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ON COMMUTATIVE SEMIGROUPS WHICH ARE UNIONS OF A FINITE NUMBER OF PRINCIPAL IDEALS

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The class of semigroups under the title includes the finitely generated commutative semigroups and noetherian commutative semigroups. We develop here some properties of noetherian semigroups related to its prime ideal structure. It is shown that archimedean semigroups which are unions of a finite number of principal ideals are noetherian semigroups and they are exactly unions of two principal ideals. We describe completely their ideal structure. One of the surprising results in this paper is that finitely generated archimedean semigroups without idempotents can have at most two generators and they cannot admit a ring structure. We shall prove also an analogue of Hilbert's basis theorem for semigroups.

Throughout this paper all semigroups under consideration are commutative. An ideal A in a semigroup S is said *to be finitely generated* if $A = \bigcup_{i=1}^{n} (x_i \cup x_i S) = \bigcup_{i=1}^{n} x_i S^1$. It is called *a principal ideal* or is *principally generated* if $A = xS^1$ for some $x \in S$. S can be trated as an ideal and every ideal different from S is called *proper*. A semigroup S is called *a noetherian semigroup* if every ascending chain of ideals terminates at a finite stage or equivalently every ideal is finitely generated. S is called *finitely generated* if there exist $x_1, x_2, \ldots, x_n$ in S such that every element is a product of powers of x_i's. An ideal A is *primary* (*prime*) if $xy \in A$ and $x \notin A$ then for some integer n $y^n \in A$ $(y \in A)$. For any ideal A in a semigroup S, $\sqrt{A} = \{x \in S : x^n \in A \text{ for some integer } n\}$. If A is a primary ideal, then $\sqrt{A}$ is a prime ideal. An ideal A is called *S-primary* if $\sqrt{A} = S$. It can be shown that every ideal in a noetherian semigroup is an intersection of finite number of primary ideals. The semigroup $S = \{x_i\}_{i \in N}$ with max multiplication is a noetherian semigroup and S is a principal ideal. But S is not a finitely generated semigroup. So one will be interested in knowing which noetherian semigroups are finitely generated. The results in sections 1 and 2 supplement the works of LEVIN [5], MCALISTER and CAROLL [6], and PETRICH [7]. Some of the properties of noetherian semigroups may be found in [9] and [10]. We follow the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [1], for all concepts not defined in this paper.

Lemma 1.1. *Let H be the collection of all ideals in a semigroup S, which are not principal (finitely generated). If $H \neq \emptyset$, then there exists a prime ideal which is not principal (finitely generated).*

Proof. We shall prove the theorem when no ideal in H is principal. Similar proof can be given for finitely generated case. Let $\{A_\alpha\}$ be a chain of ideals in H. If $\bigcup A_\alpha = xS^1$, then $A_\alpha = xS^1$ for some α, which is not true. So $\bigcup A_\alpha \in H$. Then by the application of Zorn's lemma to H (partially ordered by the inclusion relation) a maximal element P in H is guaranteed. Now the proof is completed by showing that P is a prime ideal. Suppose that P is not a prime ideal. Then there exist $a, b \notin P$ and $ab \in P$. By maximality of P, $P \cup bS^1 = xS^1$, which implies $x \in P$ or $x \in bS^1$. If $x \in P$, then $P = xS^1$, which is not true. So, if $x \in bS^1$, then $P \subseteq bS^1$. Since $ab \in P$ and $a \notin P$, $P : bS^1 = \{t : bS^1 t \subseteq P\}$ is an ideal containing P properly. Again by the maximality of P, $P : bS^1 = yS^1$. Now we assert $P = byS^1$, which is evidently a contradiction. Clearly $byS^1 \subseteq P$. Now if $t \in P$, $t \in bS^1$ and so $t = br$, since $t \neq b$. But $br \in P$, so that $r \in P : bS^1 = yS^1$. Thus $t \in byS^1$ and hence $P \subseteq byS^1$.

