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ON CONVERGENCE FIELDS OF REGULAR MATRIX TRANSFORMATIONS

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Let $T = (a_{ik})$ $(i, k = 1, 2, \ldots)$ be an infinite matrix of real (complex) numbers. The sequence $\{t_n\}_{n=1}^{\infty}$ of real (complex) numbers is said to be T-limitable (limitable by the method (T)) to the number t if $\lim_{n\to\infty} t'_n = t$, where

$$t'_n = \sum_{k=1}^{\infty} a_{nk} t_k \quad (n = 1, 2, \ldots).$$

If $\{t_n\}_{n=1}^{\infty}$ is T-limitable to the number t, we write $T - \lim_{n\to\infty} t_n = t$.

We denote by $F(T)$ the set of all T-limitable sequences. The set $F(T)$ is called *the convergence field of the method* (T) or *the convergence field of the matrix transformation defined by* T (cf. [7], p. 2,4). The method (T) defined by the matrix T is said *to be regular* provided that $F(T)$ contains all convergent sequences and $\lim_{n\to\infty} t_n = t$ implies $T - \lim_{n\to\infty} t_n = t$. If (T) is regular then T is called *a regular matrix*.

It is well-known that the method (T) is regular if and only if the matrix T satisfies the following three conditions:

(1) There exists $K > 0$ such that for each $n = 1, 2, \ldots$ we have $\sum_{k=1}^{\infty} |a_{nk}| \leqq K$;

(2) $\lim_{n\to\infty} a_{nk} = 0$ for each fixed positive integer k;

(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$

(cf. [1], p. 79; [7], p. 8).

H. STEINHAUS has shown in the year 1911 that for each regular method (T) there exists a sequence of numbers 0 and 1 which is not T-limitable (to any real number) (cf. [10]; [1], p. 93–94).

In the year 1958 F. R. KOEGH and G. M. PETERSEN have shown that if (T) is a regular method and $x = \{\xi_k\}_{k=1}^{\infty}$ is a bounded divergent sequence of real numbers then there exists a subsequence of the sequence x which is not T-limitable (cf. [5]; [1], p. 405; [7], p. 78–81).

Denote by X the set of all sequences of 0's and 1's with an infinite number of 1's. Define the function $v : X \to (0, 1\rangle$ in the following way: for $x = \{\xi_k\}_{k=1}^{\infty} \in X$ put $v(x) = \sum_{k=1}^{\infty} \xi_k 2^{-k}$. Evidently, v is a one-to-one mapping of X onto $(0, 1\rangle$. Denote by $X_1(T)$ the set of all $x \in X$ which are T-limitable. In the year 1945 J. D. HILL has shown that the set $v(X_1(T))$ is a set of the first Baire category in $(0, 1\rangle$ provided that (T) is a regular method (cf. [4]; [1], p. 239).

The purpose of this paper is to show the usefulness of the well-known theorem on discontinuity points of functions of the first Baire class in the study of the structure of convergence fields of regular matrix methods. Using this theorem we shall prove an assertion (Theorem 1.1) from which the above mentioned results of Steinhaus, Koegh-Petersen and Hill follow. Further, we shall give a new proof (based on the theorem on discontinuity points of functions of the first Baire class) of a result of C. GOFFMAN and G. M. PETERSEN (cf. [3]) on submethods of regular matrix methods (Theorem 1.2). In the second part of the paper we shall study some problems for certain T-limitable sequences defined by help of Cantor's expansions of real numbers.

1

Let M_k $(k = 1, 2, \dots)$ be a non-void set of complex numbers. Put $Y = M_1 \times \times M_2 \times \dots$ Define the function ϱ on the set $Y \times Y$ as follows: if $x = \{\xi_k\}_{k=1}^{\infty} \in Y$, $y = \{\eta_k\}_{k=1}^{\infty} \in Y$, then we put $\varrho(x, y) = 0$ for $x = y$ and

$$\varrho(x, y) = (\inf\{n; \xi_n \neq \eta_n\})^{-1}$$

for $x \neq y$.

It is easy to check that ϱ is a metric on Y and the metric space (Y, ϱ) is complete. According to the well-known Baire's theorem Y is a set of the second Baire category in (Y, ϱ).

If $T = (a_{nk})$ is an infinite matrix of numbers, then the set of all T-limitable sequences $x \in Y$ will be denoted by $Y_1(T)$.

If (M, ϱ_1) is a metric space, $x \in M$ and $\delta > 0$, then the set $S(x, \delta) = \{y \in M; \varrho_1(x, y) < \delta\}$ is called *the spherical neighbourhood of the point x in the space M*. Further, diam A denotes the diameter of the set A.

Theorem 1.1. *Let $T = (a_{nk})$, let (T) be a regular method. Let M_k $(k = 1, 2, \dots)$ be a non-void set of complex numbers. Let us suppose that*

(i) $$\sup_{k=1,2,\dots} \operatorname{diam} M_k < +\infty\,;$$

(ii) *there exist two sequences*

$$y' = \{\eta'_k\}_{k=1}^{\infty} \in Y_1(T), \quad y'' = \{\eta''_k\}_{k=1}^{\infty} \in Y_1(T)$$

such that $\{\eta''_k - \eta'_k\}_{k=1}^{\infty}$ *is a convergent sequence and* $\lambda = \lim_{k\to\infty} (\eta''_k - \eta'_k) \neq 0$.

