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Global solution of mixed problem for a certain system of nonlinear conservation laws

GLOBAL SOLUTION OF MIXED PROBLEM FOR A CERTAIN SYSTEM OF NONLINEAR CONSERVATION LAWS*)

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1. INTRODUCTION

The present paper deals with the existence of solution of a certain 2×2 nolinear system of conservation laws on the quadrant $x > 0$, $t > 0$, satisfying initial conditions for $t = 0$, $x > 0$ and a boundary condition for $x = 0$, $t > 0$. As an example of this problem can be taken equations describing the flow of a gas caused by the action of a piston. It is shown that such a mixed problem has a global generalized solution provided the given initial and boundary functions have small variation (see Theorem 2.4). This solution is obtained by means of a difference scheme with a random choice of mesh points. The scheme is similar to that used first by J. GLIMM [1] in the case of the Cauchy problem for systems of conservation laws. As well as in this case the proof of convergence of the scheme is based on a detailed investigation of certain functionals representing the norm and variation of approximate solutions (Section 7). However, there are important modifications since we have to take into account the influence of waves reflected on the boundary (Remark 7.3).

2. FORMULATION OF THE PROBLEM

We shall consider the following 2×2 system of conservation laws:

$$(2.1) \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u, v) = 0 ,$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} g(u) = 0 .$$

Let $F \equiv (f, g) \in C^{(3)}(\Omega_F)$ for a domain $\Omega_F \subset \mathbb{R}^2$; vectors from Ω_F will be denoted by $U = (u, v)$, $U_0 = (u_0, v_0)$, $\tilde{U} = (\tilde{u}, \tilde{v})$, ... We suppose that the system (2.1) is strictly hyperbolic in the sense that

(2.2)
$$f_v(U) < 0 , \quad g_u(u) < 0 \quad \text{for all} \quad U \in \Omega_F$$

(subscripts denote derivatives). Then for each $U \in \Omega_F$ the Jacobi matrix $\mathrm{d}F(U)$ of F has two real distinct eigenvalues $\lambda_1(U) < \lambda_2(U)$. We shall suppose

(2.3)
$$\lambda_1(U) < 0 < \lambda_2(U) ;$$

(this is the case e.g. for $f = f(v)$). The corresponding right eigenvectors of $\mathrm{d}F$ can be written in the form $r^{(1)} = (1, a_1)$, $r^{(2)} = (-1, a_2)$, where $a_i \equiv g_u/\lambda_i$. We shall suppose that (2.1) is genuinely nonlinear in the sense of P. D. Lax [2], i.e.

(2.4)
$$r^{(i)}(U) \operatorname{grad} \lambda_i(U) > 0 , \quad U \in \Omega_F , \quad i = 1, 2 .$$

Let us denote

$$\mathbb{R}_T^2 \equiv \{(x, t) \in \mathbb{R}^2; \ t > T\} \quad \text{for} \quad T \in \mathbb{R} , \quad \Omega \equiv \{(x, t) \in \mathbb{R}_0^2; \ x > 0\} .$$

We shall consider the following problem $(\mathscr{P})$: Given functions $U_0(x), u_1(t)$, find a function $U(x, t)$ satisfying the equations (2.1) on Ω, the initial conditions

(2.5)
$$U(x, 0) = U_0(x) , \quad x > 0 ,$$

and the boundary condition

(2.6)
$$u(0, t) = u_1(t) , \quad t > 0 .$$

Remark 2.1. The role of u and v is symmetric in the sense that instead of (2.6) we can introduce an analogous boundary condition

(2.6*)
$$v(0, t) = u_1(t) , \quad t > 0 .$$

The possibility of introducing one boundary condition follows from general considerations about correctness of boundary conditions for hyperbolic systems by which the number of boundary conditions should be equal to the number of characteristics starting on the boundary. Since we suppose (2.3), this number is one.

The form of the boundary condition is then closely related to the Riemann invariants — the functions $w = w(u, v)$, $z = z(u, v)$ with the property $r^{(1)} \operatorname{grad} w = r^{(2)} \operatorname{grad} z = 0$ (see [2], [4]). The classical solution of (2.1) is equivalent to the solution of the system

(2.7)
$$z_t + \lambda_1 z_x = 0 , \quad w_t + \lambda_2 w_x = 0 .$$

Since z is constant along the first characteristics by (2.7) and since $\lambda_1 < 0$, the values $z(0, t)$ are determined by the given initial values $z(x, 0)$. On the other hand w is constant along the second characteristics and since $\lambda_2 > 0$, we can prescribe the

values of w for $x = 0$. From these facts we can deduce the symmetry of u and v: for a given $u(0, t)$ or $v(0, t)$ it suffices to find respectively $v(0, t)$ or $u(0, t)$) such that $z(u(0, t), v(0, t)) = z(0, t)$; then we can determine the correct boundary condition for $w : w(0, t) = w(u(0, t), v(0, t))$. ∎

Remark 2.2. If F has the special form

$$F(u, v) = (kv^{-\gamma}, -u), \quad k > 0, \quad \gamma > 1,$$

then (2.1) is the system of gas dynamics describing the one-dimensional isentropic flow of an ideal polytropic gas in Lagrange's coordinates (see e.g. [4]; u means the velocity and v the specific volume) and F satisfies (2.2)−(2.4) on $\Omega_F = \{(u, v); v > 0\}$. In this case the mixed problem $(\mathscr{P})$ has the following physical interpretation: the piston moves by a given speed u_1 into the gas with the initial state u_0, v_0 and we seek for the resulting flow of the gas. The condition (2.6) would mean that we know the specific volume on the piston, i.e. the pressure on the piston, because $p = kv^{-\gamma}$. ∎

Since, as well as in the case of the Cauchy problem, the nonlinear character of equations (2.1) gives rise to discontinuities of the solution even for smooth U_0, u_1, we shall turn to the generalized solution:

Definition 2.3. Let $U_0 \in [L_{1,loc}\langle 0, \infty)]^2$, $u_1 \in L_{1,loc}\langle 0, \infty)$ and let

$$(2.8) \qquad C_0^2 \equiv \{\varphi \in C^{(2)}(\mathbf{R}^2); \text{ sup } \varphi \text{ is compact}\},$$

$$(2.9) \qquad C_{00}^2 \equiv \{\psi \in C_0^2; \text{ supp } \psi \subset \{(x, t) \in \mathbf{R}^2; \ x > 0\}\}.$$

A generalized solution of the problem $(\mathscr{P})$ is any function $U \in [L_{1,loc}(\Omega)]^2$ such that $F(U) \in [L_{1,loc}(\Omega)]^2$ and

$$(2.10) \qquad \int_\Omega [\psi_t u + \psi_x f(U)] \, dx \, dt + \int_0^\infty \psi(x, 0) \, u_0(x) \, dx = 0, \quad \psi \in C_{00}^2,$$

$$(2.11) \qquad \int_\Omega [\varphi_t v + \varphi_x g(u)] \, dx \, dt + \int_0^\infty \varphi(x, 0) \, v_0(x) \, dx +$$

$$+ \int_0^\infty \varphi(0, t) \, g(u_1(t)) \, dt = 0, \quad \varphi \in C_0^2. \qquad ∎$$

It is evident that the classical $C^{(1)}$-solution of $(\mathscr{P})$ is also a generalized solution. On the other hand if U is a generalized solution of $(\mathscr{P})$ such that $U \in C^{(1)}$, a standard argument yields that (2.1), (2.5) hold and, further, $g(u_1(t)) = g(u(0, t))$, $t > 0$. But then (2.6) immediately follows from (2.2) and so U is a classical solution, too.

In what follows we shall use the following notation:

for a vector $V = (V_1, ..., V_N) \in \mathbf{R}^N$, let $|V| \equiv \sum_{i=1}^N |V_i|$ be its norm;

for a function h, let $\|h\|$ be its L_∞-norm.

The aim of this paper is to prove the following global existence theorem:

Theorem 2.4. *Let* $F = (f, g)$ *satisfy* $(2.2)-(2.4)$. *Then for each* $\tilde{U} \in \Omega_F$ *there exist constants* $d > 0$, $c > 0$ *such that if* $U_0 \in [L_\infty \langle 0, \infty)]^2$, $u_1 \in L_\infty \langle 0, \infty)$ *satisfy*

(2.12)
$$\max \left[\|U_0 - \tilde{U}\|, \|u_1 - \tilde{u}\|, \operatorname*{Var}_{\langle 0, \infty)} U_0, \operatorname*{Var}_{\langle 0, \infty)} u_1 \right] \leq d,$$

then there exists a generalized solution $U \in [L_\infty(\Omega)]^2$ *of the problem* $(\mathscr{P})$. *This solution satisfies the inequalities*

(2.13)
$$\|U - \tilde{U}\| \leq c \left(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| + \operatorname*{Var}_{\langle 0, \infty)} U_0 + \operatorname*{Var}_{\langle 0, \infty)} u_1 \right) ;$$

(2.14)
$$\operatorname*{Var}_{\langle 0, \infty)} U(\cdot, t) \leq c \left(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| + \operatorname*{Var}_{\langle 0, \infty)} U_0 + \operatorname*{Var}_{\langle 0, \infty)} u_1 \right)$$
$$\text{for each} \quad t \geq 0 ;$$

(2.15)
$$\int_0^\infty |U(x, t_1) - U(x, t_2)| \, dx \leq$$

$$\leq c |t_1 - t_2| \left(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| + \operatorname*{Var}_{\langle 0, \infty)} U_0 + \operatorname*{Var}_{\langle 0, \infty)} u_1 \right)$$
$$\text{for each} \quad t_1 \geq 0, \quad t_2 \geq 0.$$

3. AUXILIARY RESULTS

Our construction of approximate solutions is based on simple and shock waves which are particular solutions of the so called initial Riemann problem:

$$U_t + F(U)_x = 0 \quad \text{for} \quad x \in \mathbf{R}, \quad t > 0,$$
$$U(x, 0) = U_L \quad \text{for} \quad x < 0, \quad U(x, 0) = U_R \quad \text{for} \quad x > 0,$$

where U_L, U_R are given constant vectors from Ω_F. We shall refer to this problem briefly as to $IRP(U_L, U_R)$.

The i-th simple rarefaction wave (centered at the origin), $i = 1, 2$, is such a continuous solution of $IRP(U_L, U_R)$ that the i-th Riemann invariant is constant in $\mathbf{R}_0^2$. The i-th shock wave, $i = 1, 2$, is any piece-wise constant solution of $IRP(U_L, U_R)$ with a discontinuity at a half-line $x = \sigma t$, $t > 0$, satisfying the jump conditions

$$\lambda_i(U_R) < \sigma < \lambda_i(U_L), \quad \lambda_{i-1}(U_L) < \sigma < \lambda_{i+1}(U_R).$$

(The solutions are of course taken in the generalized sense.) Both rarefaction and shock wave shall be called briefly waves. The waves are functions of x/t (the so called automodel functions); the i-th rarefaction waves are constant outside the sector

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$\left\{(x, t) \in \mathbf{R}_0^2; \lambda_i(U_L) < x/t < \lambda_i(U_R)\right\}$ (for a more detailed study of the waves see [2], [3], [4]). The following "wave parametrization" theorem is known (see [3]):

Theorem 3.1. *For each* $U_0 \in \Omega_F$ *there exist curves* $V_i(U_0, u)$, $\overline{V}_i(U_0, u)$, $i = 1, 2$, *defined and once continuously differentiable for u near to* u_0, $V_i(U_0, u_0) = \overline{V}_i(U_0, u_0) = v_0$, *such that* U_0 *and* $(u, V_i(U_0, u))$ (U_0 *and* $(u, \overline{V}_i(U_0, u))$) *can be connected from the left to the right* (*from the right to the left, respectively*) *by an i-th wave; if we define a parameter* α *by the relation*

$$(3.1) \qquad \alpha \equiv \begin{cases} u - u_0 & \text{for } i = 1, \\ u_0 - u & \text{for } i = 2, \end{cases}$$

then for $\alpha \geqq 0$ (*respectively,* $\alpha \leqq 0$) *this wave is a rarefaction one, otherwise it is a shock.*

By means of the curves V_i and the parameter α from (3.1) we shall characterize all waves connecting some vectors from the left to the right; so the *i*-th wave α is a rarefaction one iff $\alpha \geqq 0$ and a shock iff $\alpha < 0$.