An immediate consequence of 1.1 is

Corollary 1.2. *If every prime ideal including S is principal (finitely generated) in a semigroup S, then every ideal in S is principal (finitely generated).*

Theorem 1.3. *Let S be a semigroup, which is a union of a finite number of principal ideals. If every proper prime ideal is principal, then the following are true:*

a) *every ideal is an intersection of a principal ideal and an S-primary ideal.*
b) *If $S = S^2$ then every proper ideal is principal.*

Proof. We shall prove firstly that every primary ideal Q such that $\sqrt{Q} \neq S$ is a principal ideal. By hypothesis the proper prime ideal $P = \sqrt{Q}$ is of the form aS^1 for some $a \in S$. This implies that there exists a natural number r such that $a^r \in Q$. Therefore $P^r = a^r S^1 \subseteq Q$. In the case when Q is contained in every power of P we have $Q = P^r = a^r S^1$. On the other hand let there exist a natural number m such that $Q \subseteq P^m$ and $Q \nsubseteq P^{m+1}$. Since P^m is a principal ideal, $Q = P^m A$ for some ideal A, and $Q \nsubseteq P^{m+1}$ implies that $A \nsubseteq P$. Since Q is P-primary we must have that $P^m \subseteq Q$ so that $Q = P^m$ and hence Q is principal. Now by 1.2 S is noetherian and so any arbitrary ideal A is of the form $Q_1 \cap Q_2 \cap \ldots \cap Q_n$, where Q_i's are primary ideals such that $P_i = \sqrt{Q_i} \neq \sqrt{Q_j} = P_j$ for $i \neq j$. We may assume $P_i \neq S$ for $i = 1, 2, \ldots, m$ and $P_i = S$ for $m + 1 \leqq i \leqq n$. Clearly $\sqrt{(Q_{m+1} \cap \ldots \cap Q_n)} = S$ and hence $Q_{m+1} \cap \ldots \cap Q_n$ is a S-primary ideal. Now we claim that $Q_1 \cap Q_2 \cap \ldots \cap Q_m = Q_1 Q_2 \ldots Q_m$, which proves that $Q_1 \cap Q_2 \cap \ldots \cap Q_m$ is a principal ideal since every one of $Q_1, \ldots, Q_m$ is a principal ideal. This establishes (a). For this order these P_i's $1 \leqq i \leqq m$ so that we can assume without loss of generality that P_1

is maximal in $\{P_i\}_1^m$, P_2 maximal in $\{P_i\}_2^m$ and so forth. This means no $P_i \subseteq P_j$ for $i \neq j$. Now assume for $r < m$ $Q_1 \cap Q_2 \cap \ldots \cap Q_r = Q_1 Q_2 \ldots Q_r$. Then $Q_1 \cap \cap Q_2 \cap \ldots \cap Q_{r+1} = (Q_1 \cap Q_2 \cap \ldots \cap Q_r) \cap Q_{r+1} = aS^1 \cap Q_{r+1}$ for some $a \in S$, since every one of $Q_1, Q_2, \ldots, Q_r$ is principal. Let $x = ay \in Q_{r+1}$. By the choice of P_i's $a \notin P_{r+1}$ since $a \in P_{r+1}$ implies $\sqrt{(aS^1)} = \sqrt{(Q_1 \cap Q_2 \cap \ldots \cap Q_r)} = P_1 \cap \cap P_2 \cap \ldots \cap P_r$ and thus $P_i \subseteq P_{r+1}$ for $i < r+1$, which is not true. Hence $y \in Q_{r+1}$ since Q_{r+1} is a primary ideal such that $\sqrt{Q_{r+1}} = P_r$. Thus $aS^1 \cap Q_{r+1} \subseteq aS^1 . Q_{r+1}$, which implies $aS^1 \cap Q_{r+1} = aS^1 Q_{r+1}$. Therefore by induction, $Q_1 \cap Q_2 \cap \ldots \ldots \cap Q_m = Q_1 Q_2 \ldots Q_m$. The proof of this part (a) is adopted from [4].

To prove (b), it suffices to show there are no proper S-primary ideals by virtue of (a). We can write $S = \bigcup_{i=1}^{n} x_i S^1$, where $x_i \notin x_j S^1$ for $i \neq j$. Then the condition $S = S^2$ implies that $x_i \in x_i^2 S^1$ for every i and so $x_i S^1 = e_i S^1$ where e_i is an idempotent. Thus $S = \bigcup_{i=1}^{n} e_i S$. Now, if A is a proper ideal such that $\sqrt{A} = S$, then $e_i^{n_i} \in A$ for some n_i, so that $A = S$, which is not true.