Then the set $Y_1(T)$ *is a dense set (in* Y*) of the first Baire category.*

Corollary. *The set* $Y_0(T) = Y - Y_1(T)$ *of all sequences from* Y *which are not* T*-limitable to any number is a residual set of the second category in* Y.

Proof of Theorem 1.1. Let $x^0 = \{\xi^0_k\}_{k=1}^{\infty} \in Y$, $\delta > 0$. Choose a natural number s so that $1/s < \delta$ and define the sequence $y = \{\eta_k\}_{k=1}^{\infty}$ as follows: $\eta_k = \xi^0_k$ for $k \leqq s$ and $\eta_k = \eta'_k$ for $k > s$ (see the assumption (ii) of the theorem). Then $\varrho(x^0, y) < 1/s < \delta$, hence $y \in S(x^0, \delta)$ and evidently $y \in Y_1(T)$. Hence $Y_1(T)$ is a dense set in Y.

On the subspace $Y_1 = Y_1(T)$ of the space Y we shall define the functions τ, τ_n $(n = 1, 2, \ldots)$ in the following way: for $x = \{\xi_k\}_{k=1}^{\infty} \in Y_1$ we put

$$\tau_n(x) = \sum_{k=1}^{\infty} a_{nk}\xi_k \quad (n = 1, 2, \ldots),$$

$$\tau(x) = \lim_{n\to\infty} \tau_n(x).$$

We shall show that:

(a) each function τ_n $(n = 1, 2, \ldots)$ is continuous on Y_1;

(b) the function τ is discontinuous at each point of Y.

It follows from (a), (b) that τ is a function of the first Baire class on Y_1 and according to the well-known theorem on discontinuity points of functions of the first Baire class (cf. [9], p. 185) the set of discontinuity points of τ is a set of the first Baire category in Y_1. Therefore (see (b)) the set Y_1 is a set of the first category in Y_1 and hence of the first category in Y, too.

Hence it suffices to prove the assertions (a), (b).

(a) Let n be a fixed natural number, $x^0 = \{\xi^0_k\}_{k=1}^{\infty} \in Y_1$. We shall show that τ_n is continuous at x^0.

According to (i) there exists a number $K_1 > 0$ such that

$$\text{diam } M_k \leqq K_1 \quad (k = 1, 2, \ldots). \tag{4}$$

Let $\varepsilon > 0$. On account of (1) there exists a natural number m such that

$$\sum_{k=m+1}^{\infty} |a_{nk}| < \frac{\varepsilon}{K_1}. \tag{5}$$

Let $x = \{\xi_k\}_{k=1}^{\infty} \in S(x^0, 1/m)$ ($S(x^0, 1/m)$ is the spherical neighbourhood of x^0 in the space Y_1). Then it follows from the definition of the metric ϱ that $\xi_k = \xi_k^0$ for $k = 1, 2, \ldots, m$. Using (4), (5) we obtain easily the estimate

$$|\tau_n(x) - \tau_n(x^0)| = \left|\sum_{k=m+1}^{\infty} a_{nk}(\xi_k - \xi_k^0)\right| \leqq$$

$$\leqq \sum_{k=m+1}^{\infty} |a_{nk}| \operatorname{diam} M_k \leqq K_1 \frac{\varepsilon}{K_1} = \varepsilon .$$

Hence τ_n is continuous at x^0.

(b) Let $z^0 = \{\zeta_k^0\}_{k=1}^{\infty} \in Y_1$. We shall prove that the function τ is discontinuous at z^0. It suffices to prove that in each spherical neighbourhood $S(z^0, \delta)$ of the point z^0 (in the space Y_1) there are two points x^1, x^2 such that $|\tau(x^1) - \tau(x^2)| \geqq \frac{1}{4}|\lambda|$ (see the assumption (ii)).

Let $S(z^0, \delta)$ be an arbitrary spherical neighbourhood of the point z^0 in the space Y_1. Put

$$\eta_k'' - \eta_k' = \lambda + \varepsilon_k \quad (k = 1, 2, \ldots),$$

hence $\varepsilon_k \to 0$ (see (ii)). Choose a natural number m so that $1/m < \delta$ and simultaneously

(6) $$|\varepsilon_k| < \frac{|\lambda|}{4} \quad \text{for each} \quad k > m .$$

Define $x^i = \{\xi_k^i\}_{k=1}^{\infty}$ $(i = 1, 2)$ as follows:

$$\xi_k^1 = \xi_k^2 = \zeta_k^0 \quad \text{for} \quad 1 \leqq k \leqq m , \quad \xi_k^1 = \eta_k' \quad \text{and} \quad \xi_k^2 = \eta_k'' \quad \text{for} \quad k > m .$$

It is easy to see that $\varrho(x^i, z^0) < 1/m < \delta$, hence $x^i \in S(z^0, \delta)$ $(i = 1, 2)$.