Since $\lambda_1 < \lambda_2$, the first wave α_1 connecting U_L and U_M and the second wave α_2 connecting U_M abd U_R can be fitted together to form a solution of $IRP(U_L, U_R)$. We shall say that such a solution is represented by the wave $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2)$ and write $\boldsymbol{\alpha} : U_L \mapsto U_R$; the first (second) wave α can be taken as the "vector-wave" $\boldsymbol{\alpha} = (\alpha, 0)$ $(\boldsymbol{\alpha} = (0, \alpha))$.

Theorem 3.2. *For each* $U_0 \in \Omega_F$ *there exist constants* $d_3(U_0) > 0$, $d_4(U_0) > 0$ *such that for* $U_i \in \Omega_F$, $|U_i - U_0| \leqq d_3(U_0)$, $i = 1, 2$, *there exists a unique solution of* $IRP(U_1, U_2)$ *represented by the wave* $\boldsymbol{\varepsilon}$ *with* $|\boldsymbol{\varepsilon}| \leqq d_4(U_0)$.

For the proof see [2], [4].

Definition 3.3. Let
$$(3.2) \qquad \alpha^+ = \max(0, \alpha), \quad \alpha^- = \min(0, \alpha).$$

For a sequence (finite or not) of waves $\boldsymbol{\varepsilon}^1$, $\boldsymbol{\varepsilon}^2$, ... we define

$$(3.3) \qquad D(\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2, \ldots) \equiv \sum_{i < j} |\varepsilon_2^i| \, |\varepsilon_1^j| + \sum_{k=1}^2 \sum_{i \neq j} \left\{ |(\varepsilon_k^i)^-| \, |\varepsilon_k^j| + |(\varepsilon_k^i)^+| \, |(\varepsilon_k^j)^-| \right\}. \qquad \blacksquare$$

Remark 3.4. In the terminology of approaching waves (see [1]) we can write $D(\boldsymbol{\varepsilon}^1, \ldots)$ as follows:

$$D(\boldsymbol{\varepsilon}^1 . \boldsymbol{\varepsilon}^2, \ldots) = \sum \left\{ |\varepsilon_k^i| \, |\varepsilon_m^j|; \ \varepsilon_k^i \text{ and } \varepsilon_m^j \text{ approach} \right\};$$

so D "measures" the influence of possible interactions between the waves $\boldsymbol{\varepsilon}^i$. (Two waves α, β approach (each other) iff they can interact in a finite time.) $\qquad \blacksquare$

In [1] J. GLIMM has proved an important theorem concerning "composing" of Riemann problems. We give here its slight generalization suitable for our purposes:

Theorem 3.5. *Let* $U_0 \in \Omega_F$ *be fixed and let* $U_i \in \Omega_F$, $|U_i - U_0| \leqq d_3(U_0)$, $i = 1, 2, 3, 4$; $\varepsilon : U_1 \mapsto U_4$, $\alpha^i : U_{i-1} \mapsto U_i$, $i = 2, 3, 4$. *Then*

$$(3.4) \qquad \varepsilon_i = \alpha_i^2 + \alpha_i^3 + \alpha_i^4 + D(\alpha^2, \alpha^3, \alpha^4)\, O(1), \quad i = 1, 2,$$

where the function $O(1)$ *is uniform in the sense that there exists a constant* $K_1(U_0) > 0$ *such that*

$$(3.5) \qquad |O(1)| \leqq K_1(U_0) \quad \text{for all} \quad \alpha^2, \alpha^3, \alpha^4 \quad \text{considered}.$$

"Wave" solutions U of $IRP(U_L, U_R)$ are in general defined a.e. in $\overline{R}_0^2$; if we put $U(x, 0) \equiv U_L$ for $x \leqq 0$, $U(x, 0) \equiv U_R$ for $x > 0$ and $U(x, t) \equiv U(x, t - 0) = \lim_{s \to t - 0} U(x, s)$ on a possible line of discontinuity, U is defined everywhere in $\overline{R}_0^2$.

For two different points $a, b \in R^2$ we denote by $\{a - b\}$ the segment with the endpoints a, b.

Definition 3.6. Let $\{a^1 - a^2\} \subset \overline{R}_0^2$, $a^i = (a_x^i, a_t^i)$, $i = 1, 2$, and let the wave α represent the solution U of $IRP(U_L, U_R)$ for some U_L, U_R. Then the wave $\alpha \cap \{a^1 - a^2\}$ is defined as follows:

If $a_x^1/a_t^1 < a_x^2/a_t^2$ and if α coincides with an i-th wave α, $i \in \{1, 2\}$, then for $\alpha < 0$, set $\alpha \cap \{a^1 - a^2\} \equiv 0$ if $U(a^1) = U(a^2)$ and $\alpha \cap \{a^1 - a^2\} \equiv \alpha$ otherwise;

for $\alpha \geqq 0$ there exists a rarefaction wave $\beta : U(a^1) \mapsto U(a^2)$, $0 \leqq \beta \leqq \alpha$; set $\alpha \cap \{a^1 - a^2\} \equiv \beta$.

Further

$$\alpha \cap \{a^1 - a^2\} \equiv \begin{cases} \alpha \cap \{a^2 - a^1\} & \text{for} \quad a_x^1/a_t^1 > a_x^2/a_t^2, \\ 0 & \text{for} \quad a_x^1/a_t^1 = a_x^2/a_t^2; \end{cases}$$

$$\alpha \cap \{a^1 - a^2\} \equiv (\alpha_1 \cap \{a^1 - a^2\}, \alpha_2 \cap \{a^1 - a^2\}).$$

For $a, b \in \{t = 0\}$ set $\alpha \cap \{a - b\} \equiv \alpha \cap \{\tilde{a} - \tilde{b}\}$, where $\tilde{a}, \tilde{b}$ are such points that $U(\tilde{a}) = U(a)$, $U(\tilde{b}) = U(b)$.

For $\ell = \{a^0 - a^1\} \cup \{a^1 - a^2\}$, $a^i \in \overline{R}_0^2$, $i = 0, 1, 2$, such that $a_x^{i-1}/a_t^{i-1} \leqq a_x^i/a_t^i$, $i = 1, 2$, set $\alpha \cap \ell \equiv \alpha \cap \{a^0 - a^2\}$. ∎

The total variation $\operatorname{Var}_{\langle x, y \rangle} \varphi$ of a function φ over the interval $\langle x, y \rangle \subset R$ is defined as usual, moreover,

$$\operatorname*{Var}_{\langle x, \infty \rangle} \varphi \equiv \sup_{y > x} \operatorname*{Var}_{\langle x, y \rangle} \varphi,$$

For φ defined on $\mathbf{R}^2$, $\{a - b\} \subset \mathbf{R}^2$, let the variation of φ along the segment $\{a - b\}$ be

$$\operatorname*{Var}_{\{a-b\}} \varphi \equiv \operatorname*{Var}_{\langle 0,1\rangle} \varphi^*$$

where

$$\varphi^*(\xi) \equiv \varphi\big(a_x + \xi(b_x - a_x),\, a_t + \xi(b_t - a_t)\big), \quad \xi \in \langle 0, 1\rangle .$$

Lemma 3.7. *Let $U_0 \in \Omega_F$. Then there exist constants $c_i(U_0) > 0$, $i = 1, 2$, $d_8(U_0) \in \langle 0, d_3(U_0)\rangle$ with the following properties: for all $U_i \in \Omega_F$ such that $|U_i - U_0| \leqq d_3(U_0)$, $i = 1, 2$, and for the solution U of $IRP(U_1, U_2)$ given by the wave $\varepsilon : U_1 \mapsto U_2$ it holds*

(3.6) $$\big|U(x, t) - U_i\big| \leqq |\varepsilon|/c_1(U_0), \quad (x, t) \in \mathbf{R}_0^2, \quad i = 1, 2 ;$$

(3.7) $$c_1(U_0)\big|U_1 - U_2\big| \leqq |\varepsilon| \leqq c_2(U_0)\big|U_1 - U_2\big| .$$

If moreover $|U_i - U_0| \leqq d_8(U_0)$, $i = 1, 2$, then

(3.8) $$\big|U(x, t) - U_0\big| \leqq d_3(U_0) \quad \text{for all} \quad (x, t) \in \mathbf{R}_0^2$$

and for each segment $\{a - b\} \subset \overline{\mathbf{R}}_0^2$ we have

(3.9) $$c_1(U_0) \operatorname*{Var}_{\{a-b\}} U \leqq \big|\varepsilon \cap \{a - b\}\big| \leqq c_2(U_0) \operatorname*{Var}_{\{a-b\}} U .$$

We omit the proof since it is not difficult and follows from the definitions and the automodel character of U.

4. MIXED RIEMANN PROBLEM

We can pose a problem analogous to the initial-value Riemann problem: to find a solution U of the system (2.1) on Ω satisfying the initial condition $U(x, 0) = U_0$, $x > 0$, and the boundary condition $u(0, t) = u_1$, $t > 0$, where U_0, u_1 are given constants. In what follows we shall refer to such a problem as to a mixed Riemann problem or briefly $MRP(u_1, U_0)$.

Theorem 4.1. *Let $\tilde{U} \in \Omega_F$. Then there exists a constant $c^*(\tilde{U}) > 0$ such that if $U_0 \in \Omega_F$, $u_1 \in \mathbf{R}$ satisfy $|U_0 - \tilde{U}| \leqq d_3(\tilde{U})$, $|u_1 - \tilde{u}| \leqq d_3(\tilde{U})$ then there exists a solution U of $MRP(u_1, U_0)$ given by the second wave ε and satisfying the inequality*

(4.1) $$\big|v(0, t) - v_0\big| \leqq c^*(\tilde{U}) |\varepsilon|, \quad t > 0 .$$

Proof. Let $U_1 \equiv \big(u_1, \overline{V}_2(U_0, u_1)\big)$. There exists a solution U^* of $IRP(U_1, U_0)$ given by the second wave $\varepsilon = u_0 - u_1$; for U^* we have

$$u^*(0, t) = u_1, \quad t > 0 ; \quad U^*(x, 0) = U_0, \quad x > 0 .$$

Hence it is sufficient to put $U \equiv U^*|_\Omega$. (Geometrically the solution of $MRP(u_1, U_0)$ is represented by the intersection U_1 of the curve $\overline{V}_2(U_0, \cdot)$ with the line $u = u_1$.)

Since $v(0, t) = v_1$ for all $t > 0$, we have

$$\left|v(0, t) - v_0\right| = \left|\overline{V}_2(U_0, u_1) - \overline{V}_2(U_0, u_0)\right|$$

and consequently it suffices to put

$$c^*(\tilde{U}) \equiv \sup \left\{ \left|\frac{\partial \overline{V}_2}{\partial u}(U, \bar{u})\right| ; \quad \left|U - \tilde{U}\right| \leqq d_3(\tilde{U}), \left|\bar{u} - \tilde{u}\right| \leqq d_3(\tilde{U}) \right\}. \qquad \blacksquare$$

In what follows we shall consider only solutions of MRP of the type constructed in the proof of Theorem 4.1. This agreement makes our considerations unambiguous although no uniqueness is established.