Lemma 1.4. *Let S be a semigroup in which $S \neq S^2$ and every maximal ideal is principal. Then S has at most two maximal ideals and for any proper prime ideal P, either P is a principal ideal or $P = xP$ for some $x \in S$.*

Proof. Let $a \in S \setminus S^2$. Then $S \setminus a$ is a maximal ideal and so by hypothesis $S \setminus a = bS^1$. Clearly $b \neq a$. Let $b \in S^2$. Thus $S \setminus a = S^2$. If $M = cS^1$ is any maximal ideal and if $c \in S^2$, then $M \subseteq S^2$ and $M = S^2 = S \setminus a$. Now if $c \notin S^2$, then $c \notin S \setminus a$, so that $c = a$. Thus $M = aS^1$. Hence in the case when $b \in S^2$, S can have at most two maximal ideals, namely, $S \setminus a$ and aS^1. Let $b \notin S^2$. Then $S = a \cup bS^1 = a \cup b \cup S^2$. We claim that $S \setminus a$ and $S \setminus b$ are the only two maximal ideals. If $M = cS^1$ is a maximal ideal, then consider the case when $c \notin S^2$. This implies $c = a$ or b, so that $M = S \setminus a$ or $S \setminus b$. The case that $c \in S^2$ is inadmissible, since otherwise $M = S^2$, which implies that the maximal ideal S^2 is contained properly in the maximal ideal $S \setminus a$.

To prove the second part consider any proper prime ideal P. If $a \notin P$, then $P \subseteq S \setminus a = bS^1$. This implies that $P = bS^1$ if $b \in P$ and $P = bP$ if $b \notin P$ since P is a prime ideal. Let $a \in P$. If $b \in P$ also, then $P = S$. If $b \notin P$, then $P \subseteq S \setminus b$. In the first part we have proved $S \setminus b$ is a maximal ideal and so $S \setminus b = xS^1$ for some x. Then as before $P = xS^1$ or $P = xP$.

Theorem 1.5. *Let every maximal ideal in a semigroup S be principal. If $S \neq S^2$ and $\bigcap_{n=1}^{\infty} x^n S = \emptyset$ for every $x \in S$, then S is a union of two principal ideals and every ideal is an intersection of a principal ideal and an S-primary ideal.*

Proof. By 1.4, every proper prime ideal is principal. If $a \in S \setminus S^2$, then by hypothesis, the maximal ideal $S \setminus a$ is of the form bS^1 for some b in S. Therefore $S = a \cup bS^1 = aS^1 \cup bS^1$. Now the conclusion is evident from 1.3.

Theorem 1.6. *Let S be a noetherian semigroup such that $S = \bigcup_{i=1}^{n} x_i S^1$. Suppose $a \notin x_i a S^1$ for all a in S, which is not a product of powers of x_i's. Then S is finitely generated. In particular if S is a noetherian cancellative semigroup without identity, then S is finitely generated.*

Proof. Suppose there exists an element a such that a is not a product of x_i's. Then $a = x_i s_1$, where $a \neq s_1$ and s_1 is not a product of powers of x_i's. Hence $s_1 = x_j s_2$. If $s_2 \in s_1 S$ or $s_2 = s_1$, then we have $s_1 \in x_j s_1 S^1$, which is not true by hypothesis. Thus $s_1 S^1$ is properly contained in $s_2 S^1$. Proceeding in this manner, we have a non-terminating chain of ideals, $s_1 S^1 \subset s_2 S^1 \subset \dots$. This is impossible by the noetherian condition. The second assertion follows now immediately by noting that in cancellative semigroups the condition $a = ab$ implies that b is an identity.

Proposition 1.7. *Let S be a semigroup which is a union of finite numbers of ideals. Then S contains idempotents if $S = S^2$. If S is cancellative, then S contains an identity if and only if $S = S^2$.*

Proof. Let $S = \bigcup_{i=1}^{n} x_i S^1$ with $x_i \notin x_j \cup x_j S$ for $i \neq j$. Since $S = S^2$, $\bigcup_{i=1}^{n} x_i S^1 = \bigcup_{i=1}^{n} (x_i^2 \cup x_i x_j S)$ which implies $x_i = x_i^2$ or $x_i = x_i^2 s$ for every i. Thus S contains idempotents. If S is cancellative, S can have at most one idempotent, which is the identity itself. Hence the second part is evident.