Let n be an arbitrary natural number. According to the definition of x^i $(i = 1, 2)$, a simple calculation yields the estimate

(7) $$|\tau_n(x^2) - \tau_n(x^1)| = \left|\sum_{k=1}^{\infty} a_{nk}(\xi_k^2 - \xi_k^1)\right| = \left|\sum_{k=m+1}^{\infty} a_{nk}(\eta_k'' - \eta_k')\right| =$$

$$= \left|\lambda \sum_{k=1}^{\infty} a_{nk} - \lambda \sum_{k=1}^{m} a_{nk} + \sum_{k=m+1}^{\infty} a_{nk}\varepsilon_k\right| \geqq$$

$$\geqq |\lambda| \left|\sum_{k=1}^{\infty} a_{nk}\right| - |\lambda| \sum_{k=1}^{m} |a_{nk}| - \sum_{k=m+1}^{\infty} |a_{nk}| \, |\varepsilon_k| .$$

According to (3) there exists n_1 such that for each $n > n_1$ we have

(8) $$\left|\sum_{k=1}^{\infty} a_{nk}\right| > \tfrac{3}{4} .$$

Similarly, according to (2) there exists n_2 such that for each $n > n_2$ we have

(9) $$\sum_{k=1}^{m} |a_{nk}| < \tfrac{1}{4}.$$

For each $n > \max(n_1, n_2)$ we get from (7) on account of (6), (8), (9) the inequality

$$|\tau_n(x^2) - \tau_n(x^1)| \geqq \frac{|\lambda|}{4}$$

and from this by $n \to \infty$ we obtain $|\tau(x^2) - \tau(x^1)| \geqq \frac{1}{4}|\lambda|$. The proof is complete.

Now we show that the above mentioned results of Steinhaus, Koegh-Petersen and Hill (see Introduction) follow from Theorem 1.1 just proved.

(I) Theorem of Steinhaus.

Put

$$M_k = \{0, 1\}, \quad \eta'_k = 0, \quad \eta''_k = 1 \quad (k = 1, 2, \ldots).$$

The assumptions of Theorem 1.1 are evidently fulfilled. It follows from the corollary of Theorem 1.1 that the set of all sequences of 0's and 1's which are not T-limitable ((T) being a regular method) is a set of the second category in Y, hence it is non-empty. Therefore there exists a sequence of 0's and 1's which is not T-limitable.

(II) Theorem of Koegh and Petersen.

Let $a = \{\alpha_k\}_{k=1}^{\infty}$ be a bounded divergent sequence of real numbers. Hence

$$-\infty < \liminf_{k\to\infty} \alpha_k = t_1 < t_2 = \limsup_{k\to\infty} \alpha_k < +\infty.$$

It is evident that there exist two sequences of natural numbers

$$k_1 < k_2 < \ldots < k_n < \ldots,$$

$$k'_1 < k'_2 < \ldots < k'_n < \ldots$$

such that

$$k_1 < k'_1 < k_2 < k'_2 < \ldots,$$

$$\lim_{n\to\infty} \alpha_{k_n} = t_1, \quad \lim_{n\to\infty} \alpha_{k'_n} = t_2$$

and for each $n = 1, 2, \ldots$ we have $\alpha_{k_n} \neq \alpha_{k_n'}$.

Put

$$M_n = \{\alpha_{k_n}, \alpha_{k_n'}\}, \quad \eta'_n = \alpha_{k_n}, \quad \eta''_n = \alpha_{k_n'} \quad (n = 1, 2, \ldots).$$

Then the assumptions of Theorem 1.1 are fulfilled. It follows from the corollary of Theorem 1.1 that the set of all sequences $x \in Y$, $Y = M_1 \times M_2 \times \ldots$, which are not T-limitable ((T) being a regular matrix method) is a set of the second Baire category in Y. It suffices to notice that each point of Y is a subsequence of the sequence a.

(III) Theorem of Hill.

Put

$$M_k = \{0, 1\}, \quad \eta'_k = 0, \quad \eta''_k = 1 \quad (k = 1, 2, \ldots).$$

The assumptions of Theorem 1.1 are fulfilled. Denote by Y' the set of all $x = \{\xi_k\}_{k=1}^{\infty} \in Y$ which contain an infinite number of 1's and also an infinite number of 0's. Y' is considered a metric subspace of the space Y. Denote by D the set of all rational numbers of the form $k/2^n$, where n is a natural number, k is an integer, $0 < k \leqq 2^n$. It can be easily verified that the mapping $v \mid Y'$ (see Introduction) is a homeomorphic mapping of Y' onto $(0, 1\rangle - D$. If (T) is a regular matrix method then according to Theorem 1.1 the set $Y_1(T)$ is a set of the first category in Y. From this fact it follows easily that the set $Y_1'(T) = Y_1(T) \cap Y'$ is a set of the first category in Y'. Hence the set $v(Y_1'(T))$ is a set of the first category in $(0, 1\rangle - D$ and so in $(0, 1\rangle$, too. Since $Y_1'(T) \subset X_1(T)$ and $v(X_1(T)) - v(Y_1'(T))$ is countable, the set $v(X_1(T))$ is a set of the first category in $(0, 1\rangle$, too.

Let $T = (a_{n,k})$ be an infinite matrix of numbers and let $x = \sum_{k=1}^{\infty} 2^{-m_k}$ be the dyadic expansion of the number $x \in (0, 1\rangle$. Denote by $T(x)$ the matrix $(a_{m_k,k})$. In the paper [3] the following result is proved.

Theorem 1.2. *Let $T = (a_{n,k})$ be a regular matrix and $\{s_n\}_{n=1}^{\infty}$ a bounded not T-limitable sequence. Then the set M of all $x \in (0, 1\rangle$ for which the sequence $\{s_n\}_{n=1}^{\infty}$ is $T(x)$-limitable, is a set of the first category in $(0, 1\rangle$.*

We shall give now a new proof of the foregoing theorem based on the same idea as was the proof of Theorem 1.1.