Remark 4.2. Since all results from Sections 3,4 are invariant with respect to translation of the origin, we can apply them to the problem

$$U_t + F(U)_x = 0 \quad \text{on} \quad \mathbf{R}_T^2, \quad T \geqq 0 ;$$

$$U(x, T) = U_L \quad \text{for} \quad x < X , \quad U(x, T) = U_R \quad \text{for} \quad x > X , \quad X \in \mathbf{R} .$$

We shall call such a problem $IRP(U_L, U_R)$ with the center (X, T); analogously for MRP and waves arising from these problems. $\qquad \blacksquare$

5. DIFFERENCE SCHEME AND APPROXIMATE SOLUTIONS

In what follows we shall confine ourselves to certain local character of initial and boundary functions in the sense that for certain fixed $\tilde{U} \in \Omega_F$ and sufficiently small constant $d_{50} > 0$ we suppose

(5.1) $$\max \left[\|U_0 - \tilde{U}\|, \|u_1 - \tilde{u}\|, \operatorname*{Var}_{\langle 0, \infty \rangle} U_0, \operatorname*{Var}_{\langle 0, \infty \rangle} u_1 \right] \leqq d_{50} .$$

Let us start from the restriction

(5.2) $$d_{50} \leqq d_8(\tilde{U})$$

and denote $d_3 \equiv d_3(\tilde{U})$, $d_8 \equiv d_8(\tilde{U})$, $c^* \equiv c^*(\tilde{U})$. Further let

$$\lambda_0^* \equiv \sup \left\{ |\lambda_2(U)|; \left|U - \tilde{U}\right| \leqq (1 + c^*) d_3 \right\},$$

$$\lambda_0 \equiv \sup \left\{ |\lambda_j(\Phi(U, \boldsymbol{\alpha}))|; \left|U - \tilde{U}\right| \leqq d_3, |\boldsymbol{\alpha}| \leqq d_4, j = 1, 2 \right\},$$

where

$$\Phi(U, \boldsymbol{\alpha}) \equiv \left(u + \alpha_1 - \alpha_2, V_2((u + \alpha_1, V_1(U, u + \alpha_1)), u + \alpha_1 - \alpha_2) \right).$$

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We fix a constant $q_0 > 0$ such that

(5.3)
$$q_0 > 2 \max\left[\lambda_0, \lambda_0^*\right]$$

and from now on we shall suppose that $r > 0$, $s > 0$ satisfy

(5.4)
$$\frac{r}{s} = q_0$$

so that $s = s(r)$ is a function of r.

Let $N = \{1, 2, \ldots\}$ be the set of all natural numbers, $\overline{N} \equiv \{0\} \cup N$.

Now we define

$$Y \equiv \{(m, n) \in \overline{N}^2;\ m + n \text{ is even}\}$$

and for $(m, n) \in Y$ let $A_{m,n} = A_{m,n}(r) \subset \overline{\Omega}$ be the horizontal line-segment

$$A_{m,n} \equiv \{(x, ns);\ \max\left[0, (m - 1)r\right] \leq x \leq (m + 1)r\}.$$

Finally, let

$$A = A(r) \equiv \prod_{(m,n)\in Y} A_{m,n}(r).$$

An infinite vector $a = \{a_{m,n}\} \in A$ will represent a random choice of mesh points in our difference scheme.

Given $r > 0$ and $a \in A(r)$, the approximate solution $U_{r,a}$ of the mixed problem $(\mathscr{P})$ is constructed as follows:

1° Let

(5.5) $U_{0,r}(x) \equiv U_0(mr)$ for $x \in (\max\left[0, (m - 1)r\right], (m + 1)r\rangle$, $m \in \overline{N}$ even ;

$u_{1,r}(t) \equiv u_1(ns)$ for $t \in \langle(n - 1)s, (n + 1)s\rangle$, $n \in N$ odd ;

and further

(5.6)
$$U_{r,a}(x, 0) \equiv U_{0,r}(x) \quad \text{for} \quad x > 0.$$

Then (5.1) implies

(5.7)
$$\left|U_{r,a}(a_{m,n}) - \tilde{U}\right| \leq d_3$$

for $n = 0$, $m \in \overline{N}$ even.

2° Suppose that for $n \in \overline{N}$ the approximate solution $U_{r,a}$ is defined for $t \leq ns$, $x \geq 0$ and that (5.7) holds for all $m \in \overline{N}$, $m + n$ even. Let us denote

(5.8)
$$m_0 = m_0(n) \equiv \min\{m \in \overline{N};\ (m, n) \in Y\}.$$

By (5.7) and Theorem 3.2, for each $m \geq m_0 + 2$, $(m, n) \in Y$ there exists a solution $U^{m,n}$ of $IRP(U_{r,a}(a_{m-2,n}), U_{r,a}(a_{m,n}))$ with the center $((m - 1)r, ns)$ given by the wave

$$\varepsilon^{m,n} : U_{r,a}(a_{m-2,n}) \mapsto U_{r,a}(a_{m,n})$$

with the same center. We define

(5.9) $U_{r,a}(x, t) \equiv U^{m,n}(x, t)$ for $t \in \langle ns, (n + 1) s\rangle$,

$$x \in \left(\max\left[\frac{r}{2}, (m - 2) r\right], mr\right), \quad m \geqq m_0 + 2.$$

Further, by Theorem 4.1 there exists a solution $U^{m_0,n}$ of $MRP(u_{1,r}(ns), U_{r,a}(a_{m_0,n}))$
with the center $(0, ns)$ given by the second wave $\varepsilon^{m_0,n}$. Then we define

(5.10) $U_{r,a}(x, t) \equiv U^{m_0,n}(x, t)$ for $x \in \left\langle 0, \max\left[\frac{r}{2}, m_0 r\right]\right\rangle$,

$$t \in \langle ns, (n + 1) s\rangle.$$

Relations (5.9), (5.10) define $U_{r,a}$ on the strip

(5.11) $\Pi_n \equiv \{(x, t) \in \bar{\Omega}; \; t \in \langle ns, (n + 1) s\rangle\}$.

3° At the mesh points on the line $t = (n + 1) s$ we define

(5.12) $U_{r,a}(a_{m,n+1}) \equiv U_{r,a}(a'_{m,n+1}, (n + 1) s - 0)$, $(m, n + 1) \in Y$,

where $a'_{m,n}$ denotes the x-coordinate of the point $a_{m,n}$.

4° On the line $t = (n + 1) s$ we set

(5.13) $U_{r,a}(x, (n + 1) s) \equiv U_{r,a}(a_{m,n+1})$ for $x \in (\max[0, (m - 1) r], (m + 1) r\rangle$.

5° Relations (5.9)–(5.13) define $U_{r,a}$ on Π_n and we can repeat our construction
from the item 2° provided (5.7) holds for $n + 1$ and all m such that $m + n + 1$ is
even.

If (5.7) holds for all $(m, n) \in Y$ then the approximate solution $U_{r,a}$ is defined on
the whole quadrant $\bar{\Omega}$; this will be proved for sufficiently small d_{50} in the following
sections. The initial restriction (5.2) implies that $U_{r,a}$ is defined at least for $t \leqq 2s$.

Lemma 5.1. *The waves $\varepsilon^{m,n}$, $(m, n) \in Y$ do not intersect each other in the strip $\bar{\Pi}_n$
and consequently, $U_{r,a}$ is an exact solution of the system* (2.1) *in the strip Π_n.*

Proof of Lemma follows immediately from conditions (5.3), (5.4) and from the
basic properties of rarefaction and shock waves.   ∎

6. $\mathscr{I}$-CURVES

By an $\mathscr{I}$-curve we mean, roughly speaking, an infinite piece-wise linear curve
starting on the line $x = 0$ and connecting the mesh points $a_{m,n}, a_{m+1,\bar{n}}, \bar{n} = n \pm 1$.

Definition 6.1. Let $b_n \equiv \left(0, \left(n + \frac{1}{2}\right) s\right)$, $n \in \overline{N}$. For a given $n \in \overline{N}$ let the first segment of an $\mathscr{I}$-curve $\mathscr{I}$ starting at its origin b_n be one of the following segments: $\{b_n - a_{m_0(n),n}\}$, $\{b_n - a_{m_0(n+1),n+1}\}$ ($m_0(n)$ is defined by (5.8)). If we have defined a part $\mathscr{I}^*$ of $\mathscr{I}$ and

$$m = \max\{\overline{m}; \, a_{\overline{m},n} \in \mathscr{I}^*\}, \quad \text{then the next segment of } \mathscr{I} \text{ is}$$

$$\{a_{m,n} - a_{m+1,n+1}\} \quad \text{or} \quad \{a_{m,n} - a_{m+1,n-1}\}.$$

The mesh points $a_{m,n} \in \mathscr{I}$, $b_n \in \mathscr{I}$ are called vertices of $\mathscr{I}$, the set of all vertices is denoted by $V(\mathscr{I})$.

We shall confine our consideration to $\mathscr{I}$-curves lying in the strip $\overline{\Pi}_n \cup \overline{\Pi}_{n+1}$ for some $n \in \overline{N}$ and denote by $[\mathscr{I}]$ the set of all such curves.

Finally, the first $\mathscr{I}$-curve $\mathcal{O} \in [\mathscr{I}]$ is defined as

$$\mathcal{O} \equiv \{b_0 - a_{0,0}\} \cup \{a_{0,0} - a_{1,1}\} \cup \{a_{1,1} - a_{2,0}\} \cup \ldots, \quad \mathcal{O} \subset \overline{\Pi}_0. \qquad \blacksquare$$

Each $\mathscr{I}$-curve $\mathscr{I} \in [\mathscr{I}]$ divides the domain $\{(x, t) \in R^2; \, x > 0\}$ into two components; we denote $\Omega(\mathscr{I})$ the one of them for which $\Omega(\mathscr{I}) \supset \{(x, t); \, x > 0, \, t < 0\}$.

Definition 6.2. Let $\mathscr{I}, \mathscr{J} \in [\mathscr{I}]$ be two $\mathscr{I}$-curves. We say that $\mathscr{I} \leqq \mathscr{J}$ iff $\Omega(\mathscr{I}) \subset \Omega(\mathscr{J})$ (i.e. $\mathscr{J}$ lies toward the larger time then $\mathscr{I}$). $\qquad \blacksquare$

Lemma 6.3. $1°$ *The set $[\mathscr{I}]$ is partially ordered by the relation $\leqq$.*

$2°$ *The curve $\mathcal{O}$ is the least element of $[\mathscr{I}]$, i.e. $\mathcal{O} < \mathscr{J}$ for all $\mathscr{J} \in [\mathscr{I}]$, $\mathscr{J} \neq \mathcal{O}$.*

$3°$ *The set $[\mathscr{I}]$ satisfies the condition of minimality, i.e. each nonempty subset $B \subset [\mathscr{I}]$ has a minimal element — such a curve $\mathscr{I}_B \in B$ that for $\mathscr{I} \in B$ it is not $\mathscr{I} < \mathscr{I}_B$.*

Proof of the first two items of Lemma follows immediately from the definitions. As to the item $3°$, one of minimal elements of B can be constructed by induction as follows: Let

$$n_0 \equiv \min\{n \in \overline{N}; \, b_n \in \mathscr{I} \in B\}, \quad n_1 \equiv \min\{n \in \overline{N}; \, \{b_{n_0} - a_{m,n}\} \subset \mathscr{I} \in B\};$$

$$\ell_1 \equiv \{b_{n_0} - a_{m_0(n_1),n_1}\}.$$

Evidently $\ell_1 \subset \mathscr{I}$ for some $\mathscr{I} \in B$. Having constructed segments $\ell_i = \{a_{m_i-1,n_i-1} - a_{m_i,n_i}\}$ *), $i = 1, 2, \ldots, k$, we put

$$n_{k+1} \equiv \min\left\{n \in \overline{N}; \, a_{m_k+1,n} \in \mathscr{I} \in B, \, \bigcup_{i=1}^{k} \ell_i \subset \mathscr{I}\right\},$$

$$\ell_{k+1} \equiv \{a_{m_k,n_k} - a_{m_k+1,n_{k+1}}\}.$$

---

*) Here formally $a_{m_1-1,n_0} = b_{n_0}$.

