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ON THE LIMIT-3 CLASSIFICATION OF THE SQUARE OF A SECOND-ORDER, LINEAR DIFFERENTIAL EXPRESSION

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1. Introduction. In recent years the work of several mathematicians has been directed towards a study of the formal powers of the symmetric, second-order differential expression M, where, for suitably differentiable complex-valued f, M is defined by

$$M[f] = -(pf')' + qf \quad on \quad I \quad (' \equiv \mathrm{d}/\mathrm{d}x)\,. \tag{1.1}$$

Here I is an interval of the real line, and the coefficients p and q are real-valued, with $p > 0$, on I. The formal powers M^n, where $n = 1, 2, 3, \ldots$, of M are defined by $M^1 = M$ and $M^n = M[M^{n-1}]$ for $n = 2, 3, \ldots$; this definition requires certain differentiability properties of the coefficients p and q if M^n is also to be a differential expression.

The first result on the relationship between M and M^2, as symmetric differential expressions, were given by CHAUDHURI and EVERITT in [1]. Since then there have been contributions to the properties of M^n, and more general polynomials in M, from EVERITT and GIERTZ [3], [4] and [5], KAUFFMAN [6], KUMAR [7], READ [9] and ZETTL [10]. In particular [5] is a survey article on the general powers M^n of M.

For the general definition of a real-valued formally symmetric (equivalently formally self-adjoint) differential expression see [2, Ch. XIII, 2.1] or [8, section 15]. When M is given by (1.1) and the power M^n exists then M^n is also formally symmetric. In the particular case of (1.1) with $p = 1$, i.e. $p(x) = 1$ $(x \in I)$, we have

$$M[f] = -f'' + qf \quad \text{on} \quad I \tag{1.2}$$

and

$$M^2[f] = f^{(4)} - (2qf')' + (q^2 - q'')f \quad \text{on} \quad I\,. \tag{1.3}$$

Here derivatives of order greater than 2 are denoted by $f^{(3)}$ and $f^{(4)}$.

The results discussed in this paper are concerned with M and M^2, as given by (1.2) and (1.3), in the case when the interval I is the half-line $[0, \infty)$. In particular one of the results, see Theorem 1 below, answers a previously unsolved problem posed by Chaudhuri and Everitt in 1969, see [1, section 12].

Since we deal only with the half-line $[0, \infty)$ we use the abbreviations L^2 for the Hilbert function space $L^2(0, \infty)$, and $\mathrm{AC}_{\mathrm{loc}}$ for $\mathrm{AC}_{\mathrm{loc}}[0, \infty)$, i.e. those complex-valued functions defined on $[0, \infty)$ which are absolutely continuous on all compact sub-intervals of $[0, \infty)$.

Throughout the paper we assume that the coefficient q in (1.2) and (1.3) satisfies the following basic conditions:

(1.4) (i) q *is real-valued on* $[0, \infty)$

(ii) $q' \in \mathrm{AC}_{\mathrm{loc}}$,

which ensure that both M and M^2 exist as formally symmetric differential expressions on $[0, \infty)$.

In these circumstances the minimal closed symmetric operator generated by M in L^2 has deficiency indices either (1,1) or (2,2), the limit-point and limit-circle classifications at ∞, respectively, of Weyl; see [2, page 1306] or [8, section 17·5]. Similarly the deficiency indices corresponding to M^2 in L^2 are (2,2), (3,3) or (4,4) and we refer to M^2 as limit-r at ∞ when these are (r, r) for $r = 2$, 3 or 4, respectively.

The problem raised in [1, section 12] concerned the existence of coefficients q such that M, given by (1.2), is limit-point and M^2 is limit-3, both at ∞. At the time of writing of [1] it was known that M^2 is limit-4 if and only if M is limit-circle and that M^2 is frequently limit-2 when M is limit-point, but the general theory and examples available left the above problem open. Since then, several mathematicians have tried to find an example of such a coefficient q or, conversely, to prove that M^2 is limit-2 if and only if M is limit-point at ∞. The situation is further complicated by a recent result in [4] which states that if q satisfies (1.4) and, additionally, for some non-negative numbers k and X

$$q(x) \geqq -kx^2 \quad (x \in [X, \infty)) \tag{1.5}$$

then M is limit-point at ∞ (previously known, see [8, section 23]) *and* M^2 is limit-2 at ∞. This shows that if there is a coefficient q for which M is limit-point at ∞ and M^2 is limit-3 at ∞, then q will have to enjoy excursions through the $-kx^2$ barrier, for every positive number k, and yet do so in a way as to keep M in the limit-point case at ∞.