Proof of Theorem 1.2. Let D denote the set of all numbers $k/2^n$, where k, n are natural numbers, $0 < k \leqq 2^n$. Put $M' = M \cap [(0, 1\rangle - D]$. It suffices to prove that M' is a set of the first category in $(0, 1\rangle$.

We shall define on M' the functions f, f_m $(m = 1, 2, \ldots)$ as follows: for $x \in M'$,

$$x = \sum_{k=1}^{\infty} 2^{-n_k}; \quad n_1 < n_2 < \ldots,$$

we put

$$f(x) = \lim_{k \to \infty} v_{n_k}, \quad \left(v_j = \sum_{k=1}^{\infty} a_{j,k} s_k, \quad j = 1, 2, \ldots\right).$$

Let m be a fixed natural number, let again $x = \sum_{k=1}^{\infty} 2^{-n_k} \in M'$. If $m < n_1$, we put $f_m(x) = 0$. If $m \geqq n_1$, then we can choose a natural number $p = p(m)$ such that $n_p \leqq m < n_{p+1}$ and we put $f_m(x) = \sum_{l=1}^{\infty} a_{n_p,l} s_l = v_{n_p}$.

If $m \to \infty$, then $p \to \infty$ and so $\lim_{m \to \infty} f_m(x) = f(x)$ (for each $x \in M'$).

It is well-known that all irrational numbers from a given interval of the form $(s/2^m, (s+1)/2^m)$ $(0 \leqq s \leqq 2^m - 1)$ have the same digits on the first m places in their dyadic expansions. This implies that the function f_m is constant on each of the sets

$$\left(\frac{s}{2^m}, \frac{s+1}{2^m}\right) \cap M' \quad (s = 0, 1, \ldots, 2^m - 1)$$

and so f_m is continuous on M'.

We shall prove that f is discontinuous at each point of the set M'.

Since $\{s_n\}_{n=1}^{\infty}$ is not T-limitable, we have

$$-\infty < t_1 = \liminf_{n\to\infty} v_n < \limsup_{n\to\infty} v_n = t_2 < +\infty ,$$

$$v_n = \sum_{k=1}^{\infty} a_{n,k} s_k \quad (n = 1, 2, \ldots) .$$

Let us construct such sequences

$$m_1 < m_2 < \ldots < m_k < \ldots ,$$

$$r_1 < r_2 < \ldots < r_k < \ldots$$

of natural numbers that

(10) $$\lim_{k\to\infty} v_{m_k} = t_1 , \quad \lim_{k\to\infty} v_{r_k} = t_2 .$$

Let $x_0 \in M'$, $x_0 = \sum_{k=1}^{\infty} 2^{-n_k}$, $n_1 < n_2 < \ldots$ Then $f(x_0)$ differs at least from one of the numbers t_1, t_2. Let e.g. $f(x_0) \neq t_1$ and put $\varepsilon_0 = |f(x_0) - t_1| > 0$. It suffices to prove that for each natural number $m \geqq \max(n_1, m_1)$ the following assertion holds:

If

$$x_0 \in \left(\frac{s}{2^m}, \frac{s+1}{2^m}\right) \cap M' = D_m \quad (0 \leqq s \leqq 2^m - 1) ,$$

then there exists such a point $x_1 \in D_m$ that $|f(x_1) - f(x_0)| = \varepsilon_0$.

Let $m \geqq \max(n_1, m_1)$, $x_0 \in D_m$. Let us choose natural numbers p, v such that

$$n_p \leqq m < n_{p+1} , \quad m_v \leqq m < m_{v+1} .$$

Put

$$x_1 = 2^{-n_1} + \ldots + 2^{-n_p} + 2^{-m_{v+1}} + 2^{-m_{v+2}} + \ldots$$

Then, evidently, $x_1 \in D_m$ and (see (10))

$$f(x_1) = \lim_{k\to\infty} v_{m_k} = t_1 ,$$

hence $|f(x_1) - f(x_0)| = |t_1 - f(x_0)| = \varepsilon_0$.

If $f(x_0) \neq t_2$, then we proceed analogously using the second part of (10).

Hence f is a function of the first Baire class on M', discontinuous at each point of M'. From the well-known theorem on discontinuity points of functions of the first Baire class we conclude that M' is a set of the first category in M' and hence also in $\langle 0, 1\rangle$. The proof is complete.

2

Let $\{q_k\}_{k=1}^{\infty}$ be a sequence of natural numbers, $q_k > 1$ $(k = 1, 2, \ldots)$. Then each $x \in \langle 0, 1)$ is uniquely expressible in the form

$$x = \sum_{k=1}^{\infty} \frac{\varepsilon_k(x)}{q_1 \cdot q_2 \ldots q_k}, \tag{11}$$

where $\varepsilon_k(x)$ $(k = 1, 2, \ldots)$ are integers, $0 \leqq \varepsilon_k(x) \leqq q_k - 1$ $(k = 1, 2, \ldots)$ and for an infinite number of k's we have $\varepsilon_k(x) < q_k - 1$. The series on the right-hand side of (11) is called the Cantor expansion of the number x (cf. [6], p. 7–10).