Evidently, the $\mathscr{I}$-curve $\mathscr{I}_B \equiv \bigcup_{i=1}^{\infty} \ell_i$ belongs to B. If $\mathscr{I}_B$ were not a minimal element of B, then there would exist $\mathscr{I}^* \in B$ and $k^* \in \mathbb{N}$ such that $\mathscr{I}^* < \mathscr{I}_B$ and $\bigcup_{i=1}^{k^*} \ell_i \subset \mathscr{I}^*$, $\ell_{k^*+1} \not\subset \mathscr{I}^*$. But this would contradict to the definition of n_{k^*+1}. ∎

Definition 6.4. Let $\mathscr{I}, \mathscr{J} \in [\mathscr{I}]$ be two $\mathscr{I}$-curves. We say that $\mathscr{J}$ is the successor of $\mathscr{I}$ iff one of the following possibilities holds:

(6.1) $V(\mathscr{J}) \setminus V(\mathscr{I}) = a_{m,n}, \quad V(\mathscr{I}) \setminus V(\mathscr{J}) = a_{m,n-2}, \quad (m,n) \in Y;$

(6.2) $V(\mathscr{J}) \setminus V(\mathscr{I}) = \emptyset, \quad V(\mathscr{I}) \setminus V(\mathscr{J}) = a_{0,n} \quad$ or

$\qquad\qquad V(\mathscr{J}) \setminus V(\mathscr{I}) = a_{0,n}, \quad V(\mathscr{I}) \setminus V(\mathscr{J}) = \emptyset, \quad n$ even ;

(6.3) $V(\mathscr{J}) \setminus V(\mathscr{I}) = b_{n+1}, \quad V(\mathscr{I}) \setminus V(\mathscr{J}) = b_n, \quad n \in \overline{\mathbb{N}}.$ ∎

7. FUNCTIONALS F_1, F_2

We shall introduce real functions $F_i : [\mathscr{I}] \to \langle 0, \infty \rangle$, $i = 1, 2$, which, in a sense, represent apirori estimates of the variation and the norm of the approximate solution. Since in this section we shall deal only with fixed r, a we shall write briefly $U(x, t)$ instead of $U_{r,a}(x, t)$.

Definition 7.1. Let the approximate solution U be defined along the curve $\mathscr{I} \in [\mathscr{I}]$. We define the waves $\varepsilon^{m,n} \cap \mathscr{I}$, $(m, n) \in Y$ as follows:

1° For $m \geq 1$ let $\ell^* \equiv \mathscr{I} \cap [\{a_{m-2,n} - a_{m-1,n+1}\} \cup \{a_{m-1,n+1} - a_{m,n}\}]$ *). If $\ell^* \neq \emptyset$, we put $\varepsilon^{m,n} \cap \mathscr{I} \equiv \varepsilon^{m,n} \cap \ell^*$ (see Definition 3.6).

2° For $m = 1$ and $\ell^* = \emptyset$ let $\ell_1 \equiv \mathscr{I} \cap \{b_n - a_{1,n}\}$. If $\ell_1 \neq \emptyset$, we put $\varepsilon^{1,n} \cap \mathscr{I} \equiv \varepsilon^{1,n} \cap l_1 = \varepsilon^{1,n}$.

3° For $m = 0$ and n even the curve $\mathscr{I}$ can "come back" into the wave $\varepsilon^{0,n}$ and can intersect this wave even three times. Therefore we define $\varepsilon^{0,n} \cap \mathscr{I}$ as a set of one or two waves taking into account only the segments of $\mathscr{I}$ intersecting $\varepsilon^{0,n}$ in the direction of increasing x:

3a° If $\{b_n, a_{0,n}, a_{1,n+1}, a_{2,n}\} \subset V(\mathscr{I})$ and $a_{1,n+1}' < r/2$ then

$$\varepsilon^{0,n} \cap \mathscr{I} \equiv \{\varepsilon^{0,n}, \varepsilon^{0,n} \cap \{a_{1,n+1} - a_{2,n}\}\}.$$

3b° If $\{b_n, a_{1,n+1}, a_{2,n+2}\} \subset V(\mathscr{I})$ and $a_{0,n} \notin \mathscr{I}$ then

$$\varepsilon^{0,n} \cap \mathscr{I} \equiv \{\varepsilon^{0,n} \cap \{b_n - a_{1,n+1}\}\}.$$

---

*) Here formally $a_{-1,n} = b_n$.

3c° If $\{b_{n+1}, a_{1,n+1}, a_{2,n}\} \subset V(\mathcal{I})$ or $\{b_{n-1}, a_{0,n}, a_{1,n+1}, a_{2,n}\} \subset V(\mathcal{I})$ and if $a_{1,n+1}^{\backprime} < r/2$ then

$$\varepsilon^{0,n} \cap \mathcal{I} \equiv \{\varepsilon^{0,n} \cap \{a_{1,n+1} - a_{2,n}\}\} \,.$$

3d° If $\{b_n, a_{0,n}, a_{1,n-1}\} \subset V(\mathcal{I})$, $a_{1,n+1}^{\backprime} \geq r/2$ or if $\{b_n, a_{0,n}, a_{1,n-1}\} \subset V(\mathcal{I})$ or if $\{b_n, a_{1,n+1}, a_{2,n}\} \subset V(\mathcal{I})$, $a_{0,n} \notin \mathcal{I}$ or if $\{b_n, a_{0,n}, a_{1,n+1}, a_{2,n+2}\} \subset V(\mathcal{I})$ then

$$\varepsilon^{0,n} \cap \mathcal{I} \equiv \{\varepsilon^{0,n}\} \,.$$

4° In all other cases except 1°–3° we take $\varepsilon^{m,n} \cap \mathcal{I}$ as an empty (nonexisting) wave (e.g. if $\varepsilon^{m,n}$ is not defined or if $\mathcal{I} \cap \varPi_n = \emptyset$). ∎

For $\mathcal{I} \in [\mathcal{I}]$ we denote by $\varepsilon(\mathcal{I})$ the set of all nonempty waves $\varepsilon^{m,n} \cap \mathcal{I}$ intersecting the curve $\mathcal{I}$. We arrange the countable set $\varepsilon(\mathcal{I})$ into a sequence $\{\alpha^i\}_{i=1}^{\infty} = \varepsilon(\mathcal{I})$ in such a way that $i > j$ iff α^i intersects $\mathcal{I}$ after (toward the larger x than) α^j.

Definition 7.2. Let U be defined along $\mathcal{I} \in [\mathcal{I}]$, $\varepsilon(\mathcal{I}) = \{\alpha^i\}_{i=1}^{\infty}$, $\alpha^i = (\alpha_1^i, \alpha_2^i)$. We define the following functionals:

$$(7.1) \qquad L(\mathcal{I}) \equiv \sum_{i=1}^{\infty} |\alpha^i| \,;$$

$$(7.2) \qquad Q(\mathcal{I}) \equiv D(\alpha^1, \alpha^2, \ldots) + \sum_{i<j} \{|(\alpha_1^i)^+| \, |\alpha^j| + |(\alpha_1^i)^-| \, (|\alpha_1^j| + |(\alpha_2^j)^+|)\} \,;$$

$$(7.3) \qquad F_1(\mathcal{I}) \equiv L(\mathcal{I}) + 14 K_1 Q(\mathcal{I}) \,;$$

where D has been introduced in Definition 3.3 and $K_1 = K_1(\tilde{U}) > 0$ is the constant from Theorem 3.5. ∎

Remark 7.3. As follows from the proof of Lemma 7.4, the functional L represents the variation of the approximate solution U. However, suitable estimates cannot be proved directly for L, which is why certain "correction" $14 K_1 Q$ is added to L.

The functional Q can be interpreted as the influence of the interactions of waves from the approximate solution: D represents the waves which can interact directly (see Remark 3.4) while the other terms on the right-hand side of (7.2) represent the influence of waves reflecting on the boundary $x = 0$ (i.e. on the head of the piston in the physical interpretation of Remark 2.2). Since $\lambda_1 < 0$, the waves of the first family intersect the line $x = 0$ at a finite time $T > 0$; for $t > T$ the solution of the mixed problem has the character of a wave of the second family of the opposite kind — i.e., a rarefaction wave if the initial wave is a shock and a shock wave if the initial wave is a rarefaction one (see e.g. [4] for the equations of gas dynamics). To the interaction of reflected waves, we can then apply again Remark 3.4. ∎

Lemma 7.4. *Let U be defined along $\mathcal{I} \in [\mathcal{I}]$. Then*

$$(7.4) \qquad\qquad F_1(\mathcal{I}) \leq [1 + 28 K_1 L(\mathcal{I})] L(\mathcal{I}) \,.$$

If, moreover, $\mathscr{I} \subset \bar{\Pi}_n$ for some $n \in \bar{\mathbb{N}}$ and

(7.5)
$$\left| U(b) - \tilde{U} \right| \leqq d_8 \quad \text{for all} \quad b \in V(\mathscr{I}),$$

then

(7.6)
$$\operatorname*{Var}_{\langle 0, \infty \rangle} U(\cdot, t) \leqq \frac{1}{c_1} F_1(\mathscr{I}), \quad ns \leqq t \leqq (n + 1) s.$$

Finally,

(7.7)
$$F_1(\mathscr{O}) \leqq c_{11} \Big[\operatorname*{Var}_{\langle 0, \infty \rangle} U_0 + \left\| U_0 - \tilde{U} \right\| + \left\| u_1 - \tilde{u} \right\| \Big],$$

where

(7.8)
$$c_{11} \equiv (2 + c_2)(1 + 84 K_1 (2 + c_2) d_{50}).$$

$(c_i \equiv c_i(\tilde{U}), i = 1, 2,$ *are the constants from Lemma 3.7.*$)$

Proof. Evidently $D(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \ldots) \leqq \sum_{i,j=1}^{\infty} |\boldsymbol{\alpha}^i| \, |\boldsymbol{\alpha}^j| = [L(\mathscr{I})]^2$ and analogously for the other terms in (7.2). Therefore $Q(\mathscr{I}) \leqq 2[L(\mathscr{I})]^2$ and we immediately obtain (7.4). Further, if $\mathscr{I} \subset \bar{\Pi}_n$, $n \in \bar{\mathbb{N}}$, then

(7.9)
$$L(\mathscr{I}) \geqq \sum_m \left| \varepsilon^{m,n} \right|.$$

Using Lemma 3.7 we get for $t \in \langle ns, (n + 1) s \rangle$

(7.10)
$$\left| \varepsilon^{m,n} \right| \geqq c_1 \operatorname*{Var}_{\langle \max[0, (m-2)r], mr \rangle} U(\cdot, t).$$

Since $L(\mathscr{I}) \leqq F(\mathscr{I})$, (7.9) and (7.10) imply (7.6). Finally

$$L(\mathscr{O}) \leqq 2 \left| \varepsilon^{0,0} \right| + \sum_{k=1}^{\infty} \left| \varepsilon^{2k,0} \right|$$

and since $\left| \varepsilon^{0,0} \right| = \left| u_0(0) - u_1(s) \right|$, $U(\cdot, 0) = U_{0,r}$, we obtain by means of Lemma 3.7

$$L(\mathscr{O}) \leqq (2 + c_2) \Big[\operatorname*{Var}_{\langle 0, \infty \rangle} U_{0,r} + \left| u_0(0) - u_1(s) \right| \Big]$$

and (7.7) follows by direct computation. (Let us observe that the inequality in (7.9) can hold iff n is even and $\varepsilon^{0,n} \cap \mathscr{I} \neq \{ \varepsilon^{0,n} \}$, so in all other cases we have moreover $L(\mathscr{I}) \leqq c_2 \operatorname*{Var}_{\langle 0, \infty \rangle} U(\cdot, t).$) ∎

Definition 7.5. Let U be defined along $\mathscr{I} \in [\mathscr{I}]$. For $a_{m_0,n_0} \in \mathscr{I}$ let $\varepsilon(a_{m_0,n_0}; \mathscr{I})$ be the set of all waves $\boldsymbol{\alpha} \cap \ell$ where $\boldsymbol{\alpha} \in \varepsilon(\mathscr{I})$ and $\ell \subset \mathscr{I}$ is a segment of the form

$\{a_{m-1,n*} - a_{m,n}\}$ or $\{b_{n*} - a_{m,n}\}$, $m \leqq m_0$. For $b_n \in \mathscr{I}$ let $\varepsilon(b_n; \mathscr{I}) \equiv \emptyset$. Further let $b \in V(\mathscr{I})$. We put

(7.11)
$$L(b; \mathscr{I}) \equiv \sum_{\alpha \in \varepsilon(b; I)} |\alpha| ,$$

(7.12)
$$F_2(b; \mathscr{I}) \equiv c_9 |U(b) - \tilde{U}| + L(b; \mathscr{I}) + 14K_1 \, Q(\mathscr{I}) ,$$

where

(7.13)
$$c_9 \equiv \min \left[c_1, (1 + c^*)^{-1} \right]$$

Finally let

(7.14)
$$F_2(\mathscr{I}) \equiv \sup_{b \in V(I)} F_2(b; \mathscr{I}) . \qquad \blacksquare$$