An answer to this problem has now been obtained and is given in

Theorem 1. *There exist coefficients q which satisfy the basic conditions* (1.4) *such that when the differential expressions M and M^2 on $[0, \infty)$ are defined by* (1.2) *and* (1.3) *then*

(a) *M is limit-point at ∞*

(b) *M^2 is limit-3 at ∞.*

Proof. This is given in sections 2 and 3 below.

In the proof of Theorem 1 two particular results are used which are themselves of interest and these are stated here separately since they throw some light on the nature of the integrable-square solutions of the differential equations associated with M and M^2. These equations are

$$(1.6) \qquad M[y] = 0 \quad \text{on} \quad [0, \infty) \quad \text{and} \quad M^2[y] = 0 \quad \text{on} \quad [0, \infty);$$

both regular at all points of $[0, \infty)$ but with singular points at ∞.

Theorem 2. *Assume that* (1.4) *holds and that the equation* $M^2[y] = 0$ *on* $[0, \infty)$ *has exactly* 3 *linearly independent solutions which are of integrable-square on* $[0, \infty)$, *i.e. solutions in* $L^2(0, \infty)$; *then* M^2 *is limit*-3 *at* ∞, M *is limit-point at* ∞ *and the equation* $M[y] = 0$ *on* $[0, \infty)$ *has exactly one linearly independent solution in* $L^2(0, \infty)$.

Theorem 3. *Assume that* (1.4) *holds*; *then the following two statements are equivalent*:

(1) *The equation* $M^2[y] = 0$ *on* $[0, \infty)$ *has exactly three linearly independent solutions in* $L^2(0, \infty)$.

(2) *The equation* $M[y] = 0$ *on* $[0, \infty)$ *has real-valued solutions* φ *and* $\psi \neq 0$ *such that*

$$\varphi \notin L^2(0, \infty), \quad \psi F \in L^2(0, \infty)$$

where $F(x) = \int_0^x \varphi^2 \quad (x \in [0, \infty))$.

We outline the contents of the paper. Section 2 contains proofs of Theorems 2 and 3 and associated results. The proof of Theorem 1 is given in section 3. In section 4 there are some remarks about the extent of the oscillations in such coefficients q as determined by the construction in the proof of Theorem 1. There is a list of references.

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2. Proof of Theorem 2 and Theorem 3. From now on we assume that q satisfies the basic conditions (1.4). It is then a standard result that the number of linearly independent solutions in L^2 of the eigenvalue problem

$$(2.1) \qquad M^2[y] = \lambda y \quad \text{on} \quad [0, \infty)$$

does not depend on λ as long as λ is a complex but non-real number, and also that this number is r if and only if M^2 is limit-r. The situation is more complicated when λ is real. However, the following statements are known to hold true, see e.g. [5]:

(a) When M^2 is limit-r and λ is real, then (2.1) has at most r linearly independent solutions which are of integrable square on $[0, \infty)$, that is in L^2.

(b) M^2 is limit-4 if and only if M is limit-2 (that is, in the limit-circle condition) in which case all solutions of (2.1) are in L^2, also for every real λ.

Theorem 2 is an almost direct consequence of (a) and (b). In fact assume, as in Theorem 2, that the equation $M^2[y] = 0$ on $[0, \infty)$ has exactly 3 linearly independent solutions in L^2. Then M^2 can not be limit-2 according to (a), and it can not be limit-4 in view of (b). Thus M^2 must be limit-3 and, again from (b), M must be limit-1 so that the equation $M[y] = 0$ may have at most one linearly independent solution in L^2 according to (a). But if it has no such solution, then $M^2[y] = 0$ can have at most two linearly independent solutions in L^2, since every solution of $M[y] = 0$ is also a solution of $M^2[y] = 0$. This completes the proof of Theorem 2.