2. ARCHIMEDEAN SEMIGROUPS

We begin with a well-known result [7; 148].

Lemma 2.1. *A semigroup S is archimedean if and only if S has no prime ideals except S.*

Theorem 2.2. *If S is an archimedean semigroup, then every proper ideal is principal and S is a union of at most two principal ideals if either one of the following conditions is satisfied*

i) *S is a union of a finite number of principal ideals.*
ii) *S contains a maximal ideal which is finitely generated.*

Proof. Assume (i). Let $S = \bigcup_{i=1}^{n} x_i S^1$. If H is the collection of all proper ideals which are not principal and if $\{A_\alpha\}$ is a chain of ideals in H, then $S \neq \bigcup A_\alpha$, since otherwise $x_i \in A_{i_r}$ and so all $x_i \in A_j$ where $j = \max\{1_r, 2_r, \dots, i_r\}$. Thus $A_j = S$, which is impossible. If $A = aS^1$, then there exists A_i containing a. Hence $A_i = aS^1$, which is not true. Thus $\bigcup A_\alpha \in H$. Then by Zorn's lemma there exists a maximal element $P \in H$. As in lemma 1.1, P can be shown to be a prime ideal, which contradicts 2.1. Thus every proper ideal is principal. Let $S = \bigcup_{i=1}^{n} x_i S^1$ with $x_i \notin x_j S^1$ for $i = j$. If $n > 2$, $S \neq x_1 S^1 \cup x_2 S^1$. Hence from the first part $x_1 S^1 \cup x_2 S^1$ is a principal ideal and so $x_1 S^1 \subseteq x_2 S^1$ or $x_2 S^1 \subseteq x_1 S^1$. This contradicts the choice of x_i's. Thus $n \leqq 2$. In the second case if S contains a maximal ideal M which is finitely generated, then $S = M \cup a \cup aS$, $a \notin M$. So S is a union of finite number of principal ideals. Hence this conclusion follows from (i).

Combining 2.2 and 1.6, we have

Theorem 2.3. *Let S be an archimedean semigroup with $S = \bigcup_{i=1}^{n} x_i S^1$. Suppose $a \notin x_i a S^1$ for all a in S, which is not a product of powers of x_i's. Then S is finitely generated.*

A semigroup S is said to be *rational* if, for each $a, b \in S$, there exist natural numbers m, n such that $a^m = b^n$.

Theorem 2.4. *Let S be an archimedean semigroup without idempotents. Then S is finitely generated if and only if S is finitely generated as an ideal. In this case S is a rational semigroup with at most two generators.*

Proof. By virtue of 2.3, the 'if and only if' condition is evident from the fact that $a \neq ab$ for any a and b in an archimedean semigroup without idempotents [6; 136]. From 2.3 follows also S has at most two generators. A result of Levin [5; 370] then asserts that S is a rational semigroup.

Theorem 2.5. *Let $S = xS^1$ be a semigroup in which $\bigcap_{n=1}^{\infty} x^n S = \emptyset$. Then S is an infinite cyclic semigroup generated by x.*

Proof. Suppose there exists an y in S, which is not a power of x. Then $y = xs_1 = x^2 s_2 = \dots$. Therefore $y \in \bigcap_{n=1}^{\infty} x^n S = \emptyset$.

Theorem 2.6. *Let S be a noetherian cancellative semigroup with no idempotents, in which every maximal ideal is principal. Then*

i) *S is finitely generated archimedean semigroup with at most two generators if S has no proper prime ideals.*
ii) *If S has proper prime ideals, then every proper ideal is an intersection of a prinpal ideal and a S-primary ideal.*

Proof. If S has no proper prime ideals, then S is archimedean by 2.1. Then (i) is evident from 2.4. Suppose that S has proper prime ideals. By showing that every proper prime ideal is principal, we obtain the desired result by virtue of 1.2 and 1.3. If P is a proper prime ideal and if P is not a maximal ideal, then by noetherian condition $P \subseteq xS^1$, where xS^1 is maximal ideal. If $P \neq xS^1$, then $x \notin P$, which implies $P = xP$. Since S is noetherian, $P = \bigcup_{i=1}^{n} x_i S^1$. We may assume that $x_i \in \in x_j \cup x_j S^1$ for $i \neq j$. Now $\bigcup_{i=1}^{n} x_i S^1 = \bigcup_{i=1}^{n} x x_i S^1$. This implies that $x_i = x x_i$ or $x_i = = x x_i s$. Then, by the cancellative property x or xs is an idempotent, which is a contradiction. Thus $P = xS^1$.