Lemma 7.6. *Let U be defined along $\mathscr{I} \in [\mathscr{I}]$. Then*

(7.15)
$$\left| U(b) - \tilde{U} \right| \leqq \frac{1}{c_9} F_2(\mathscr{I})$$

for each $b \in V(\mathscr{I})$. Moreover, if (7.5) holds then (7.15) is true for all $b \in \mathscr{I}$. Finally

(7.16)
$$F_2(0) \leqq c_{12}\left(\| U_0 - \tilde{U} \| + \| u_1 - \tilde{u} \| + \operatorname*{Var}_{\langle 0, \infty \rangle} U_0 \right),$$

where

(7.17)
$$c_{12} \equiv c_9 + 2c_{11} .$$

Proof. Inequality (7.15) for $b \in V(\mathscr{I})$ follows directly from the definition of F_2. Let $(x, t) \in \mathscr{I}$ and suppose $(x, t) \in \ell$, where $\ell = \{a_{m-1,n'} - a_{m,n}\}$ or $\ell = \{b_{n'} - a_{m,n}\}$ $|n - n'| \leq 1$. Let e.g. (for the sake of definiteness) $n' = n + 1$ and put $\alpha \equiv \varepsilon^{m,n} \cap \ell$. Then

$$\left| U(x, t) - U(a_{m,n}) \right| \leqq \operatorname*{Var}_{\ell} U \leqq \frac{1}{c_1} |\alpha|$$

by Lemma 3.7 and therefore

(7.18)
$$\left| U(x, t) - \tilde{U} \right| \leqq \frac{1}{c_1} |\alpha| + \left| U(a_{m,n}) - \tilde{U} \right| .$$

Evidentely $\alpha \in \varepsilon(a_{m,n}; \mathscr{I})$ *) so that (7.18) implies

$$c_9 |U(x, t) - \tilde{U}| \leqq L(a_{m,n}; \mathscr{I}) + c_9 |U(a_{m,n}) - \tilde{U}| \leqq F_2(a_{m,n}; \mathscr{I})$$

---

*) For $m = 0$, $\alpha \notin \varepsilon(a_{m,n}; \mathscr{I})$ may occur, but in this case $\varepsilon^{0,n} \in \varepsilon(a_{m,n}; \mathscr{I})$, $|\varepsilon^{0,n}| \geqq |\alpha|$ and so it suffices to replace α by $\varepsilon^{0,n}$.

and consequently (7.15) holds for $b = (x, t)$. Finally, since $d_{50} \leq d_8$ we can apply the preceding estimates to the curve $\mathcal{O}$, in particular for $b = a_{m,1}$ we have $c_9 |U(a_{m,1}) - \tilde{U}| \leq F_2(a_{m+1,0}; \mathcal{O})$ and therefore

$$F_2(\mathcal{O}) \leq c_9 \sup_{k \in \mathbb{N}} |U(a_{2k,0}) - \tilde{U}| + 2 F_1(\mathcal{O})$$

because $L(b; \mathcal{O}) \leq L(\mathcal{O})$. Then (7.16) is obtained by means of Lemma 7.4. $\blacksquare$

Lemma 7.7. *Let $\mathcal{J} \in [\mathcal{I}]$ be a successor of $\mathcal{I} \in [\mathcal{I}]$ of the type* (6.1). *Let U be defined along $\mathcal{I}$ so that* (7.5) *holds and let further*

$$(7.19) \qquad\qquad F_1(\mathcal{I}) \leq c_3(\mathcal{I}) d_{50}, \quad c_3(\mathcal{I}) > 0,$$

$$(7.20) \qquad\qquad d_{50} \leq \min\left[d_8, (c_3(\mathcal{I}) 14 K_1)^{-1}\right].$$

Then U is defined also along $\mathcal{J}$ and

$$(7.21) \qquad\qquad F_i(\mathcal{J}) \leq F_i(\mathcal{I}), \quad i = 1, 2.$$

Proof. The curves $\mathcal{I}, \mathcal{J}$ differ in the diamond Λ with the vertices $a_{m,n}, a_{m,n-2}$, $a_{m-1,n-1}, a_{m+1,n-1}$. From (7.5) it follows that there exists a solution of $IRP(U(a_{m-1,n-1}), U(a_{m+1,n-1}))$ and so U is defined along $\mathcal{J}$.

While the upper part $\mathcal{J} \setminus \mathcal{I}$ of Λ is intersected by one wave $\varepsilon^{m+1,n-1}$, the lower part $\mathcal{I} \setminus \mathcal{J}$ of Λ can be intersected by two or three waves. However, the proofs are similar in both cases. Let us consider the case $m = 2$, $a'_{1,n-1} < r/2$, i.e. the case of three waves. Denote $\varepsilon(\mathcal{I}) = \{\alpha^i\}$, $\varepsilon(\mathcal{J}) = \{\beta^i\}$ and $\gamma \equiv \varepsilon^{0,n-2} \cap \{a_{1,n-1} - a_{2,n-2}\}$, $\delta \equiv \varepsilon^{4,n-2} \cap \{a_{2,n-2} - a_{3,n-1}\}$, $\beta \equiv \varepsilon^{3,n-1}$, $\varepsilon \equiv \varepsilon^{2,n-2}$; $D \equiv D(\gamma, \varepsilon, \delta)$.

Comparing $\varepsilon(\mathcal{J})$ and $\varepsilon(\mathcal{I})$ we see that $\beta = \beta^2$ and

$$(7.22) \qquad L(\mathcal{J}) = L(\mathcal{I}) + |\beta| - |\gamma| - |\varepsilon| - |\delta|,$$

$$
\begin{aligned}
(7.23) \qquad Q(\mathcal{J}) = {}& Q(\mathcal{I}) - D + |\beta_2^1| \left(|\beta_1| - |\varepsilon_1| - |\delta_1|\right) + \\
& + |(\beta_2^1)^-| \left(|\beta_2| - |\gamma_2| - |\varepsilon_2| - |\delta_2|\right) + \\
& + |(\beta)_2^{1+}| \left(|(\beta_2)^-| - |(\gamma_2)^-| - |(\varepsilon_2)^-| - |(\delta_2)^-|\right) + \\
& + \left(|(\beta_1)^-| - |(\varepsilon_1)^-| - |(\delta_1)^-|\right) \sum_{k=3}^{\infty} |\beta_1^k| + \\
& + \left(|(\beta_1)^+| - |(\varepsilon_1)^+| - |(\delta_1)^+|\right) \sum_{k=3}^{\infty} |(\beta_1^k)^-| + \\
& + \left(|\beta_2| - |\gamma_2| - |\varepsilon_2| - |\delta_2|\right) \sum_{k=3}^{\infty} |\beta_1^k| + \\
& + \left(|(\beta_2)^-| - |(\gamma_2)^-| - |(\varepsilon_2)^-| - |(\delta_2)^-|\right) \sum_{k=3}^{\infty} |\beta_2^k| +
\end{aligned}
$$

$$+ \left(\left| (\beta_2)^+ \right| - \left| (\gamma_2)^+ \right| - \left| (\varepsilon_2)^+ \right| - \left| (\delta_2)^+ \right| \right) \sum_{k=3}^{\infty} \left| (\beta_2^k)^- \right| +$$

$$+ \left(\left| (\beta_1)^+ \right| - \left| (\varepsilon_1)^+ \right| - \left| (\delta_1)^+ \right| \right) \sum_{k=3}^{\infty} \left| \beta^k \right| +$$

$$+ \left(\left| (\beta_1)^- \right| - \left| (\varepsilon_1)^- \right| - \left| (\delta_1)^- \right| \right) \sum_{k=3}^{\infty} \left(\left| \beta_1^k \right| + \left| (\beta_2^k)^- \right) +$$

$$+ \left(- \left| (\varepsilon_1)^+ \right| \left| \delta \right| - \left| (\varepsilon_1)^- \right| \left[\left| \delta_1 \right| + \left| (\delta_2)^- \right| \right] \right) \equiv$$

$$\equiv Q(\mathscr{I}) - D + q_1 + q_2 + \ldots + q_{11} \,.$$

By Theorem 3.5 we obtain $\beta_i = \gamma_i + \varepsilon_i + \delta_i + D\, O(1)$, $i = 1, 2$, and consequently

$$(7.25) \qquad \Delta_i \equiv \left| \beta_i \right| - \left| \gamma_i \right| - \left| \varepsilon_i \right| - \left| \delta_i \right| \leqq DK_1 \,, \quad i = 1, 2 \,.$$

Further if $(\beta_i)^- \neq 0$ then

$$q_i' \equiv \left| (\beta_i)^- \right| - \left| (\gamma_i)^- \right| - \left| (\varepsilon_i)^- \right| - \left| (\delta_i)^- \right| = -\beta_i - \left| (\gamma_i)^- \right| - \ldots =$$

$$= -\gamma_i - \varepsilon_i - \delta_i - D\, O(1) - \left| (\gamma_i)^- \right| - \ldots \leqq$$

$$\leqq \left| (\gamma_i)^- \right| + \left| (\varepsilon_i)^- \right| + \left| (\delta_i)^- \right| + D \left| O(1) \right| - \left| (\gamma_i)^- \right| - \ldots$$

and therefore
$$(7.26) \qquad q_i' \leqq DK_1 \,, \quad i = 1, 2 \,,$$

since evidently $q_i' \leqq 0$ for $(\beta_i)^- = 0$. Analogously (it is $\gamma_1 = 0$)

$$(7.27) \qquad \left| (\beta_1)^+ \right| - \left| (\varepsilon_1)^+ \right| - \left| (\delta_1)^+ \right| \leqq DK_1 \,.$$

Inequalities $(7.25)-(7.26)$ imply

$$q_j \leqq DK_1 \, L(\mathscr{I}) \,, \quad j = 1, 2, \ldots, 10 \,,$$

and since $q_{11} \leqq 0$, we get

$$(7.28) \qquad Q(\mathscr{I}) \leqq Q(\mathscr{I}) + \left(10K_1 \, L(\mathscr{I}) - 1 \right) D \,.$$

Then we obtain from (7.25), (7.28)

$$(7.29) \quad \Delta_1 + \Delta_2 + 14K_1 \, Q(\mathscr{I}) \leqq \left[2 + 140K_1 \, L(\mathscr{I}) - 14 \right] K_1 D + 14K_1 \, Q(\mathscr{I}) \,.$$

In virtue of (7.19), (7.20)

$$(7.30) \qquad 2 + 140K_1 \, L(\mathscr{I}) - 14 \leqq 2 + 10 - 14 < 0 \,,$$

and therefore

$$(7.31) \qquad \left| \beta \right| - \left| \gamma \right| - \left| \varepsilon \right| - \left| \delta \right| + 14K_1 \, Q(\mathscr{I}) \leqq 14K_1 \, Q(\mathscr{I}) \,.$$

Similarly $Q(\mathscr{J}) \leqq Q(\mathscr{I}) + D[10K_1 \, c_3(\mathscr{I}) \, d_{50} - 1] \leqq Q(\mathscr{I}) + D[(10/14) - 1]$ and so

(7.32) $$Q(\mathscr{J}) \leqq Q(\mathscr{I}) \,.$$

Now (7.21) for $i = 1$ immediately follows from (7.22) and (7.31).