To prove Theorem 3 we begin by considering certain solutions of $M^2[y] = 0$ given in terms of solutions of $M[y] = 0$. Let f_1 and f_2 be any two linearly independent real-valued functions which satisfy $M[y] = 0$, and are normalised so that $f_1 f_2' - f_1' f_2 = 1$. Define f_3 and f_4 by

$$f_3(x) = f_1(x) \int_0^x f_1 f_2 - f_2(x) \int_0^x f_1^2 \quad (x \in [0, \infty)),$$

and

$$f_4(x) = f_1(x) \int_0^x f_2^2 - f_2(x) \int_0^x f_1 f_2 \quad (x \in [0, \infty)).$$

A direct calculation verifies that

$$M[f_3] = f_1 \quad \text{and} \quad M[f_4] = f_2,$$

and it follows that f_i ($i = 1, 2, 3, 4$) are all solutions of $M^2[y] = 0$. They are linearly independent, since if

$$f = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 = 0$$

then

$$M[f] = a_3 f_1 + a_4 f_2 = 0,$$

which shows that $a_3 = a_4 = 0$ and thus also $a_1 = a_2 = 0$.

In one direction, the proof is now essentially contained in the following lemma concerning properties of certain pairs of functions in L^2_{loc}.

Lemma 1. *Let f and g be two functions which are defined and locally of integrable-square on* $[0, \infty)$. *Assume that*

$$f \notin L^2 \quad \text{and} \quad gF \in L^2 \quad \text{where} \quad F(x) = \int_0^x |f|^2 \quad (x \in [0, \infty))\,;$$

then

$$g \in L^2 \quad \text{and} \quad fG \in L^2 \quad \text{where} \quad G(x) = \int_x^\infty |g|^2 \quad (x \in [0, \infty))\,,$$

$$fg \in L \quad \text{and} \quad fH \in L^2 \quad \text{where} \quad H(x) = \int_x^\infty |fg| \quad (x \in [0, \infty))\,.$$

Proof. Let f and g satisfy the conditions of the lemma. The assumption that f is not in L^2 implies that $F(x) = \int_0^x |f|^2$ tends to infinity with x and thus that there exists a y in $(0, \infty)$ for which $F(y) = 1$ and $F(x) \geqq 1$ $(x > y)$. The assumption $gF \in L^2$ then implies $g \in L^2$ and $g\sqrt{F} \in L^2$, and it follows that also $f\sqrt{G}$ is in L^2, where $G(x) = \int_x^\infty |g|^2$, since a partial integration and the fact that F is increasing give

$$\int_0^x |f^2 G| = F(x) \int_x^\infty |g|^2 + \int_0^x |g^2 F| \leqq \int_0^\infty |g\sqrt{F}|^2 < \infty\,.$$

Since G is continuous and bounded this proves that fG is in L^2.

The inequality, valid for $0 \leqq y < x < \infty$,

$$\left[\int_y^x |fg|\right]^2 = \left[\int_y^x |gFf/F|\right]^2 \leqq \int_y^x |gF|^2 \int_y^x \{F'/F^2\} = [1 - 1/F(x)] \int_y^x |gF|^2$$

proves that fg is in L.

The analogous inequality

$$H^2(x) = \left[\int_x^\infty |fg|\right]^2 \leqq [1/F(x)] \int_x^\infty |gF|^2$$

shows that $FH^2(x) \to 0$ $(x \to \infty)$ and since

$$\int_0^x |fH|^2 = (FH^2)(x) + 2\int_0^x |fgFH| \leqq (FH^2)(x) + 2\left\{\int_0^x |gF|^2 \int_0^x |fH|^2\right\}^{1/2}$$

also that $fH \in L^2$. This completes the proof of Lemma 1.

Now assume that the statement (2) in Theorem 3 holds true, with φ and ψ normalised so that $\varphi\psi' - \varphi'\psi = 1$. Then φ and ψ satisfy the conditions of Lemma 1 with $f = \varphi$ and $g = \psi$. Thus $\psi \in L^2$, $\varphi\psi \in L$ and the functions defined by

$$\varphi(x)\int_x^\infty \psi^2\,, \quad \varphi(x)\int_x^\infty \varphi\psi \quad \text{and} \quad \psi(x)\int_0^x \varphi\psi \quad (x \in [0, \infty))$$

are all in L^2. With

$$f_3(x) = \varphi(x)\int_0^x \varphi\psi - \psi(x)\int_0^x \varphi^2 \tag{2.2}$$

$$f_4(x) = \varphi(x)\int_0^x \psi^2 - \psi(x)\int_0^x \varphi\psi \tag{2.3}$$

it follows that ψ, $f_3 - \varphi \int_0^\infty \varphi\psi$ and $f_4 - \varphi \int_0^\infty \psi^2$ are three linearly independent L^2-solutions of $M^2[y] = 0$. Since φ is a fourth solution which is not in L^2, it is clear that (1) is satisfied.