Many of McAlister's results in [6], for finitely generated semigroups do also hold good for semigroups which are unions of a finite number of principal ideals. We shall briefly mention these results.

Theorem 2.7. *Let S be an archimedean semigroup, finitely generated as an ideal. Then*

i) *Kernel of* $S = \bigcap_{n=1}^{\infty} S^n$,

ii) *S has idempotents if and only if,* $\bigcap_{n=1}^{\infty} S^n \neq \emptyset$.

iii) *S is a group if and only if* $S^2 = S$.

Proof. Assume that $S = \bigcup_{i=1}^{n} a_i S^1$, where $a_i \notin a_j S^1$ for $i \neq j$. Let $x \in \bigcap_{n=1}^{\infty} S^n$. By the archimedean property there exists an integer N such that $a_i^N = x y_i$ for $i = 1, 2, \ldots, n$. Since $x \in \bigcap_{n=1}^{\infty} S^n$, for sufficiently large n we can have $x = a_i^N z$. Then the same proof as in Theorem 4.1 of [6], proves (i). The rest can be proved as in the corollaries 1 and 2 of 4.1 in [6].

Theorem 2.8. *Let S be an archimedean semigroup with an idempotent. Then S is a union of finite number of principal ideals if and only if $S/\mathrm{Ker}\, S$ is a finite nilpotent semigroup, where* Ker S *is the kernel of* S.

Proof. Let $S = \bigcup_{i=1}^{n} x_1 S^1$. If $S \neq \mathrm{Ker}\, S$ and if $x_1, x_2, \ldots, x_m \notin \mathrm{Ker}\, S$, then $S/\mathrm{Ker}\, S$ is generated, as an ideal, by the corresponding images of $x_1, x_2, \ldots, x_m$. Clearly the images of $x_1, \ldots, x_m$ are nilpotent. Therefore $S/\mathrm{Ker}\, S$ is a finite nilpotent semigroup. Conversely if $S/\mathrm{Ker}\, S$ contains a finite number of elements $y_1, \ldots, y_n$, then $S = = eS \cup t_1 S^1 \cup \ldots \cup t_n S^1$ where e is the unique idempotent in S and t_i's are the inverse images of y_i's under the canonical mapping $S \twoheadrightarrow S/\mathrm{Ker}\, S$. Now the result is true if $S = \mathrm{Ker}\, S$.

Some of the results so far proved enable us now to assert that some classes of semigroups do not admit ring structure. By combining the results in 1.3, 1.5 and 2.2 of this paper with corollary 1.2 and theorem 1.5 of [8], we have

Theorem 2.9. *Let S^0 be a semigroup S without an identity but adjoined with 0. Then S^0 cannot admit any ring structure in the following cases:*

i) $S = S^2$; *S is a union of finite number of principal ideals and every proper prime ideal is principal,*

ii) $S \neq S^2$; *every maximal ideal in S is principal and* $\bigcap_{n=1}^{\infty} x^n S = \emptyset$, *for every* $x \in S$,

iii) *S is an archimedean semigroup which is a union of finite number of principal ideals.*

iv) *S is an archimedean semigroup containing a finitely generated maximal ideal.*

3. AN ANALOGUE OF HILBERT BASIS THEOREM

Given a noetherian ring, one can construct a larger noetherian ring containing the previous one as a subring. This is the essence of Hilbert's basis theorem. In [3] we can find Bourne's proof on an analogue of Hilbert's basis theorem for finitely generated free commutative semigroups. In this section we shall prove a close analogue of Hilbert's basis theorem for semigroups. This is suggested by the following example of a semigroup $T = \{x_1, x_2, \ldots\}$ with max. multiplication. If $S = \{x_2, x_3, \ldots\}$ then $T = x_1 S^1$. T contains S and every ideal in T and S are finitely generated (in fact principal).

