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A GENERALIZATION OF HAMILTONIAN CYCLES FOR TREES

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If G is a graph, then we denote by $V(G)$ and $E(G)$ the vertex set of G and the edge set of G, respectively. If G_1 and G_2 are graphs, then we denote by $G_1 \cup G_2$ the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If G is a connected graph, and $u, v \in V(G)$, then we denote by $d_G(u, v)$ the distance between u and v in G. For terms not defined here, see BEHZAD and CHARTRAND [1].

Let G be a connected graph of order $p \geqq 3$. We denote by $\mathscr{C}(G)$ the set of all cycles C with the property $V(C) = V(G)$. If $C \in \mathscr{C}(G)$, then we denote

$$\varphi_G(C) = \max_{uv \in E(C)} d_G(u, v)$$

and

$$\psi_G(C) = \sum_{uv \in E(C)} d_G(u, v) .$$

Moreover, we denote

$$\varphi_G = \min_{C \in \mathscr{C}(G)} \varphi_G(C) , \quad \psi_G = \min_{C \in \mathscr{C}(G)} \psi_G(C)$$

and

$$\mathscr{C}_\varphi(G) = \{C \in \mathscr{C}(G) \mid \varphi_G(C) = \varphi_G\} , \quad \mathscr{C}_\psi(G) = \{C \in \mathscr{C}(G) \mid \psi_G(C) = \psi_G\} .$$

Let G be a connected graph of order $p \geqq 3$. It is clear that the following three statements are equivalent: (i) G is hamiltonian, (ii) $\varphi_G = 1$, and (iii) $\psi_G = p$. The sets $\mathscr{C}_\varphi(G)$ and $\mathscr{C}_\psi(G)$ represent two distinct generalizations of the set of hamiltonian cycles of G. M. SEKANINA [5] proved that $\varphi_G \leqq 3$ for every connected graph G of order $p \geqq 3$. H. FLEISCHNER [2] proved that $\varphi_G \leqq 2$ for every 2-connected graph G. A characterization of the trees T with $\varphi_T = 2$ follows immediately from [4]. In the present paper we shall study the set $\mathscr{C}_\psi(T)$ for trees T of order $p \geqq 3$.

Let T be a tree. If $r, s \in V(T)$, then we denote by $P_T(r, s)$ the $r - s$ path in T. If $u, v, w \in V(T)$, then there is precisely one vertex t such that $t \in V(P_T(u, v)) \cap \cap V(P_T(v, w)) \cap V(P_T(w, u))$; see [3], Section 1.1; the vertex t will be denoted by $R_T(u, v, w)$.

Lemma 1. *Let T be a tree of order $p \geqq 3$, $u, v \in V(T)$, $C \in \mathscr{C}(T)$, and let P be a $u - v$ path in C. Then for every $e \in E(P_T(u, v))$ there is a pair of vertices u_0 and v_0 such that $u_0v_0 \in E(P)$ and $e \in E(P_T(u_0, v_0))$.*

Proof. There are vertices u' and v' such that $u'v' = e$ and that $u' = R_T(u, u', v')$. Denote

$$P_u = \{r \in V(P) \mid R_T(u, r, v) \in V(P_T(u, u'))\}$$

and

$$P_v = \{s \in V(P) \mid R_T(u, s, v) \in V(P_T(v', v))\} .$$

Since $u \in P_u$ and $v \in P_v$, there are $u_0 \in P_u$ and $v_0 \in P_v$ such that $u_0v_0 \in E(P)$. It is easy to see that $u'v' \in E(P_T(u_0, v_0))$.

Let C be a cycle of length at least four, and let $u, v, w \in V(C)$ be such that $uv, vw \in E(C)$. We denote by $C \lhd v$ the cycle C' with the property that $V(C') = V(C) - \{v\}$ and $E(C') = (E(C) - \{uv, vw\}) \cup \{uw\}$.