Conversely, assume that the statement (1) holds true. Then, from Theorem 2, M must be in the limit-point condition at ∞ and $M[y] = 0$ must have exactly one linearly independent L^2-solution. Let φ and ψ be real-valued solutions of $M[y] = 0$ which satisfy $\varphi \notin L^2$, $\psi \in L^2$ and $\varphi\psi' - \varphi'\psi = 1$, and let f_3 and f_4 be defined by (2.2) and (2.3) so that $\{\varphi, \psi, f_3, f_4\}$ is a basis for the solutions of $M^2[y] = 0$. According to (1) there exist linearly independent vectors (a_2, a_3, a_4) and (b_2, b_3, b_4) in R^3 for which $a_2\varphi + a_3 f_3 + a_4 f_4$ and $b_2\varphi + b_3 f_3 + b_4 f_4$ are both in L^2. It follows that $f_3 + a\varphi \in L^2$ for some unique real number a (eliminating f_4 above in case both $a_4 \neq 0$ and $b_4 \neq 0$). Put

$$F(x) = \int_0^x \varphi^2 \quad \text{and} \quad H(x) = a + \int_0^x \varphi\psi \quad (x \in [0, \infty)).$$

Then $\varphi H - \psi F = f_3 + a\varphi \in L^2$, and the identity $(f_3 + a\varphi)^2 = (\varphi H)^2 + (\psi F)^2 - F(H^2)'$ gives

$$\int_0^x (f_3 + a\varphi)^2 = 2\int_0^x (\varphi H)^2 + \int_0^x (\psi F)^2 - (FH^2)(x) \quad (x \in [0, \infty)), \tag{2.4}$$

after a partial integration of the last term.

We shall show that $\psi F \in L^2$, so that the statement (2) follows, by obtaining a contradiction from (2.4) in case ψF is not in L^2.

If $\psi F \notin L^2$ it is clear from (2.4) that

$$U(x) = (FH^2)(x) - 2\int_0^x (\varphi H)^2 \to +\infty \quad (x \to \infty), \tag{2.5}$$

and since the function V defined by

$$V(x) = F^{-2}(x)\int_0^x (\varphi H)^2 \quad (x \in (0, \infty))$$

satisfies $V' = \varphi^2 F^{-3} U$ it follows from (2.5) and the definition of F that V' (as well as V) is non-negative for sufficiently large x. Thus $V(x) > \frac{1}{2}C^2$, say, that is

$$\int_0^x (\varphi H)^2 > \tfrac{1}{2}C^2\, F^2(x)\,,$$

for some constant $C > 0$ and large x. Returning to (2.5) we obtain

$$0 < U(x) < (FH^2)(x) - C^2\, F^2(x)\,,$$

that is, $H(x) > C\sqrt{F(x)}$ for all large x. Now this inequality gives us the required contradiction. In fact, let X be so large that

$$\int_X^\infty \psi^2 < C^2/4\,.$$

Then for $x > X$,

$$H(x) = a + \int_0^x \varphi\psi \leqq a + \int_0^X |\varphi\psi| + \left[\int_X^x \psi^2 \int_X^x \varphi^2\right]^{1/2} \leqq$$
$$\leqq \left(C/2 + \left(a + \int_0^X |\varphi\psi|\right)\Big/\sqrt{F(x)}\right)\sqrt{F(x)}$$

where the term in ( ) is $< C$ when x is large enough, since $F(x) \to \infty$ $(x \to \infty)$.

Thus $\psi F \in L^2$ and the proof of Theorem 3 is complete.

3. Proof of Theorem 1. To prove Theorem 1 it is sufficient, according to Theorem 2, to produce differential expressions M for which (1) of Theorem 3 holds true. To achieve this it is not only sufficient but also necessary for us to ensure that the equation $M[y] = 0$ on $[0, \infty)$ has solutions φ and ψ which satisfy (2). As it turns out, such solutions must necessarily be of an oscillatory nature on $[0, \infty)$, with an unbounded sequence of discrete and simple zeros. In the following lemma we give additional conditions on the zeros of the linearly independent L^2-solution ψ which ensure that M^2 is, indeed, in the limit-3 case.