Taking account of (7.31), (7.32) one can easily show that

(7.33) $$F_2(b; \mathscr{J}) \leqq F_2(b; \mathscr{I}) \quad \text{for} \quad b \in V(\mathscr{I}) \cap V(\mathscr{J}) \,.$$

If we denote $\varphi^1 \equiv \varepsilon^{3,n-1} \cap \{a_{1,n-1} - a_{2,n}\}$, $\varphi^2 \equiv \varepsilon^{3,n-1} \cap \{a_{2,n} - a_{3,n-1}\}$, then

$$L(a_{2,n}; \mathscr{J}) = L(a_{3,n-1}; \mathscr{J}) + |\varphi^1| - |\gamma| - |\varepsilon| - |\delta| \,,$$

$$|U(a_{2,n}) - \tilde{U}| \leqq |U(a_{3,n-1}) - \tilde{U}| + |\varphi^2|/c_1 \,.$$

Since $|\varphi^1| + |\varphi^2| = |\beta|$, we obtain by means of (7.31)

(7.34) $$F_2(a_{2,n}; \mathscr{J}) \leqq c_9 |U(a_{3,n-1}) - \tilde{U}| + L(a_{3,n-1}; \mathscr{J}) +$$

$$+ |\beta| - |\gamma| - |\varepsilon| - |\delta| + 14K_1 \, Q(\mathscr{J}) \leqq F_2(a_{3,n-1}; \mathscr{J}) \,.$$

Now (7.21) for $i = 2$ follows from (7.33), (7.34). ∎

Lemma 7.8. *Let $\mathscr{J} \in [\mathscr{I}]$ be a successor of $\mathscr{I} \in [\mathscr{I}]$ of the type (6.2) and let U be defined along $\mathscr{I}$. Then U is defined also along $\mathscr{J}$ and (7.21) holds.*

Proof. Since $V(\mathscr{J}) \subset V(\mathscr{I})$ or $V(\mathscr{J}) = V(\mathscr{I}) \cup \{a_{0,n}\}$, $\mathscr{J} = \{b_{n-1} - a_{1,n-1}\} \cup [\mathscr{J} \cap \mathscr{I}]$, U is defined along $\mathscr{J}$. Comparing $\varepsilon(\mathscr{I})$, $\varepsilon(\mathscr{J})$ we see that $\varepsilon(\mathscr{I}) = \varepsilon(\mathscr{J}) \cup \{\varepsilon^{0,n} \cap \{a_{0,n} - a_{1,n+1}\}\}$ for $V(\mathscr{J}) \setminus V(\mathscr{I}) = \emptyset$, $a`_{1,n+1} < r/2$, otherwise $\varepsilon(\mathscr{J}) = \varepsilon(\mathscr{I})$. Hence the rest of the proof is easy. ∎

Lemma 7.9. *Let $\mathscr{J} \in [\mathscr{I}]$ be a successor of $\mathscr{I} \in [\mathscr{I}]$ of the type (6.3) for n even and let U be defined along $\mathscr{I}$ such that*

(7.35) $$|U(a_{1,n+1}) - \tilde{U}| \leqq d_3 \,.$$

Then U is defined also along $\mathscr{J}$ and (7.21) holds.

Proof. We have $\mathscr{J} \setminus \mathscr{I} = \{b_{n+1} - a_{1,n+1}\}$, $\mathscr{I} \setminus \mathscr{J} = \{b_n - a_{1,n+1}\}$, n even. Since by Theorem 4.1 there exists a solution of $MRP(u_1((n+1)s), U(a_{1,n+1}))$, U is defined along $\mathscr{J}$. Denote $\varepsilon(\mathscr{I}) = \{\alpha^i\}_{i=1}^{\infty}$, $\varepsilon(\mathscr{J}) = \{\beta^i\}_{i=1}^{\infty}$ and

$$\gamma \equiv \left\langle \begin{array}{l} \alpha^2 = \varepsilon^{2,n} \cap \{b_n - a_{1,n+1}\} \quad \text{if} \quad a`_{1,n+1} > r/2 \,, \\ 0 \quad \text{otherwise} \,. \end{array} \right.$$

Evidently $\alpha^1 = \varepsilon^{0,n} \cap \{b_n - a_{1,n+1}\}$, $\beta^1 = \varepsilon^{1,n+1}$. Comparing $\varepsilon(\mathscr{I})$, $\varepsilon(\mathscr{J})$ we see that

(7.36)
$$L(\mathscr{J}) = L(\mathscr{I}) + |\beta^1| - |\alpha^1| - |\gamma|,$$

(7.37)
$$Q(\mathscr{J}) = Q(\mathscr{I}) - D(\alpha^1, \gamma) + [|(\beta_2^1)^-| - |(\alpha_2^1)^-| - |(\gamma_2)^-|] \sum_{k=2}^{\infty} |\beta^k| +$$
$$+ [|(\beta_2^1)^+| - |(\alpha_2^1)^+| - |(\gamma_2)^+|] \sum_{k=2}^{\infty} [|\beta_1^k| + |(\beta_2^k)^-|] -$$
$$- |(\gamma_1)^-| \sum_{k=2}^{\infty} [2|\beta_1^k| + |(\beta_2^k)^-|] -$$
$$- |(\gamma_1)^+| \sum_{k=2}^{\infty} [|(\beta_1^k)^-| + |\beta^k|] =$$
$$\equiv Q(\mathscr{I}) - D(\alpha^1, \gamma) + q_1 + q_2 + q_3 + q_4.$$

Since $a'_{1,n+1} \leq r/2$ implies $\alpha^1 = \beta^1$ and $\gamma = 0$, we can confine ourselves to the case $a'_{1,n+1} > r/2$. By the definition of U, u_{1r} we have $u(b_{n+1}) = u(b_n) = u_1((n+1)s)$. Hence
$$\alpha_2^1 = u(b_n) - u(a_{0,n}), \quad \beta_2^1 = u(b_n) - u(a_{1,n+1})$$
and further
$$\gamma_1 = u(r, (n+1/2)s) - u(a_{0,n}), \quad \gamma_2 = u(r, (n+1/2)s) - u(a_{1,n+1}).$$

Therefore $\beta_2^1 = \alpha_2^1 - \gamma_1 + \gamma_2$ and

(7.38)
$$|(\beta_2^1)^-| = \max(0, -\beta_2^1) \leq |(\alpha_2^1)^-| + |(\gamma_1)^+| + |(\gamma_2)^-|,$$

(7.39)
$$|(\beta_2^1)^+| = \max(0, \beta_2^1) \leq |(\alpha_2^1)^+| + |(\gamma_1)^-| + |(\gamma_2)^+|,$$

(7.40)
$$|\beta^1| \leq |\alpha^1| + |\gamma|.$$

Now (7.38), (7.39) imply $q_1 + q_4 \leq 0$, $q_2 + q_3 \leq 0$ and hence

(7.41)
$$Q(\mathscr{J}) \leq Q(\mathscr{I}).$$

Since $L(\mathscr{J}) \leq L(\mathscr{I})$ by (7.36) and (7.40), (7.41) yields $F_1(\mathscr{J}) \leq F_1(\mathscr{I})$. Further $L(b; \mathscr{J}) = L(b; \mathscr{I}) + |\beta^1| - |\alpha^1| - |\gamma|$ for $b \in V(\mathscr{I}) \cap V(\mathscr{J})$ and therefore using (7.40), (7.41) we see that

(7.42)
$$F_2(b; \mathscr{J}) \leq F_2(b; \mathscr{I})$$

for $b \in V(\mathscr{I}) \cap V(\mathscr{J})$. Since $L(b_{n+1}; \mathscr{J}) = L(a_{1,n+1}; \mathscr{J}) - |\alpha^1| - |\gamma|$, we obtain from (4.1)

(7.43)
$$F_2(b_{n+1}; \mathscr{J}) \leq c_9 |U(a_{1,n+1}) - \tilde{U}| + c_9(1 + c^*)|\beta^1| +$$
$$+ L(a_{1,n+1}; \mathscr{J}) - |\alpha^1| - |\gamma| + 14K_1 Q(\mathscr{J}).$$

But $c_9(1 + c^*) \leq 1$ and so (7.40), (7.41), (7.43) imply (7.42) for $b = b_{n+1}$. Hence $F_2(\mathscr{J}) \leq F_2(\mathscr{I})$ and the proof is complete. ∎

Lemma 7.10. *Let $\mathcal{J} \in [\mathcal{J}]$ be a successor of $\mathcal{I} \in [\mathcal{I}]$ of the type (6.3) for n odd and let U be defined along $\mathcal{I}$ such that*

(7.44)
$$|U(a_{0,n+1}) - \tilde{U}| \leqq d_3$$

and (7.19), (7.20) *hold. Then U is defined also along $\mathcal{J}$ and*

(7.45)
$$F_i(\mathcal{J}) \leqq F_i(\mathcal{I}) + 2|u_1((n+2)s) - u_1(ns)|, \quad i = 1, 2.$$

Proof. Since $\mathcal{J} \setminus \mathcal{I} = \{b_{n+1} - a_{0,n+1}\}$, $\mathcal{I} \setminus \mathcal{J} = \{b_n - a_{0,n+1}\}$, n odd and since (7.44) holds, Theorem 4.1 implies that U is defined at b_{n+1} and hence along $\mathcal{J}$. Let $\gamma \equiv \varepsilon^{1,n} \cap \{b_n - a_{0,n}\}$, $\varepsilon(\mathcal{I}) = \{\alpha^i\}$, $\varepsilon(\mathcal{J}) = \{\beta^i\}$;evidently $\varepsilon^{0,n+1} = \beta^1$. Then

(7.46)
$$\beta_2^1 = \gamma_2 + u(b_{n+1}) - u(b_n).$$

Further, $L(\mathcal{J}) = L(\mathcal{I}) + |\beta^1| - |\gamma|$ and

$$Q(\mathcal{J}) = Q(\mathcal{I}) + \sum_{k=2}^{\infty} [(|\beta_2^1| - |\gamma_2|) |\beta_1^k| + (|(\beta_2^1)^-| - |(\gamma_2)^-|) |\beta_2^k| +$$
$$+ (|(\beta_2^1)^+| - |(\gamma_2)^+|) |(\beta_2^k)^-|].$$

From (7.46) it follows

(7.47)
$$|\beta_2^1| - |\gamma_2| \leqq |u(b_{n+1}) - u(b_n)|,$$
$$|(\beta_2^1)^{\pm}| \leqq |(\gamma_2)^{\pm}| + |u(b_{n+1}) - u(b_n)|.$$

Then

(7.48)
$$L(\mathcal{J}) \leqq L(\mathcal{I}) + |u(b_{n+1}) - u(b_n)|,$$
$$Q(\mathcal{J}) \leqq Q(\mathcal{I}) + |u(b_{n+1}) - u(b_n)| L(\mathcal{I}),$$

and using (7.19), (7.20) we obtain

(7.49)
$$F_1(\mathcal{J}) \leqq F_1(\mathcal{I}) + [1 + 14K_1 L(\mathcal{I})] |u(b_{n+1}) - u(b_n)| \leqq$$
$$\leqq F_1(\mathcal{I}) + 2|u(b_{n+1}) - u(b_n)|.$$

Since $u(b_n) = u(ns)$, $u(b_{n+1}) = u((n+2)s)$, (7.49) implies (7.45) for $i = 1$.