From Theorem 1,1 we can easily deduce the following

Theorem 2.1. *Let T be a regular matrix, let there exist $\lim_{k\to\infty} q_k = q$ (it may be $q = +\infty$). Then the set $P = P(T; q_1, q_2, \ldots)$ of all $x \in \langle 0, 1)$ for which the sequence $\{\varepsilon_k(x)/q_k\}_{k=1}^{\infty}$ (see (11)) is T-limitable, is a set of the first Baire category in $\langle 0, 1)$.*

Proof. Using the notation used in the proof of Theorem 1.1 put

$$M_k = \left\{0, \frac{1}{q_k}, \ldots, \frac{q_k - 1}{q_k}\right\}, \quad \eta_k' = 0, \quad \eta_k'' = 1 - \frac{1}{q_k} \quad (k = 1, 2, \ldots).$$

It is easy to check that the assumptions of Theorem 1.1 are fulfilled. Following Theorem 1.1 the set $Y_1(T)$ of all T-limitable sequences $z = \{\zeta_k\}_{k=1}^{\infty} \in Y$, $Y = M_1 \times \times M_2 \times \ldots$, is a set of the first category in Y.

Denote by H the set of all $z = \{\zeta_k\}_{k=1}^{\infty} \in Y$ with the following property: there exists $m = m(z)$ such that for each $k \geqq m$ we have $\zeta_k = 0$ or for each $k \geqq m$ we have $\zeta_k = (q_k - 1)/q_k$. Evidently H is a countable set. Put $Y^* = Y - H$. Then Y^* is a set of the second category in Y^* (Y^* is considered a subspace of Y) and $Y_1^*(T) = Y_1(T) \cap \cap Y^*$ is a set of the first category in Y^*. For $z = \{\zeta_k\}_{k=1}^{\infty} \in Y^*$ we put

$$\psi(z) = \sum_{k=1}^{\infty} \frac{q_k \zeta_k}{q_1 \cdot q_2 \ldots q_k}.$$

It is easy to check that ψ is a homeomorphic mapping of Y^* onto $P^* = \langle 0, 1) - Q$, where Q denotes the countable set of all numbers $v \in \langle 0, 1)$ of the form

$$v = \sum_{k=1}^{n} \frac{\beta_k}{q_1 \cdot q_2 \ldots q_k},$$

n is a natural number, β_k are integers, $0 \leqq \beta_k \leqq q_k - 1$ $(k = 1, 2, \ldots, n)$. Hence $\psi(Y_1^*(T))$ is a set of the first category in P^* and therefore also in $\langle 0, 1)$. But $P = \psi(Y_1^*(T)) \cup Q$ and so P is a set of the first category in $\langle 0, 1)$. The proof is complete.

Theorem 2.1 is related to an earlier result of the author. From Theorem 3 of the paper [13] it follows that in the case $\lim_{k \to \infty} q_k = +\infty$ the following assertion holds for almost all $x \in \langle 0, 1)$:

The set of all accumulation points of the sequence

$$\left\{\frac{\varepsilon_k(x)}{q_k}\right\}_{k=1}^{\infty} \tag{12}$$

coincides with the interval $\langle 0, 1\rangle$.

Hence the set C of all $x \in \langle 0, 1)$ for which the sequence (12) converges, is a null-set. Let $T = (a_{nk})$ be the matrix of convergence, i.e. $a_{nn} = 1$ and $a_{nk} = 0$ for $n \neq k$ $(n, k = 1, 2, \ldots)$. Then according to Theorem 2.1 the set C is a set of the first category in $\langle 0, 1)$ and the analogous result holds also in the case of an arbitrary regular matrix.

In the paper [4] some metric problems connected with the limitable sequences of digits in dyadic expansions of real numbers are studied. Since the dyadic expansions are special cases of Cantor expansions, the natural question arises whether this study can be extended to the study of analogous problems for Cantor expansions. The situation for Cantor series will be more complicated, e.g. in the case $\limsup_{k \to \infty} q_k = +\infty$ the sequence $\{\varepsilon_k(x)\}_{k=1}^{\infty}$ of the digits of x (see (11)) can be unbounded.

In what follows $T = (a_{nk})$ denote the Cesàro matrix, i.e. for each n we have $a_{nk} = 1/n$ if $k \leqq n$ and $a_{nk} = 0$ if $k > n$. The method defined by the Cesàro matrix will be denoted by $(C, 1)$.

Denote by $S = S(q_1, q_2, \ldots)$ the set of all $x \in \langle 0, 1)$ for which the sequence $\{\varepsilon_k(x)\}_{k=1}^{\infty}$ is $(C, 1)$-limitable. Further, let $S^* = S^*(q_1, q_2, \ldots)$ denote the set of all $x \in \langle 0, 1)$ for which the sequence

$$\{\sigma_n(x)\}_{n=1}^{\infty}, \quad \sigma_n(x) = \frac{\varepsilon_1(x) + \ldots + \varepsilon_n(x)}{n} \quad (n = 1, 2, \ldots),$$

is bounded. Obviously we have

$$S(q_1, q_2, \ldots) \subset S^*(q_1, q_2, \ldots).$$

If $A \subset \{1, 2, \ldots, n, \ldots\} = N$, then we put

$$h(A) = \limsup_{n \to \infty} \frac{A(n)}{n},$$

where $A(n) = \sum_{a \leqq n, a \in A} 1$. The number $h(A)$ is called the upper asymptotic density of the set A. Evidently $h(A) \in \langle 0, 1\rangle$ for each set $A \subset N$.