Lemma 2. *Let φ and ψ be two real-valued functions in $C^4[0, \infty)$ which satisfy $\varphi\psi' - \varphi'\psi = 1$ on $[0, \infty)$. Assume that ψ has a denumerable increasing sequence $(x_n)_{n=0}^\infty$ of zeros, with $x_0 = 0$ and $x_n \to \infty$ $(n \to \infty)$, and that*

(i) $$\sum_{n=1}^\infty \{(x_n - x_{n-1})^3\, x_n^2\} \quad \textit{converges}\,,$$

(ii) *there exist positive numbers A, B and C such that*

$$\int_{x_{n-1}}^{x_n} \psi^2 < A(x_n - x_{n-1})^3 \quad \textit{and} \quad Bx_n < \int_0^{x_n} \varphi^2 < Cx_n$$

for all positive integers n.

Then $(\psi''/\psi)(x)$ *tends to a finite limit as x tends to a zero of* ψ; *the coefficient* q *defined by* $q(x) = (\psi''/\psi)(x)$ $(x \in [0, \infty))$ *is in* $C^2[0, \infty)$ (where, of course, q is defined by continuity at the zeros of ψ); *and* φ *and* ψ *satisfy* (2) *of Theorem* 3 *with* M *defined by* $M[f] = -f'' + qf$.

Proof. Let φ and ψ satisfy the conditions of the lemma. The facts that $(\psi''/\psi)(x) = q(x)$ tends to a finite limit as x tends to a zero of ψ and that $q \in C^2[0, \infty)$ follow directly from the assumptions $\varphi, \psi \in C^4[0, \infty)$ and $\varphi\psi' - \varphi'\psi = 1$, which imply that $\varphi(x) \neq 0$ when $\psi(x) = 0$, and that $\varphi\psi'' = \varphi''\psi$. It is also clear from the last equality that $M[\varphi] = M[\psi] = 0$.

The lower bound in the assumption (ii) implies that φ is not in L^2. On the other hand with $F(x) = \int_0^x \varphi^2$ $(x \in [0, \infty))$ the upper bounds give, for x in $[x_{N-1}, x_N)$,

$$\int_0^x (\psi F)^2 < \sum_{n=1}^{N} \left\{ \int_{x_{n-1}}^{x_n} (\psi F)^2 \right\} < AC^2 \sum_{n=1}^{N} \{(x_n - x_{n-1})^3 x_n^2\} .$$

In view of the assumption (i) it follows that $\psi F \in L^2$, that is, φ and ψ satisfy (2) of Theorem 3. This completes the proof of Lemma 2.

We now prove Theorem 1 by displaying functions φ and ψ which satisfy the conditions of Lemma 2. We shall construct such functions in terms of a real-valued function f in $C^\infty[0, 1]$ with the properties that

(i) f *is infinitely differentiable, positive and convex upwards with*

$$f(0) = 0\,, \quad f'(0) = k > 0\,, \quad f(1) = B > 0 \quad \textit{and} \quad f'(1) = 0\,,$$

and that, for some number $r \in (0, \frac{1}{2})$,

(ii) $f(x) = kx$ $(x \in [0, r])$ *and* $f(x) = B$ $(x \in [1 - r, 1])$,

(iii) $\int_r^1 (1/f)^2 = 1/(k^2 r)$.

Then we shall show that functions f with the above properties do endeed exist, provided $k/B \in (1, 2)$ and r is small enough.

Assuming that f has the properties (i)–(iii), define g on $[0, 1]$ by

$$g(x) = -f(x) \int_x^1 (1/f)^2 \quad (x \in [0, 1])\,,$$

where $g(0)$ is defined by continuity.

Since f is positive this function g is negative, and since, by a direct calculation,

$$fg' - f'g = 1 \quad \textit{and} \quad fg'' = f''g \quad \textit{on} \quad [0, 1]\,, \tag{3.1}$$

we infer from (ii) that

$$g \quad \textit{is convex downwards with} \quad g(0) = -1/k \quad \textit{and} \quad g(1) = 0\,. \tag{3.2}$$

Near the left end point of $[0, 1]$ we obtain from (iii)

$$g(x) = -kx\left[\int_x^r (kt)^{-2}\,dt + \int_r^1 f^{-2}\right] = -1/k \quad (x \in [0, r]), \tag{3.3}$$

and near the right end point we have

$$g(x) = -B\int_x^1 B^{-2} = (x - 1)/B \quad (x \in [1 - r, 1]). \tag{3.4}$$

It is clear from (i) that f is increasing and satisfies $Bx \leqq f(x) \leqq B$ $(x \in [0, 1])$ and it follows from (3.2) and (3.3) that $|g(x)| \leqq 1/k$ on $[0, 1]$. Thus

$$B^2/3 < \int_0^1 f^2 < B^2 \quad \text{and} \quad \int_0^1 g^2 < 1/k^2. \tag{3.5}$$