Lemma 2. *Let T be a tree of order $p \geqq 3$, $u, v \in V(T)$, $u \neq v$. Then there is $C \in \mathscr{C}(T)$ such that $uv \in E(C)$ and $\psi_T(C) = 2(p - 1)$.*

Proof. The case $p \leqq 4$ is obvious. Let $p = n \geqq 5$; assume that for $p = n - 1$, the statement is proved. The subcase when T has no end-vertex different from u or v is simple. Assume that there is a vertex s of degree 1 in T and such that $u \neq s \neq v$. Let r be the vertex adjacent to s in T. By the assumption, there in $C' \in \mathscr{C}(T - s)$ such that $uv \in E(C')$ and $\psi_{T-s}(C') = 2(p - 2)$. Obviously, there is a vertex t adjacent to r in C' and different from u and v. Let $C \in \mathscr{C}(T)$ be such that $rs, st \in E(C)$ and $C \lhd s = C'$. Obviously, $uv \in E(C)$. It is clear that $\psi_T(C) = \psi_{T-s}(C') - d_T(r, t) + d_T(r, s) + d_T(s, t) = \psi_{T-s}(C') + 2 = 2(p - 1)$. Hence the lemma follows.

Theorem 1. *Let T be a tree of order $p \geqq 3$. Then $\psi_T = 2(p - 1)$.*

Proof. Let $C \in \mathscr{C}(T)$. From Lemma 1 it follows that for every $e \in E(T)$, there are distinct edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of C such that $e \in E(P_T(u_1, v_1)) \cap E(P_T(u_2, v_2))$. This implies that $\psi_T(C) \geqq 2|E(T)| = 2(p - 1)$.

The statement of the theorem follows from Lemma 2.

Corollary 1. *Let G be a connected graph of order $p \geqq 3$ which contains a cycle of length $k \geqq 3$. Then $\psi_G \leqq 2p - k$.*

Proof. It is clear that G contains a spanning unicyclic subgraph G_0 with a cycle C of length k. There are trees $T_1, \ldots, T_k$ with disjoint vertex sets such that

$$V(G) = \bigcup_{i=1}^{k} V(T_i)$$

and

$$|V(T_j) \cap V(C)| = 1 \quad \text{for each} \quad j = 1, \ldots, k .$$

If T is a tree of order $p = 1$ or 2, we put $\psi_T = 2(p - 1)$. Clearly,

$$\psi_G \leqq \sum_{i=1}^{k} (\psi_{T_i} + 1) = 2p - k .$$

Corollary 2. *Let T be a tree of order $p \geqq 3$, $u, v \in V(T)$, $u \neq v$. Then there is $C \in \mathscr{C}_\psi(T)$ such that $uv \in E(C)$.*

Proof follows immediately from Theorem 1 and Lemma 2.

Corollary 3. *Let T be a tree of order $p \geqq 3$. Then $\mathscr{C}_\psi(T) = \mathscr{C}_\varphi(T)$ if and only if T is a star. Otherwise, $\mathscr{C}_\psi(T) - \mathscr{C}_\varphi(T) \neq \emptyset$.*

Proof. If T is a star, then it is clear that $\mathscr{C}_\psi(T) = \mathscr{C}_\varphi(T)$. If the diameter of T is at least four, then it follows from Corollary 2 that $\mathscr{C}_\psi(T) - \mathscr{C}_\varphi(T) \neq \emptyset$. Assume that the diameter of T is three. It is not difficult to see that $\varphi_T = 2$. Hence $\mathscr{C}_\psi(T) - \mathscr{C}_\varphi(T) \neq \emptyset$.

Theorem 2. *Let T be a tree of order $p \geqq 3$ such that $\varphi_T = 2$. Then $\mathscr{C}_\varphi(T) \subseteq \mathscr{C}_\psi(T)$.*

Proof. The case $p = 3$ is obvious. Let $p = n > 3$; assume that for $p = n - 1$, the statement is proved. Let $C \in \mathscr{C}_\varphi(T)$. Consider a vertex v of degree 1 in T. Let u and w be distinct vertices adjacent to v in C. Since $\varphi_T(C) = 2$, we shall assume without loss of generality that $d_T(u, v) = 2$. It is easily seen that $d_T(u, w) \leqq 2$ and that $d_T(u, w) = 1$ if and only if $d_T(v, w) = 1$. Clearly, $\varphi_{T-v}(C \lhd v) = 2$. By the assumption, $C \lhd v \in \mathscr{C}_\psi(T - v)$ and thus $\psi_{T-v}(C \lhd v) = 2(p - 2)$. This means that $\psi_T(C) = 2(p - 2) + 2 = 2(p - 1)$. Hence the theorem follows.

Remark. For the tree T in Fig. 1, $\varphi_T = 3$. The cycle with the edges 12, 23, ..., 67, 71 belongs to $\mathscr{C}_\varphi(T)$ but does not belong to $\mathscr{C}_\psi(T)$.