Theorem 2.2. *Let $A \subset N$, $A = \{k_1 < k_2 < \ldots\}$ have the following properties:*

(i) $$h(A) > 0\,,$$

(ii) $$\lim_{n \to \infty} q_{k_n} = +\infty\,.$$

Then $S^*(q_1, q_2, \ldots)$ is a set of the first category in $\langle 0, 1)$.

Corollary. *Under the assumptions of Theorem 2.2 the set $S(q_1, q_2, \ldots)$ is a set of the first Baire category in $\langle 0, 1)$.*

Proof of Theorem 2.2. Let $m \in N$. Denote by B_m the set of all $x \in \langle 0, 1)$ for which $\sigma_n(x) \leqq m$ $(n = 1, 2, \ldots)$. Then $S^* = \bigcap_{m=1}^{\infty} B_m$ and hence it suffices to prove that each of the sets B_m $(m = 1, 2, \ldots)$ is a nowhere-dense set in $\langle 0, 1)$.

Let $m \in N$. Let I be an interval, $I \subset \langle 0, 1)$. It suffices to prove that there exists an interval $I' \subset I$ such that $I' \cap B_m = \emptyset$.

Let us choose such integers $s \geqq 1$, l, $0 \leqq l \leqq q_1 \cdot q_2 \ldots q_s - 1$ that

$$i_s^{(l)} = \left\langle \frac{l}{q_1 \cdot q_2 \ldots q_s}, \frac{l+1}{q_1 \cdot q_2 \ldots q_s} \right) \subset I\,.$$

It is well-known from the construction of Cantor expansions that for a fixed $n \geqq 1$ the whole interval $\langle 0, 1)$ consists of $q_1 \cdot q_2 \ldots q_n$ pairwise disjoint intervals

$$i_n^{(v)} = \left\langle \frac{v}{q_1 \cdot q_2 \ldots q_n}, \frac{v+1}{q_1 \cdot q_2 \ldots q_n} \right)$$

$(v = 0, 1, \ldots, q_1 \cdot q_2 \ldots q_n - 1)$ of the n-th order. The number $v/q_1 \cdot q_2 \ldots q_n$ can be uniquely expressed in the form

$$\frac{v}{q_1 \cdot q_2 \ldots q_n} = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_1 \cdot q_2 \ldots q_i}\,,$$

ε_i being integers, $0 \leqq \varepsilon_i \leqq q_i - 1$ $(i = 1, 2, \ldots, n)$. For each $x \in i_n^{(v)}$, $x = \sum_{k=1}^{\infty} \varepsilon_k(x)/q_1 \cdot q_2 \ldots q_k$ (see (11)) we have $\varepsilon_k(x) = \varepsilon_k$ $(k = 1, 2, \ldots, n)$. Briefly we say that $i_n^{(v)}$ is associated with the (finite) sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$.

Put $h(A) = 2d$, $d > 0$. Choose $K > 0$ such that

(13) $$dK > m + 1\,.$$

According to (ii) there exists n_0 such that for each $n > n_0$ we have

(14) $$q_{k_n} > K + 1\,.$$

In virtue of the definition of $h(A)$ there exists g such that

$$(15) \qquad \frac{A(s+g)}{s+g} > d\,.$$

The number g can be chosen in such a way that even

$$(16) \qquad \frac{n_0 K}{s+g} < 1$$

holds.

Let the interval $i_s^{(l)}$ be associated with the sequence $\varepsilon_1^0, \varepsilon_2^0, \ldots, \varepsilon_s^0$, put $r = A(s+g)$. Let us construct the interval $i_{s+g}^{(u)}$ of the $(s+g)$-th order associated with the sequence $\varepsilon_1^0, \ldots, \varepsilon_s^0, \varepsilon_{s+1}^0, \ldots, \varepsilon_{s+g}^0$, where $\varepsilon_{s+j}^0 = 0$ if $s+j \notin \{k_1, k_2, \ldots, k_r\}$ and $\varepsilon_{s+j} = q_{k_i} - 1$ if $s+j = k_i$, $1 \leqq i \leqq r$. Then $i_{s+g}^{(u)} \subset i_s^{(l)} \subset I$ and for $x \in i_{s+g}^{(u)}$ we have

$$\sigma_{s+g}(x) \geqq \frac{\sum_{i=1}^{r} (q_{k_i} - 1)}{s+g} \geqq \frac{\sum_{i=n_0+1}^{r} (q_{k_i} - 1)}{s+g}\,.$$

Hence using (13)–(16) we obtain

$$\sigma_{s+g}(x) \geqq \frac{(A(s+g) - n_0)K}{s+g} > dK - \frac{n_0 K}{s+g} > m + 1 - 1 = m\,.$$

Hence $i_{s+g}^{(u)} \cap B_m = \emptyset$. The proof is complete.

In what follows we shall prove some metric results on the $(C, 1)$-limitable sequences of digits of real numbers in Cantor expansions.