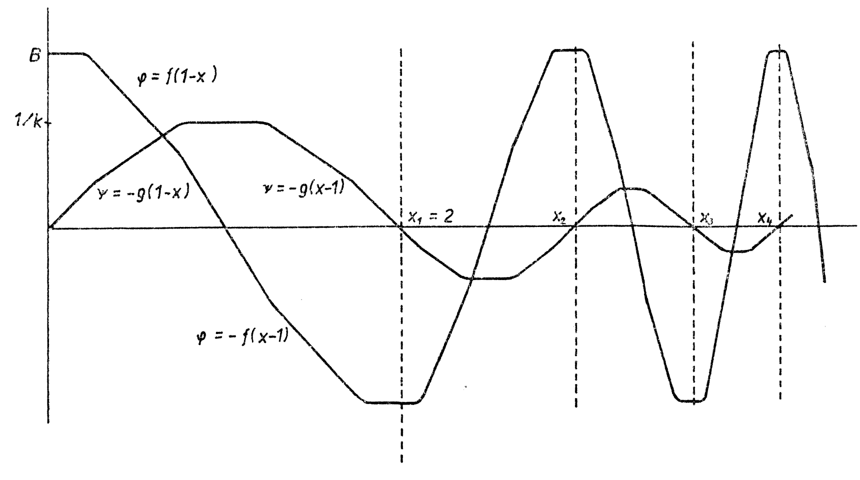

Fig. 1.

In the interval $[x_0, x_1] = [0, 2]$ $\varphi(x) = -f_1(x)$ and $\psi(x) = -g_1(x)$. In the intervals $[x_{n-1}, x_n]$ $\varphi(x) = (-1)^n f_1(2(x - x_{n-1})/(x_n - x_{n-1}))$, $\psi(x) = (-1)^n \frac{1}{2}(x_n - x_{n-1})$ $g_1(2(x - {}_{n-1}) :$ $: (x_n - x_{n-1}))$. In this particular example, $L_n = 2/n$ and $B = 1$, $k = \frac{5}{3}$, and $r = \frac{1}{5}$; as we shall see later the set of functions f satisfying (i)—(iii) is non-empty for these values of B, k, and r.

Let $(x_n)_{n=0}^{\infty}$ be a sequence of real numbers tending monotonically to infinity with n, with $x_0 = 0$ and $x_1 = 2$, and put $L_n = x_n - x_{n-1}$. Define, first f_1 and g_1 on $[0, L_1] = [0, 2]$ by $f_1(x) = -f(1 - x)$, $g_1(x) = g(1 - x)$ $(x \in [0, 1])$ and $f_1(x) = f(x - 1)$, $g_1(x) = g(x - 1)$ $(x \in [1, 2])$ and then, for each integer $n > 1$, f_n and g_n on $[0, L_n]$ by

$$f_n(x) = f_1(2x/L_n) \quad \text{and} \quad g_n(x) = (L_n/2)\, g_1(2x/L_n) \quad (x \in [0, L_n]). \tag{3.6}$$

The property (ii) of f ensures that f_1 is infinitely differentiable on its interval of definition and takes the constant value $-B$ near the left end point and the value B near the right end point of this interval. Similarly, (3.3) shows that g_1 is also infinitely differentiable and (3.4) that g_1 vanishes linearly at the end points of its interval of definition, with slope $-1/B$ near the left end point and slope $1/B$ near the right one. It follows from (3.1) that $f_1 g_1' - f_1' g_1 = 1$ on this interval. Clearly, from the definition (3.6), the functions f_n and g_n inherit these properties for all integers $n > 1$. But this means that we may patch the f_n : s together, and also the g_n : s, to obtain two functions φ and ψ in $C^\infty[0, \infty)$ of the form shown in Fig. 1 by defining, for each interval $I_n = [x_{n-1}, x_n)$,

$$\varphi(x) = (-1)^n f_n(x - x_{n-1}) \quad \text{and} \quad \psi(x) = (-1)^n g_n(x - x_{n-1}) \quad (x \in I_n)\,.$$