If $b \in V(\mathcal{I}) \cap V(\mathcal{J})$ then $L(b; \mathcal{J}) \leqq L(b; \mathcal{I}) + |\beta^1| - |\gamma| \leqq L(b; \mathcal{I}) + |u(b_{n+1}) - u(b_n)|$ and by means of (7.48), (7.19), (7.20) we get

(7.50)
$$F_2(b; \mathcal{J}) \leqq F_2(b; \mathcal{I}) + 2|u(b_{n+1}) - u(b_n)|.$$

Using (4.1) we see that $|U(b_{n+1}) - \tilde{U}| \leqq |U(a_{0,n+1}) - \tilde{U}| + (1 + c^*)|\beta^1|$ and since $L(b_{n+1}; \mathcal{J}) = L(a_{0,n+1}; \mathcal{J}) - |\gamma|$, the inequality

$$F_2(b_{n+1}; \mathcal{J}) \leqq c_9 |U(a_{0,n+1}) - \tilde{U}| + c_9(1 + c^*)|\beta^1| - |\gamma| +$$
$$+ L(a_{0,n+1}; \mathcal{J}) + 14K_1 Q(\mathcal{J})$$

holds. Then (7.47), (7.48), (7.19) and (7.20) imply (7.50) for $b = b_{n+1}$ and hence (7.45) holds for $i = 2$, too. ∎

8. ESTIMATES FOR THE APPROXIMATE SOLUTIONS

Let us define

$$(8.1) \qquad \bar{c}_{11} \equiv (2 + c_2)(1 + 84K_1(2 + c_2)d_8), \quad \bar{c}_{12} \equiv c_9 + 2\bar{c}_{11};$$

$$(8.2) \qquad\qquad c_{14} \equiv \max\left[\frac{\bar{c}_{11} + 2}{c_1}, \frac{\bar{c}_{12} + 2}{c_9}\right].$$

The constants $d_8, c_9, \bar{c}_{11}, \bar{c}_{12}, c_{14}$ do not depend on d_{50} but only on the function $F = (f, g)$ and the fixed vector $\tilde{U} \in \Omega_F$. Since $d_{50} \leq d_8$ it is evident (compare (7.8), (7.17)) that

$$(8.3) \qquad\qquad c_{11} \leq \bar{c}_{11}, \quad c_{12} \leq \bar{c}_{12}.$$

Theorem 8.1. *Let* (5.1) *hold with*

$$(8.4) \qquad\qquad d_{50} \leq \min\left[d_8, \frac{d_8}{4c_{14}}, \frac{1}{56K_1(2 + \bar{c}_{11})}\right] \equiv d_{51}.$$

Then for each $a \in A$, $r > 0$ the approximate solution $U_{r,a}$ is defined on the whole set $\bar{\Omega}$ and for $t \in \langle ns, (n + 1)s\rangle$, $n \in \bar{N}$, $x \geq 0$ it satisfies

$$(8.5) \qquad\qquad |U_{r,a}(x, t) - \tilde{U}| \leq$$

$$\leq c_{14}\left(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| + \mathop{\mathrm{Var}}_{\langle 0,\infty\rangle} U_0 + \mathop{\mathrm{Var}}_{\langle 0,(n+1)s\rangle} u_1\right) \leq d_8,$$

$$(8.6) \qquad\qquad \mathop{\mathrm{Var}}_{\langle 0,\infty\rangle} U_{r,a}(\cdot, t) \leq$$

$$\leq c_{14}\left(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| + \mathop{\mathrm{Var}}_{\langle 0,\infty\rangle} U_0 + \mathop{\mathrm{Var}}_{\langle 0,(n+1)s\rangle} u_1\right) \leq d_8.$$

Proof. For $\mathscr{I} \in [\mathscr{I}]$ starting at b_n, $n \in \bar{N}$ we put $n(\mathscr{I}) \equiv n$. Further we fix $a \in A$, $r > 0$ and denote $U = U_{r,a}$.

Let $[\mathscr{I}]^0$ be the set of all $\mathscr{I}$-curves $\mathscr{I} \in [\mathscr{I}]$ wit the properties

$$(8.7) \qquad\qquad U \text{ is defined along } \mathscr{I};$$

$$(8.8) \qquad\qquad F_1(\mathscr{I}) \leq \bar{c}_{11}\left(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| + \right.$$

$$\left. + \mathop{\mathrm{Var}}_{\langle 0,\infty\rangle} U_0 + 2\sum_{i=1}^{n(\mathscr{I})} |u_{1,r}(is) - u_{1,r}((i - 1)s)|\right),$$

(8.9)
$$F_2(\mathcal{J}) \leqq \bar{c}_{12}(\|U_0 - \tilde{U}\| + \|u_1 - \tilde{u}\| +$$
$$+ \operatorname*{Var}_{\langle 0,\infty \rangle} U_0 + 2 \sum_{i=1}^{n(\mathcal{J})} |u_{1,r}(is) - u_{1,r}((i-1)s)|) \,.$$

Now it is sufficient to show

(8.10)
$$[\mathcal{J}]^0 = [\mathcal{J}] \,.$$

Indeed, each $(x, t) \in \bar{\Omega}$ lies in a strip Π_n, $n \in \overline{N}$ and there exists an $\mathcal{J}$-curve $\mathcal{J} \subset \overline{\Pi}_n$. Since $\mathcal{J} \in [\mathcal{J}]^0$ by (8.10), (8.7) implies that U is defined on $\overline{\Pi}_n$, in particular at (x, t). Further $n(\mathcal{J}) = n$ and

(8.11)
$$\sum_{i=1}^{n(\mathcal{J})} |u_{1,r}(is) - u_{1,r}((i-1)s)| \leqq \operatorname*{Var}_{\langle 0,(n+1)s \rangle} u_1 \leqq d_{50} \,.$$

Taking into account (8.11), (8.9) and (7.15) we see that $|U(b) - \tilde{U}| \leqq d_8$ for $b \in V(\mathcal{J})$. Therefore we can apply Lemma 7.6 to the point $b = (x, t)$ and with the help of (8.9), (8.11) we immediately obtain (8.5). Further $\operatorname{Var} U(\cdot, t) \leqq F_1(\mathcal{J})/c_1$ by Lemma 7.4 which yields (8.6) by means of (8.8), (8.11).

Thus it remains to prove (8.10). Since the set $[\mathcal{J}]$ is partially ordered and satisfies the condition of minimality (see Lemma 6.3), it is sufficient to prove the following implication:

(8.12)   If $\mathcal{J} \in [\mathcal{J}]$ and if $\mathcal{I} \in [\mathcal{J}]^0$ for each $\mathcal{I} \in [\mathcal{J}]$ such that $\mathcal{I} < \mathcal{J}$, then
$\mathcal{J} \in [\mathcal{J}]^0$.

Indeed, (8.12) implies (8.10) by the principle of Noether induction.

In particular (8.12) requires $\mathcal{O} \in [\mathcal{J}]^0$. But this is an immediate consequence of Lemmas 7.4, 7.6 and (8.3). Further let $\mathcal{J} \in [\mathcal{J}]$, $\mathcal{J} \neq \mathcal{O}$. Then there exists $\mathcal{I} \in [\mathcal{J}]$ such that $\mathcal{J}$ is a successor of $\mathcal{I}$. Consequently $\mathcal{I} < \mathcal{J}$ and therefore $\mathcal{I} \in [\mathcal{J}]^0$ by the induction assumption. Then again $|U(b) - \tilde{U}| \leqq d_8$ for $b \in V(\mathcal{I})$ and $F_1(\mathcal{I}) \leqq 4(\bar{c}_{11} + 2) d_{50}$ by (8.8), (8.11). (5.1). Hence if we put $c_3(\mathcal{I}) \equiv 4(\bar{c}_{11} + 2)$, the assumptions of Lemmas 7.7–7.10 are satisfied and consequently (8.7) holds. Further, if $n(\mathcal{J}) = n(\mathcal{I})$, then (8.8), (8.9) follow from (7.21), Lemma 7.7 or 7.8. For $n(\mathcal{J}) = n(\mathcal{I}) + 1$, $n(\mathcal{I})$ even (7.21) holds by Lemma 7.9 and since $u_{1,r}(n(\mathcal{J}) s) = = u_{1,r}(n(\mathcal{I}) s)$ in this case, (7.21) again implies (8.8), (8.9). Finally if $n(\mathcal{J}) = = n(\mathcal{I}) + 1$, $n(\mathcal{I})$ odd, we deduce (8.8), (8.9) from (7.45), Lemma 7.10, and from the definition of $u_{1,r}$. Since obviously $0 \leqq n(\mathcal{J}) - n(\mathcal{I}) \leqq 1$, the proof is complete.   ∎

Theorem 8.2. Let (5.1) hold with

(8.13)
$$d_{50} \leqq \min \left[\frac{d_{51}}{8c_{14}}, \frac{d_{51}}{4(1 + c_{14})} \right] \equiv d_{52}$$

(d_{51} being introduced in (8.4)) and let

(8.14)
$$c_{16} \equiv (1 + 17c_{14}) c_{14} q_0 \,.$$

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Then for all $t_1 \geqq 0$, $t_2 \geqq 0$, $r \geqq 0$, $a \in A$ *it holds*

(8.15)
$$\int_0^\infty \left|U_{r,a}(x, t_1) - U_{r,a}(x, t_2)\right| \mathrm{d}x \leqq$$

$$\leqq c_{16}\left(\left|t_1 - t_2\right| + 4s\right)\left(\left\|U_0 - \tilde{U}\right\| + \left\|u_1 - \tilde{u}\right\| + \underset{\langle 0, \infty\rangle}{\mathrm{Var}}\, U_0 + \underset{\langle 0, \infty\rangle}{\mathrm{Var}}\, u_1\right).$$

Proof. Fix $r_0 > 0$, $a^0 \in A$ and put $U \equiv U_{r_0, a^0}$. By Theorem 8.1 U is defined on $\overline{\Omega}$ and satisfies (8.5), (8.6). We can suppose $t_1 < t_2$ without loss of generality. Let $n_0 \equiv \sup\left\{n \in \overline{N}; ns \leqq t_1\right\}$ and let q be the integer part of $(t_2/s) - n_0$ (so that $sq \leqq t_2 - t_1 + s$).

Now let us consider $m \in \overline{N}$ such that $(m, n_0 + q + 1) \in Y$. Taking into account the construction of the approximate solutions, one can easily see that for each $x \in \Delta_m^0 \equiv \left((m-1)\,r, (m+1)\,r\right\rangle \cap \langle 0, \infty\rangle$ the values $U(x, t_i)$, $i = 1, 2$, depend only on $U(\xi, n_0 s)$ for $\xi \in \Delta_m \equiv \langle(m-q-2)\,r, (m+q+2)\,r\rangle \cap \langle 0, \infty\rangle$ and further on $u_1(t)$, $t \geqq n_0 s$ in the case $m - q - 2 < 0$. Let us define

$$\tilde{U}^* \equiv U(mr, n_0 s);$$

$$U_0^*(\xi) \equiv \begin{cases} U(\xi, n_0 s) & \text{for } \xi \in \Delta_m, \\ \tilde{U}^* & \text{for } \xi \in \langle 0, \infty\rangle \setminus \Delta_m; \end{cases}$$

$$u_1^*(\tau) \equiv \begin{cases} u_1(\tau) & \text{for } \tau \geqq n_0 s, \quad m - q - 2 < 0, \\ u(mr, n_0 s) & \text{for } \tau \geqq n_0 s, \quad m - q - 2 \geqq 0 \end{cases}$$

and consider the approximate solutions $U_{r,a}^*$ of the problem

$$U_t + F(U)_x = 0, \quad x \geqq 0, \quad t \geqq n_0 s;$$

$$U(x, n_0 s) = U_0^*(x), \quad x > 0; \quad u(0, t) = u_1^*(t), \quad t \geqq n_0 s.$$

(These solutions are constructed in the same way as before, only in the case n_0 odd we replace Y by $Y^* \equiv \left\{(m, n) \in N^2; m + n \text{ is odd}\right\}$.) Using (8.5), (8.6) we obtain

(8.16)
$$\left\|U_0^* - \tilde{U}\right\| \leqq \left\|U(\cdot, n_0 s) - \tilde{U}\right\| \leqq d_{51},$$

$$\underset{\langle 0, \infty\rangle}{\mathrm{Var}}\, U_0^* = \underset{\Delta_m}{\mathrm{Var}}\, U(\cdot, n_0 s) \leqq d_{51},$$

and evidently $\mathrm{Var}\, u_1^* \leqq d_{51}$, $\left\|u_1^* - \tilde{u}\right\| \leqq d_{51}$. Thus we can apply Theorem 8.1 to $U_{r,a}^*$ and hence $U_{r,a}^*$ are defined for all x, t; evidently

(8.17)
$$U_{r_0, a^0}^*(x, t_i) = U(x, t_i), \quad x \in \Delta_m^0, \quad i = 1, 2.$$

Further, (8.5), (8.16), (8.13) yield

$$\left\|U_0^* - \tilde{U}^*\right\| \leqq 2\left\|U(\cdot, n_0 s) - \tilde{U}\right\| \leqq 8c_{14}d_{50} \leqq d_{51},$$

$$\left\|u_1^* - \tilde{u}^*\right\| \leqq \left\|u_1^* - \tilde{u}\right\| + \left\|U(\cdot, n_0 s) - \tilde{U}\right\| \leqq 4(1 + c_{14})\,d_{50} \leqq d_{51}.$$