Let us recall the notion of a homogeneous set. A set $M \subset \langle 0, 1)$ is said to be homogeneous if there exists such a number d, $0 \leqq d \leqq 1$ that for each interval $I \subset \langle 0, 1)$ we have $|M \cap I|_e / |I| = d$, where $|H|_e$ and $|H|$ denote the outer Lebesgue measure and Lebesgue measure of the set H, respectively. It is well-known that if $M \subset \langle 0, 1)$ is a measurable homogeneous set then M has the measure 0 or 1 (cf. [14]).

In the following we put

$$E = E(q_1, q_2, \ldots) = \{x \in \langle 0, 1);\ \lim_{n \to \infty} \sigma_n(x) = +\infty\}\,.$$

Lemma 2.1. *The set $S^*(q_1, q_2, \ldots)$ is a homogeneous $G_{\delta\sigma}$-set in $\langle 0, 1)$. Each of the sets $S(q_1, q_2, \ldots)$, $E(q_1, q_2, \ldots)$ is a homogeneous $G_{\delta\sigma\delta}$-set in $\langle 0, 1)$.*

Proof. It follows from the definition of the set S^* that

$$(17) \qquad S^* = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} C_{m,n}\,,$$

where $C_{m,n} = \{x \in \langle 0, 1);\ \varepsilon_n(x) \leqq m\}$. The set $C_{m,n}$ is the union of some intervals of the n-th order. Therefore it is a G_δ-set in $\langle 0, 1)$ and it follows from (17) that S^* is a $G_{\delta\sigma}$-set in $\langle 0, 1)$.

We have

$$(18) \qquad S = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{p=1}^{\infty} D_{k,m,p} ,$$

where

$$D_{k,m,p} = \left\{ x \in \langle 0, 1);\ \left|\sigma_{m+p}(x) - \sigma_{m+1}(x)\right| < \frac{1}{k} \right\}.$$

Obviously, $D_{k,m,p}$ is the union of some intervals of the $(m + p)$-th order and so it is a G_δ-set. It follows from (18) that S is a $G_{\delta\sigma\delta}$-set in $\langle 0, 1)$.

Analogously we can deduce from the relation

$$E = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m+1}^{\infty} \{x \in \langle 0, 1);\ \sigma_n(x) > k\}$$

that E is a $G_{\delta\sigma\delta}$-set in $\langle 0, 1)$.

It remains to prove that each of the sets S, S^*, E is homogeneous. It can be done by help of the following criterion of homogeneity (see [12]):

Let a measurable set $B \subset \langle 0, 1)$ satisfy the following condition: For each $n = 1, 2, \ldots$ and at each fixed n, for each two integers k, k', $0 \leqq k, k' \leqq q_1 \cdot q_2 \ldots \ldots q_n - 1$ we have

$$\left|B \cap i_n^{(k)}\right| = \left|B \cap i_n^{(k')}\right| .$$

Then B is a homogeneous set (and therefore its measure is 0 or 1).

It is easy to check that if B is any one of the sets S, S^*, E and k, k' are integers, $0 \leqq k, k' \leqq q_1 \cdot q_2 \ldots q_n - 1$, and further,

$$\frac{k}{q_1 \cdot q_2 \cdots q_n} = \sum_{j=1}^{n} \frac{\varepsilon_j}{q_1 \cdot q_2 \cdots q_j},$$

$$0 \leqq \varepsilon_j \leqq q_j - 1) \quad (j = 1, 2, \ldots, n),$$

$$\frac{k'}{q_1 \cdot q_2 \cdots q_n} = \sum_{j=1}^{n} \frac{\varepsilon'_j}{q_1 \cdot q_2 \cdots q_j},$$

$$0 \leqq \varepsilon'_j \leqq q_j - 1 \quad (j = 1, 2, \ldots, n)$$

then

$$(19) \qquad B \cap i_n^{(k')} = (B \cap i_n^{(k)}) + \sum_{j=1}^{n} \frac{\varepsilon'_j - \varepsilon_j}{q_1 \cdot q_2 \cdots q_j},$$

where $H + t$ denotes the set of all numbers $x + t$, $x \in H$. From (19) we conclude that $\left|B \cap i_n^{(k')}\right| = \left|B \cap i_n^{(k)}\right|$. On account of the mentioned criterion of homogeneity each of the sets S, S^*, E is homogeneous. The proof is complete.

Lemma 2.2. *Let us suppose that there exists such a set* $H \subset \langle 0, 1)$ *with a positive measure and such a number* $\delta < 1$ *that for each* $x \in H$ *we have* $\sum_{r=0}^{\infty} h_r(x) \leqq \delta$, *where*

$$h_r(x) = \limsup_{n \to \infty} \frac{N_n(r, x)}{n}, \quad N_n(r, x) = \sum_{k \leqq n, \varepsilon_k(x) = r} 1 .$$

Then $|E(q_1, q_2, \ldots)| = 1$.

Corollary. *Under the assumptions of Lemma* 2.2 *we have* $|S(q_1, q_2, \ldots)| = |S^*(q_1, q_2, \ldots)| = 0$.

Proof of Lemma 2.2. It suffices to prove that

$$H \subset E(q_1, q_2, \ldots) = E . \tag{20}$$

Indeed, it follows from (20) that $|E(q_1, q_2, \ldots)| \geqq |H| > 0$ and so $|E| = 1$ (see Lemma 2,1).