These functions satisfy $\varphi\psi' - \varphi'\psi = 1$ on $[0, \infty)$ and since

$$\int_{I_n} \varphi^2 = (L_n/2) \int_0^2 f_1^2 = L_n \int_0^1 f^2 \quad \text{and} \quad \int_{I_n} \psi^2 = (L_n^3/4) \int_0^1 g^2$$

it follows from (3.5) that

$$\int_{x_{n-1}}^{x_n} \psi^2 < L_n^3/(2k)^2 \quad \text{and} \quad B^2 L_n/3 < \int_{x_{n-1}}^{x_n} \varphi^2 < B^2 L_n\,.$$

Thus φ and ψ satisfy the conditions of Lemma 2 provided we select the sequence $(L_n)_{n=1}^\infty$ so that $L_1 = 2$ and

$$\sum_{n=1}^\infty \{L_n^3 x_n^2\} = \sum_{n=1}^\infty \{L_n^3 (\sum_{k=1}^n L_k)^2\} \quad \text{converges}\,. \tag{3.7}$$

To obtain examples of such sequences we may choose $L_n = 2n^{-\alpha}$ with $\alpha \in (\frac{3}{5}, 1]$; the fact that these satisfy (3.7) is easily verified on using

$$\sum_{k=1}^n L_k < 2\left(1 + \int_1^n t^{-\alpha}\, \mathrm{d}t\right) < \begin{cases} 2(1-\alpha)^{-1}\, n^{1-\alpha} & (\alpha \in (\frac{3}{5}, 1)) \\ 2 + 2\log n & (\alpha = 1)\,. \end{cases}$$

It remains to verify that there exist functions f with the above properties (i)–(iii).

Intuitively, it seems clear that there are functions which have the properties stated in (i) and (ii) when $1 < k/B < 2$, but for lack of a suitable reference we sketch a proof of

Lemma 3. *Let a, b, c and d be positive real numbers satisfying $b < c/d < a$. Then there are functions in $C^\infty[0, d]$ which are convex upwards and satisfy $f(0) = 0, f'(0) = a, f(d) = c, f'(d) = b$ and $f^{(n)}(0) = f^{(n)}(d) = 0$ $(n \geqq 2)$.*

Proof. Define $P : [0, 1] \to [0, 1]$ by

$$P(x) = K \int_0^x \exp\left\{-\frac{1}{s} - \frac{1}{1-s}\right\} ds \quad (x \in [0, 1]), \tag{3.8}$$

where the constant K is determined by the requirement $P(1) = 1$. It is straightforward to verify that P is infinitely differentiable and increases monotonically from $P(0) = 0$ to $P(1) = 1$, with all derivatives vanishing at $x = 0$ and at $x = 1$, and satisfies $P(x) + P(1 - x) = 1$ $(0 \leqq x \leqq \frac{1}{2})$ so that $\int_0^1 P = \frac{1}{2}$.

Now the assumption $bd < c < ad$ implies that $l = (c - bd)/(ad - bd)$ satifies $0 < l < 1$, and thus in turn that in a $(s\text{-}t)$-plane, the line $t + \frac{1}{2}s = l$ has a nonempty intersection with the half-square determined by $0 \leqq t < t + s \leqq 1$. For each (s, t) in this intersection, define $Q_{st} : [0, d] \to [0, a - b]$ by

$$Q_{st}(x) = \begin{cases} 0 & (0 \leqq x \leqq td) \\ (a - b)\, P\left(\dfrac{x - td}{sd}\right) & (td < x < (t + s)\, d) \\ a - b & ((t + s)\, d \leqq x \leqq d)\,. \end{cases}$$

Then Q_{st} is in $C^\infty[0, d]$ and

$$\int_0^d Q_{st} = (ad - bd)\, s \int_0^1 P + (ad - bd)(1 - s - t) = (ad - bd)(1 - t - \tfrac{1}{2}s) =$$
$$= (ad - bd)(1 - l) = ad - c\,.$$

Thus each function $f : [0, d] \to [0, c]$ defined by

$$f(x) = ax - \int_0^x Q_{st} \quad (x \in [0, d])$$

satisfies $f(d) = c$. A direct computation shows that f has the other properties stated in the lemma as well.