Theorem 3.1. *Let S be a subsemigroup of a semigroup T and $T = xS^1$ for some x in T. Then T is noetherian if S is noetherian.*

Proof. Let A' be a proper ideal of T. Set $A = \{a \in S : xa \in A'\}$, then A is an ideal in S and hence A is of the form $\bigcup_{i=1}^{m} a_i S^1$, $a_i \in S$. Set $d_i = xa_i$, $i = 1, 2, \ldots, m$. Let $B = \bigcup_{i=1}^{m} d_i S^1$. We claim $A' = B$, since, if $y \in A'$ and $y \neq x$, then $y = xs$. Clearly $s \in A$ and hence $s = a_i$ or $a_i s_1$, $s_1 \in S$. Therefore $y = xa_i$ or $xa_i s_1$, which implies that $y = d_i$ or $d_i s_1$. Hence $A' \subseteq B$. If $y \in B$, then $y = d_i$ or $d_i s$, so that $y = xa_i$ or $xa_i s$. Therefore xa_i or $xa_i s \in A'$ since a_i or $a_i s \in A$. Hence $B \subseteq A'$. Since A' is an ideal in T and $d_i \in A'$, we have

$$\bigcup_{i=1}^{m} d_i T^1 \subseteq A' = \bigcup_{i=1}^{m} d_i S^1 \subseteq \bigcup_{i=1}^{m} d_i T^1 .$$

Hence $A' = \bigcup_{i=1}^{m} d_i T^1$.

Theorem 3.2. *Let S be a subsemigroup of a semigroup T. Suppose $T = \bigcup_{i=1}^{n} x_i S$; $x_i x_j \in x_i S$ or $x_j S$ for $i \neq j$ and $S \subseteq x_i S$ for every i. Then T is noetherian, if S is noetherian.*

Proof. We prove the theorem by induction on the number of generators of T, i.e., the number of x_i's. By 3.1 for $n = 1$ the result is evident. Suppose that the theorem is true for all T with the number of generators $\leqq n - 1$. Let A' be an ideal in T. Set $A = \{a \in S : x_1 a \in A'\}$. A is an ideal in S and hence $A = \bigcup_{i=1}^{m} a_i S^1$, $a_i \in S$. Let $d_i = x_1 a_i$ for $i = 1, 2, \ldots, m$. If $T_1 = \bigcup_{i=2}^{n} x_i S$ and $L_1 = A' \cap T$, then T_1 is a subsemigroup of T containing S since $x_i x_j \in x_i S$ or $x_j S$. L_1 is clearly an ideal in T_1. Then by induction hypothesis, $L_1 = \bigcup_{i=1}^{r} b_i T_1^1$. We claim first $A' = B \cup L_1$ where $B = \bigcup_{i=1}^{m} d_i S^1$. If $x \in B$, then $x = d_i$ or $d_i s$, $s \in S$. Thus $x = x_1 a_i$ or $x_i a_i s$. Hence $x \in A'$. Clearly $L_1 \subseteq A'$ and hence $B \cup L_1 \subseteq A'$. Let $x \in A'$. If $x \in T_1$, then $x \in A' \cap T = L_1$. If $x \notin T_1$, then $x = x_1 s$, $s \in S$. Now $x_1 s \in A'$ implies by definition of A, $s \in A$. So $s = a_i t$ or a_i, where $t \in S$. Thus $x = x_1 a_i$ or $x_1 a_i t$ i.e., $x = d_i$ or $d_i t$, i.e., $x \in B$. Hence $A' = B \cup L_1$ and so

$$A' = \bigcup_{i=1}^{m} d_i S^1 \bigcup_{i=1}^{r} b_i T_1^1 = \bigcup_{i=1}^{m} d_i S^1 \cup (b_i \cup b_i x_j S) .$$

Since d_i, $b_i \in A'$ and A' is an ideal in T,

$$\bigcup_{i=1}^{m} d_i T^1 \bigcup_{i=1}^{r} b_i T^1 \subseteq A' \subseteq \bigcup_{i=1}^{m} d_i S^1 \cup (b_i \cup b_i x_j S) ,$$

which is a subset of $\bigcup_{i=1}^{m} d_i T^1 \bigcup_{i=1}^{r} b_i T^1$. Thus A' is finitely generated.

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