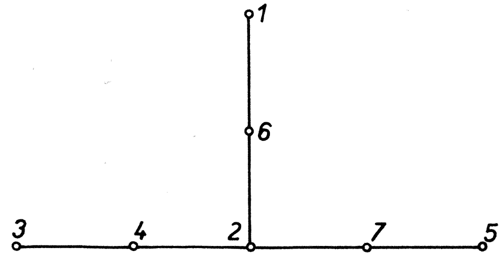

Fig. 1.

A graph which can be embedded into the plane so that all vertices lie on the boundary of the exterior region is referred to as an outerplanar graph. For outerplanar graphs, see [1], pp. 100–102. In the present paper the concept of an outerplanar graph will be used for a characterization of $\mathscr{C}_\psi(T)$.

Proposition. *A graph G with a hamiltonian cycle C is outerplanar if and only if it holds that*

if $r, s, t, u \in V(G)$ such that $rs, tu \in E(G) - E(C)$ and $\{r, s\} \cap \{t, u\} = \emptyset$, then the vertices r and s belong to the same component of the graph $C - t - u$.

Proof follows from the definition of outerplanar graphs.

Lemma 3. *Let T be a tree, and C a cycle such that $V(T) \subseteq V(C)$. Then the following two statements are equivalent:*

(1) *$T \cup C$ is outerplanar;*

(2) *if $r, s, t, u \in V(T)$ are such that r and s belong to distinct components of the graph $C - t - u$, then $P_T(r, s)$ and $P_T(t, u)$ have at least one vertex in common.*

Proof. (I) Let (1) hold. Assume that there are $r, s, t, u \in V(T)$ such that $V(P_T(r, s) \cap V(P_T(t, u)) = \emptyset$ and that the vertices r and s belong to distinct components of $C - t - u$. Then there is an edge $t_0u_0 \in E(P_T(r, s))$ such that the vertices t_0 and u_0 belong to distinct componets of $C - t - u$. This implies that t and u belong to distinct components of the graph $C - t_0 - u_0$. Similarly, there is an edge $r_0s_0 \in E(P_T(t, u))$ such that r_0 and s_0 belong to distinct components of $C - t_0 - u_0$. As follows from Proposition, $T \cup C$ is not outerplanar, which is a contradiction. Thus (2) holds.

(II) Let (2) hold. It follows from Proposition that (1) holds.

The following theorem gives a characterization of $\mathscr{C}_\psi(T)$:

Theorem 3. *Let T be a tree of order $p \geqq 3$, and let $C \in \mathscr{C}(T)$. Then $C \in \mathscr{C}_\psi(T)$ if and only if the graph $T \cup C$ is outerplanar.*

Proof. (I) Let $C \in \mathscr{C}_\psi(T)$. Assume that $T \cup C$ is not outerplanar. Since C is a hamiltonian cycle of $T \cup C$, it follows from Proposition that there are $r, s, t, u \in V(T)$ such that $rs, tu \in E(T)$, $\{r, s\} \cap \{t, u\} = \emptyset$, and that the vertices r and s belong to distinct components of $C - t - u$. Without loss of generality we assume that $s, t \in V(P_T(r, u))$. It is clear that there is an edge a of T such that

$$a \in E(P_T(r, u)) \cap E(P_T(r, t)) \cap E(P_T(s, t)) \cap E(P_T(s, u)) .$$

Lemma 1 implies that there are vertices $r_0, r_1, s_0, s_1, t_0, t_1, u_0$, and u_1 such that u_0r_1, r_0t_1, t_0s_1, and s_0u_1 are distinct edges of C, and

$$a \in E(P_T(u_0, r_1)) \cap E(P_T(r_0, t_1)) \cap E(P_T(t_0, s_1)) \cap E(P_T(s_0, u_1)) .$$

Since T has $p - 1$ edges, $\psi_T(C) \geqq 2(p - 1) + 2$, which is a contradiction. Hence $T \cup C$ is outerplanar.

(II) Let $T \cup C$ be outerplanar. We shall prove that $C \in \mathscr{C}_\psi(T)$. The case $p = 3$ is obvious. Let $p = n > 3$; assume that for $p = n - 1$, the statement is proved.

