otimes t)$ and hence $t \otimes \ldots \otimes t = 0$ since $\otimes$ preserves direct sums and g is of infinite p-height. Consequently, $p^{n(l_{i-1}-k_{i-1})}\alpha_i^n(t_i \otimes \ldots \otimes t_i) = 0$, $i = 2, \ldots, m$, so that $n(l_{i-1} - k_{i-1}) \geqq l_{i-1} + k_i - k_{i-1}$, $t_i \otimes \ldots \otimes t_i$ having the order $p^{l_{i-1}+k_i-k_{i-1}}$. Thus $(n-1)(l_{i-1} - k_{i-1}) - k_i \geqq 0$ and a has the (p, n)-property.

Finally, let G be p-reduced. It is easy to see that G satisfies Conditions (α), (β) and it therefore splits (see also [5], Theorem 2.2), $G = T \dotplus H$. Then $a = t + h$, $t \in T$, $h \in H$, $|t| = p^k$. Further, $h_p^G(h) = h_p^{\bar{G}}(\bar{a}) = l$ yields $h_p^G(p^k a) = h_p^G(p^k h) = k + l$. Thus if $\{k_i, l_i\}_{i=0}^{\infty}$ is the p-height sequence of a then there are j such that $l_j - k_j = l$. Let m be the smallest such integer. By Lemma 8, G decomposes into $G = U \dotplus V \dotplus \{a_m\}_{\pi \div p}^G$. As in the preceding part, $a = p^{l_m - k_m} a_m - t$, $t = \sum_{i=2}^{m} p^{l_{i-1} - k_{i-1}} t_i' = \sum_{i=2}^{m} p^{l_{i-1} - k_{i-1}} \alpha_i t_i$, $(\alpha_i, p) = 1$. However, from Lemma 9 one easily derives (similarly as above) that $p^{n(l_{i-1} - k_{i-1})} \alpha_i^n (t_i \otimes \ldots \otimes t_i) = 0$, $i = 2, \ldots, m$. So $(n-1)(l_{i-1} - k_{i-1}) - k_i \geqq 0$, $i = 2, \ldots, m$. Further, $l_m - k_m > l_{m-1} - k_{m-1}$ yields $0 \leqq n(l_{m-1} - k_{m-1}) - k_m < (n-1)(l_m - k_m) - k_m = nl - l_m$ since $k_m = k_{m+1} = \ldots$, $l_m = l_{m+1} = \ldots$. Thus a has the (p, n)-property and the proof of Theorem is complete.

Example. In [5] it .was shown that the groups A_σ generated by the elements $a_0, a_1, \ldots$ with respect to the relations $p^{(\sigma-1)i} a_i = p^{(\sigma-2)i} a_0$ have the splitting length σ. Clearly, $(n-1)(l_i - k_i) - k_{i+1} = (n-1)i - (\sigma-2)(i+1) = (n - \sigma + 1)i - (\sigma - 2)$ and a_0 has the (p, n)-property if and only if $n \geqq \sigma$.

References

[1] *L. Bican:* Mixed abelian groups of torsion free rank one, Czech. Math. J. *20 (95)*, (1970), 232—242.

[2] *L. Bican:* A note on mixed abelian groups, Czech. Math. J. *21 (96)*, (1971), 413—417.

[3] *L. Fuchs:* Abelian groups, Budapest, 1958.

[4] *L. Fuchs:* Infinite abelian groups I, Academic Press, New York and London, 1970.

[5] *I. M. Irwin, S. A. Khabbaz, G. Rayna:* The role of the tensor product in the splitting of abelian groups, J. Alg. *14*, (1970), 423—442.

Author 's cddress: 186 C0 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).