Now let $S = S(B, k, r)$ be the subset of $C^\infty[0, 1]$ containing real-valued functions which satisfy (i) and (ii). According to Lemma 3, with $a = k$, $b = 0$, $c = B$ and $d = 1$, this set is nonempty, and it follows from the convexity requirement in (i) that each f in S is bounded by $l(x) \leqq f(x) \leqq m(x)$ in the interval $[r, 1 - r]$, where the graph of

$$l(x) = kr + \frac{B - kr}{1 - 2r}(x - r) \quad (x \in [r, 1 - r])$$

is the line segment connecting the points (r, kr) and $(1 - r, B)$, and

$$m(x) = \begin{cases} kx & (x \in [0, B/k)) \\ B & (x \in [B/k, 1])\,. \end{cases}$$

For each s in the open interval $(r, B/k)$, let l_s be the line parallel to l which intersects the graph of m at the points (s, ks) and (t, B), say. On rounding off the corners following the recipe in Lemma 3 we obtain, for any sufficiently small positive number ε and all s in $(r + \varepsilon, Bk^{-1} - \varepsilon)$, functions in S with graphs coinciding with that of m in $(0, s - \varepsilon) \cup (t + \varepsilon, 1)$ and with l_s in $(s + \varepsilon, t - \varepsilon)$. A continuity argument shows that S contains functions for which $\int_r^1 (1/f)^2$ takes any prescribed value in the open interval

$$\left(\int_r^{1-r} (1/m)^2 + r/B^2 \, , \int_r^{1-r} (1/l)^2 + r/B^2\right).$$

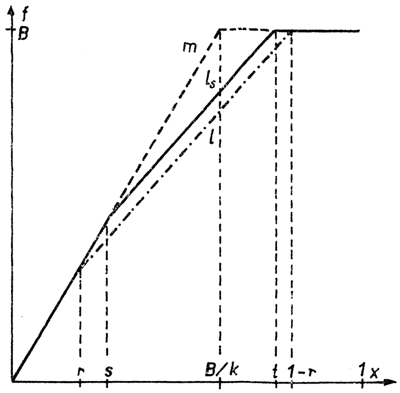

Fig. 2.

In particular, it contains functions which satisfy (iii) when $1/(k^2 r)$ lies in this interval. Putting $k/B = b$ we find after some elementary calculations that this condition takes the form

$$(1 + b^2 r - 2br)/b^2 r < 1/b^2 r < (b + b^2 r^2 - 2br)/b^2 r \, ,$$

or equivalently

$$b < 2 < b + (1 - br)^2 \, ,$$

which is satisfied for $b \in (1, 2)$ provided r is small enough — in the two figures above we have used $B = 1$, $k = \frac{5}{3}$ and $r = \frac{1}{5}$.

4. Comments on the above examples. All examples constructed by the method used in section 3 have the common property that there exist arbitrarily large x for which $q(x) = (\varphi''/\varphi)(x) < -x^3$. In fact, the property (i) for f implies that f''/f must be strictly negative at some points in $(r, 1 - r)$. Let y be such a point with, say, $(f''/f)(y) = -c^2$. Then at the points $y_n = x_{n-1} + (L_n/2)\, y$ we have

$$q(y_n) = (2/L_n)^2 \, (f_1''/f_1)(y) = -(2c/L_n)^2 \, .$$

Thus if $q(y_n) \geqq -y_n^3$ for $n > N_0$ then

$$(2c/L_n)^2 \leqq (x_{n-1} + (L_n/2)\, y)^3 < x_n^3 \quad (n > N_0),$$

so that for $N > N_0$

$$\sum_{n=N_0}^{N} \{L_n^3 x_n^2\} > 4c^2 \sum_{n=N_0}^{N} \{L_n/x_n\} = 4c^2 \sum_{n=N_0}^{N} \{L_n / \sum_{k=1}^{n} L_k\}.$$

Here the last sum diverges since $\sum^{\infty} L_k$ is divergent, just as $\int^\infty \{h(x)/\int^x h\}\, dx$ diverges when $\int^\infty h$ is divergent. This contradicts (3.7) and so $q(y_n) < -y_n^3$ for arbitrarily large n. On the other hand, given any positive number ε the method in section 3 yields coefficients which satisfy $q(x) \geqq -x^{3+\varepsilon}$ $(x \in [0, \infty))$. These $q : s$ result from functions φ and ψ constructed by means of functions f which hold close to $\sin(\pi x/2)$ on $[0, 1]$, with very small intervals of linearity near the end-points.

Thus the results of this paper still leave open the question whether the condition (1.5) for M^2 to be limit-2 at ∞ is best possible or not in the case when M is limit-point at ∞.

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