We prove (20). Let $0 < \varepsilon < 1 - \delta$. Let K be an arbitrarily chosen positive number. Let $0 < \varepsilon < 1 - \delta$. Let us construct the sequence

$$0, 1, \ldots, s , \tag{21}$$

where $s = [K/(1 - \delta - \varepsilon) + 1]$ ($[u]$ denotes the greatest integer $\leqq u$). On account of the definition of the numbers $h_j(x)$ $(0 \leqq j \leqq s)$ there exists such a natural number n_0 that for each $n > n_0$ the number of k's, $k \leqq n$, for which $\varepsilon_k(x)$ is a member of the sequence (21) is not greater than

$$\sum_{i=0}^{s} \left(h_i(x) + \frac{\varepsilon}{2^{i+1}} \right) n < n(\varepsilon + \sum_{i=0}^{\infty} h_i(x)) \leqq n(\varepsilon + \delta) .$$

Therefore the sequence $\varepsilon_1(x), \ldots, \varepsilon_s(x)$ contains more than $n(1 - \delta - \varepsilon)$ numbers which are greater than s. From this fact we obtain easily

$$\sigma_n(x) = \sum_{k=1}^{n} \frac{\varepsilon_k(x)}{n} \geqq \frac{n(1 - \delta - \varepsilon) \left[\dfrac{K}{1 - \delta - \varepsilon} + 1 \right]}{n} > K$$

for each $n > n_0$. Hence $\lim_{n \to \infty} \sigma_n(x) = +\infty$, $x \in E$. The proof is complete.

Now we prove two metric results which guarantee under certain, general enough assumptions on the sequence $\{q_k\}_{k=1}^{\infty}$ that the measure of the set $S(q_1, q_2, \ldots)$ is equal to 0.

Theorem 2.3. *Let* $\sum_{k=1}^{\infty} (1/q_k) < +\infty$. Then $|S^*(q_1, q_2, \ldots)| = 0$.

Corollary. *Under the assumption of Theorem* 2.3 *we have* $|S(q_1, q_2, \ldots)| = 0$.

Proof of Theorem 2.3. If $\sum_{k=1}^{\infty}(1/q_k) < +\infty$, then $|E(q_1, q_2, \ldots)| = 1$ (cf. [2]). From this fact the assertion follows immediately.

Theorem 2.4. *Let the sequence $\{q_k\}_{k=1}^{\infty}$ satisfy the following conditions:*

(a) $$\sum_{k=1}^{\infty} \frac{1}{q_k} = +\infty\,;$$

(b) $$\lim_{k\to\infty} q_k = +\infty\,;$$

(c) $$\sum_{k=1}^{n} \frac{1}{q_k} = o(n)\ (n \to \infty)\,.$$

Then $|E(q_1, q_2, \ldots)| = 1$.

Corollary. *Under the assumptions of Theorem* 2.4 *we have* $|S(q_1, q_2, \ldots)| = |S^*(q_1, q_2, \ldots)| = 0$.

Proof of Theorem 2.4. It is proved in [8] that if (a), (b) holds then there exists such a set $M \subset \langle 0, 1)$ that $|M| = 1$ and for each $x \in M$ we have

$$\lim_{n\to\infty} \frac{N_n(r, x)}{\sum_{k=1}^{n} \frac{1}{q_k}} = 1 \quad (r = 0, 1, \ldots)\,.$$

From this according to (c) we get (for $x \in M$ and each integer $r \geqq 0$)

$$h_r(x) = \limsup_{n\to\infty} \frac{N_n(r, x)}{n} = \limsup_{n\to\infty} \frac{N_n(r, x)}{\sum_{k=1}^{n} \frac{1}{q_k}} \frac{\sum_{k=1}^{n} \frac{1}{q_k}}{n} = 0\,.$$

The assertion follows now directly from Lemma 2.2.

Let us remark that the assumptions of Theorem 2.4 are satisfied if $q_k = k + 1$ $(k = 1, 2, \ldots)$. Hence the set $S(2, 3, \ldots)$ is a null-set and therefore also the set $S^0(2, 3, \ldots)$ of all $x \in \langle 0, 1)$ for which $\{\varepsilon_k(x)\}_{k=1}^{\infty}$ is $(C, 1)$-limitable to 0, is a null-set. Concerning the last set, we shall prove that even its Hausdorff dimension is 0.

Theorem 2.5. $\dim S^0(2, 3, \ldots) = 0$.

Proof. Let us consider that

(22) $$\sum_{k=1}^{n} \varepsilon_k(x) = \sum_{r=0}^{\infty} r\, N_n(r, x) \geqq \sum_{n=1}^{\infty} N_n(r, x)\,.$$

If $x \in S^0(2, 3, \ldots)$ then (22) implies

$$\lim_{n \to \infty} \frac{\sum_{r=1}^{\infty} N_n(r, x)}{n} = 0 .$$

Since obviously $N_n(0, x) = n - \sum_{n=1}^{\infty} N_n(r, x)$, we obtain $\lim_{n \to \infty} (N_n(0, x)/n) = 1$. Therefore we have

$$S^0(2, 3, \ldots) \subset \left\{x \in \langle 0, 1); \lim_{n \to \infty} \frac{N_n(0, x)}{n} = 1\right\} = G . \tag{23}$$

It follows from the corollary to Theorem 6 of the paper [11] that $\dim G = 0$ and hence (see (23))

$$\dim S^{\circ} (2, 3, \ldots) = 0 .$$

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