Consequently, we can replace $\tilde{U}$ by $\tilde{U}*$ in Theorem 8.1 applied to $U^*_{r_0,a^0}$ and so (8.15), (8.17) imply

$$\left| U(x, t_1) - U(x, t_2) \right| \leqq 2c_{14}\left(\left\| U_0^* - \tilde{U}* \right\| + \left\| u_1^* - \tilde{u}* \right\| + \operatorname*{Var}_{\langle 0,\infty\rangle} U_0^* + \operatorname*{Var}_{\langle n_0s,\infty\rangle} u_1^* \right).$$

Let $d_0 \equiv \left\| U_0 - \tilde{U} \right\| + \left\| u_1 - \tilde{u} \right\| + \operatorname{Var} U_0 + \operatorname{Var} u_1$. Since

$$\left\| U_0^* - \tilde{U}* \right\| \leqq \operatorname*{Var}_{\Delta_m} U(\cdot, n_0s) ; \quad \operatorname*{Var}_{\langle 0,\infty\rangle} U_0^* = \operatorname*{Var}_{\Delta_m} U(\cdot, n_0s) ;$$

$$c_{15}(m) \equiv \left\| u_1^* - \tilde{u}* \right\| + \operatorname*{Var}_{<n_0s,\infty>} u_1^* \left\{ \begin{array}{l} = 0 \quad \text{if} \quad m - q - 2 \geqq 0 , \\ \leqq (1 + c_{14})\, d_0 \quad \text{if} \quad m - q - 2 < 0 \end{array} \right.$$

and $\Delta_m \cap \Delta_{m^*} = \emptyset$ if $\left| m - m^* \right| \geqq 2(q + 2)$, we have

$$\sum_m c_{15}(m) \leqq (q + 3)(1 + c_{14})\, d_0 ,$$

$$\sum_m \operatorname*{Var}_{\Delta_m} U(\cdot, n_0s) \leqq 2(q + 3) \operatorname*{Var}_{\langle 0,\infty\rangle} U(\cdot, n_0s) ,$$

and hence

(8.18)

$$\int_0^\infty \left| U(x, t_2) - U(x, t_1) \right| \mathrm{d}x = \sum_m \int_{\Delta_m^0} \ldots \mathrm{d}x \leqq 8rc_{14} \sum_m \left[\operatorname*{Var}_{\Delta_m} U(\cdot, n_0s) + c_{15}(m) \right] \leqq$$

$$\leqq r(q + 3)(1 + 17c_{14})\, c_{14}d_0 .$$

But $r(q + 3) \leqq q_0(t_2 - t_1 + 4s)$ and thus (8.15) follows immediately from (8.18). $\blacksquare$

9. CONVERGENCE OF THE APPROXIMATE SOLUTIONS

From now on we shall suppose that d_{50} satisfies (8.13), i.e. $d_{50} \leqq d_{52}$. Therefore Theorems 8.1, 8.2 hold, in particular $U_{r,a}$ are defined on $\bar{\Omega}$ for all $r > 0$, $a \in A$. Let us denote

$$\mathfrak{M} \equiv \left\{ \varphi : \bar{\Omega} \to \mathrm{R}; \ \varphi \in L_\infty(\Omega), \ \operatorname{supp} \varphi \text{ is compact} \right\}.$$

For $\varphi \in \mathfrak{M}$, $r > 0$, $a \in A$ we define

$$(9.1) \quad \Delta(r, a, \varphi, n) \equiv \int_0^\infty \varphi(x, ns) \left[U_{r,a}(x, ns - 0) - U_{r,a}(x, ns) \right] \mathrm{d}x , \quad n \in \mathrm{N} ;$$

$$(9.2) \qquad\qquad \Delta(r, a, \varphi) \equiv \sum_{n=1}^\infty \Delta(r, a, \varphi, n)$$

(the expressions in (9.1), (9.2) are vectors $\Delta = (\Delta_1, \Delta_2)$).

Lemma 9.1. Let $\varphi \in \mathfrak{M}$, $r > 0$, $a \in A$, $n \in \mathrm{N}$. Then

$$(9.3) \qquad\qquad \left| \Delta(r, a, \varphi, n) \right| \leqq r\|\varphi\|\, 8c_{14}d_{50} .$$

Proof follows from the definition (5.13) of $U_{r,a}(\cdot, ns)$ and from Theorem 8.1 (see also $[1]$). ∎

Let $P = \prod\limits_{(m,n)\in Y} \langle 0, 1\rangle$ be the probabilistic space equipped with the product measure dp generated by the Lebesgue measure dx on $\langle 0, 1\rangle$. For $r > 0$ we define the mapping $Z_r : P \to A(r)$ as follows: $Z_r(\{p_{m,n}\}) = \{a_{m,n}\}$ where $p_{m,n} \in \langle 0, 1\rangle$ and $a_{m,n} = ((m - 1 + 2p_{m,n}) r, ns)$ for $m > 0$; $a_{0,n} = (p_{m,n}r, ns)$. Further, we put

$$\Delta_0(r, p, \varphi, n) \equiv \Delta(r, Z_r(p), \varphi, n),$$

$$\Delta_0(r, p, \varphi) \equiv \Delta(r, Z_r(p), \varphi) = \sum_{n=1}^{\infty} \Delta_0(r, p, \varphi, n)$$

and denote by $\mathfrak{M}(r)$, $r > 0$, the set of all functions $\varphi \in \mathfrak{M}$ such that for each $(m, n) \in Y$, φ is constant on the diamond

$$T_{m,n} \equiv \Big\{(x, t) \in \bar{\Omega}; \ x \in ((m - 1) r, (m + 1) r\rangle, \ -\frac{s}{r}[x - (m - 1) r] + ns \leq t <$$

$$< \frac{s}{r}[x - (m - 1) r] + ns \quad \text{for} \quad x \leq mr; \ \frac{s}{r}[x - (m + 1) r] + ns \leq t <$$

$$< -\frac{s}{r}[x - (m + 1) r] + ns \quad \text{for} \quad x > mr\Big\}.$$

Lemma 9.2. *Let* $\varphi \in \mathfrak{M}(r)$, $r > 0$, $n_1 \neq n_2$, $n_i \in N$, $i = 1, 2$. *Then*

$$(9.4) \qquad \int_P \Delta_0(r, p, \varphi, n_1)\, \Delta_0(r, p, \varphi, n_2)\, dp = 0\,.$$

Proof follows the same ideas as in the case of the Cauchy problem (see $[1]$) and so is omitted here. ∎

For $\varphi \in \mathfrak{M}$, $r > 0$ let us denote $n(r, \varphi) \equiv \inf\{n \in N; \ \varphi \equiv 0 \text{ for } t > ns\}$. From now on we confine ourselves to r of dyadic form, i.e. $r = 2^{-k}$, $k \in N$. Then $n(r, \varphi) = r^{-1} n(1, \varphi)$ and if $\varphi \in \mathfrak{M}(r)$ then $\varphi \in \mathfrak{M}(r^*)$ for all $r^* < r$. Further, Lemma 9.2 implies

$$\|\Delta_0(r, \cdot, \varphi)\|^2_{L_2(P)} = \sum_{n=1}^{\infty} \|\Delta_0(r, \cdot, \varphi, n)\|^2_{L_2(P)}$$

and using (9.3) we get the inequality

$$(9.5) \qquad \|\Delta_0(r, \cdot, \varphi)\|^2_{L_2(P)} \leq r\, n(1, \varphi)\, (\|\varphi\|\, 8c_{14}d_{50})^2\,.$$

Consequently

$$(9.6) \qquad \lim_{r\to 0} \Delta_0(r, \cdot, \varphi) = 0 \quad \text{in} \quad [L_2(P)]^2\,.$$

Thus for each $\varphi \in \mathfrak{M}(r)$ we can choose a subsequence $\{r_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \varDelta_0(r_i, p, \varphi) = 0$ for a.a. $p \in \mathrm{P}$. Applying the diagonal process we can achieve this for a countable set of φ. Since there exists a sequence $\{\varphi_i\} \subset \bigcup_{k\in\mathrm{N}} \mathfrak{M}(2^{-k})$ dense in C_0^2 in the $L_{\infty}(\overline{\Omega})$-norm, we see that the following lemma holds:

Lemma 9.3. *There exist a sequence $\{k_i\}_{i=1}^{\infty} \subset \mathrm{N}$ and $N \subset \mathrm{P}$ such that* mes $(N) =$ $= 0$, $\lim_{i\to\infty} k_i = \infty$ *and*

$$(9.7) \qquad \lim_{i\to\infty} \varDelta_0(2^{-k_i}, p, \varphi) = 0 \quad \text{for all} \quad p \in \mathrm{P} \setminus N , \quad \varphi \in C_0^2 .$$

Proof of Theorem 2.4. Let $d = d_{52}$ and let (2.12) hold. For fixed $p^0 \in \mathrm{P} \setminus N$ we put

$$U_i \equiv U_{r_i, Z_{r_i}(p^0)} , \quad i \in \mathrm{N}$$

$(N, r_i = 2^{-k_i}$ are the set and the sequence from Lemma 9.3). By Theorem 8.1 we have

$$(9.8) \qquad \|U_i(\cdot, t) - \tilde{U}\| \le d_8 , \quad \underset{\langle 0,\infty\rangle}{\mathrm{Var}} \, U_i(\cdot, t) \le d_8 , \quad t \ge 0 , \quad i \in \mathrm{N} .$$

Using Helly's theorem one can find a subsequence of $\{U_i(\cdot, t)\}$, $t \ge 0$, converging in $[L_{1,loc}\langle 0, \infty)]^2$. Applying the diagonal process we can achieve this for all rational $t > 0$; let us denote this subsequence again by $\{U_i\}$. Now using Theorem 8.2 we can prove by a standard argument (see $[1]$) that $U_i(\cdot, t)$ converge in $[L_{1,loc}\langle 0, \infty)]^2$ uniformly for bounded $t \ge 0$; let the limit function be U. Then $\|U - \tilde{U}\| \le d_8$ by (9.8) and hence $U \in [L_{\infty}(\Omega)]^2$, $U(x, t) \in \Omega_F$ for a.a. $(x, t) \in \Omega$ and $F(U) \in [L_{\infty}(\Omega)]^2$. Further, it is seen that $F(U_i) \to F(U)$ in $[L_{1,loc}(\Omega)]^2$. Therefore we obtain with the help of Lemma 5.1

$$(9.9) \qquad \int_{\Omega} [\psi_t u + \psi_x f(U)] \, dx \, dt + \int_0^{\infty} \psi(x, 0) \, u_0(x) \, dx =$$

$$= \lim_{i\to\infty} \int_{\Omega} [\psi_t u_i + \psi_x f(U_i)] \, dx \, dt + \int_0^{\infty} \psi(x, 0) \, u_0(x) \, dx =$$

$$= \lim_{i\to\infty} \left\{ \varDelta_1(r_i, Z_{r_i}(p^0), \psi) + \int_0^{\infty} \psi(x, 0) \left[u_0(x) - u_{0,r_i}(x)\right] dx \right\} , \quad \psi \in C_{00}^2 ;$$

$$(9.10) \qquad \int_{\Omega} [\varphi_t v + \varphi_x g(u)] \, dx \, dt + \int_0^{\infty} \varphi(x, 0) \, v_0(x) \, dx + \int_0^{\infty} \varphi(0, t) \, g(u_1(t)) \, dt =$$

$$= \lim_{i\to\infty} \left\{ \varDelta_2(r_i, Z_{r_i}(p^0), \varphi) + \int_0^{\infty} \varphi(x, 0) \left[v_0(x) - v_{0,r_i}(x)\right] dx + \right.$$

$$\left. + \int_0^{\infty} \varphi(0, t) \left[g(u_1(t)) - g(u_{1,r_i}(t))\right] dt \right\} , \quad \varphi \in C_0^2 .$$

Now $\lim \Delta_i = 0$, $i = 1, 2$ by Lemma 9.3 and since the remaining limits in (9.9), (9.10) are evidently zero, U is a generalized solution of the mixed problem $(\mathscr{P})$. The inequalities $(2.13)-(2.15)$ folow immediately from (8.5), (8.6), (8.15) (with $c = \max\left(c_{14}, c_{16}\right)$) and hence the proof is complete. ∎

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