We denote by e the minimum integer f such that there are $u, v \in V(T)$ with the properties that $\deg_T u = 1$, $uv \in E(T)$, and $d_C(u, v) = f$. Clearly, $e \geqq 1$. Consider vertices $r, t \in V(T)$ such that $\deg_T r = 1$, $rt \in E(T)$, and $d_C(r, t) = e$. Assume that $e > 1$. Let P_0 be an $r - t$ path of length e in C. Since $e > 1$, there is $s \in V(T) - \{r, t\}$ such that s lies on P_0. Since T is a tree, there is a vertex w of degree 1 in T such that $R_T(w, s, t) = s$. Hence $r \neq w \neq t$. Lemma 3 imlies that $V(P_T(w, t)) \subseteq V(P_0)$. We denote by w' the vertex adjacent to w in T. Since $w' \in V(P_0)$, it is $d_C(w, w') < e$, which is a contradiction.

We have proved that there are $u, v \in V(T)$ such that $\deg_T u = 1$ and $uv \in E(T) \cap E(C)$. Let v' be a vertex different from v such that $uv' \in E(C)$. It is clear that the graph $(T - u) \cup (C \lhd u)$ is outerplanar. By the assumption, $C \lhd u \in \mathscr{C}_\psi(T - u)$. We have $\psi_T(C) = \psi_{T-u}(C \lhd u) - d_T(v, v') + d_T(v, u) + d_T(u, v') = \psi_{T-u}(C \lhd u) + 2$. Since $\psi_T(C) = 2(p - 2) + 2$, $C \in \mathscr{C}_\psi(T)$. Hence the theorem follows.

The following theorem gives one more characterization of $\mathscr{C}_\psi(T)$:

Theorem 4. *Let T be a tree of order $p \geqq 3$, and let $C \in \mathscr{C}(T)$. Then $C \in \mathscr{C}_\psi(T)$ if and only if*

(3) *for every pair of vertices u and v adjacent in C, the graph $P_T(u, v) \cup C$ is outerplanar.*

Proof. (I) Let $C \in \mathscr{C}_\psi(T)$. By Theorem 3, $T \cup C$ is outerplanar. Since every subgraph of an outerplanar graph is outerplanar, the statement (3) holds.

(II) We shall prove that (3) implies $C \in \mathscr{C}_\psi(T)$. The case $p = 3$ is obvious. Let $p = n \geqq 4$; assume that for $p = n - 1$, the statement is proved. Let (3) hold. We denote by h the minimum integer i such that there are vertices $u', v' \in V(T)$ with the property that $u'v' \in E(T)$ and $d_C(u', v') = i$. Consider $r, s \in V(T)$ such that $rs \in E(T)$ and $d_C(r, s) = h$. Let P be an $r - s$ path of length h in C. By Lemma 1, there are $u_0, v_0 \in V(P)$ such that $u_0v_0 \in E(P)$ and $rs \in E(P_T(u_0, v_0))$. Since $P_T(u_0, v_0) \cup C$ is outerplanar, Lemma 3 implies that $V(P_T(u_0, v_0)) \subset V(P)$. Assume that $h > 1$. Then there are $u, v \in V(P_T(u_0, v_0))$ such that $r \neq u \neq s$ and that $uv \in E(P_T(u_0, v_0))$. Hence $d_C(u, v) < h$, which is a contradiction. Thus $h = 1$. We have $rs \in E(T) \cap E(C)$.

We denote by r' and s' vertices such that $r' \neq s$, $s' \neq r$, and that $r'r, ss' \in E(C)$. Consider the graph $C \cup P_T(r', s)$. If $s = R_T(r', s, r)$, then $P_T(r', s)$ is a subgraph of $P_T(r, s)$; thus $P_T(r', s) \cup C$ is outerplanar. Let $s \neq R_T(r', s, r)$. Then $r = R_T(r', s, r)$. Since $P_T(r', r) \cup C$ is outerplanar and $rs \in E(C)$, it is easily seen from Lemma 3 that $P_T(r', s) \cup C$ is outerplanar. Similarly, $P_T(r, s') \cup C$ is outerplanar.

We denote by T' and C' the graph obtained from T and C, respectively, by identifying the vertices r and s. Let t be the new vertex in C' and T'. Since $P_{T'}(r', t) \cup C'$ and $P_{T'}(t, s') \cup C$ are outerplanar, (3) holds for T' and C'. By the assumption, $C' \in \mathscr{C}_\psi(T')$. As follows from Theorem 3, the graph $T' \cup C'$ is outerplanar. Assume that $T \cup C$ is not outerplanar. It is not difficult to see from Lemma 3 that there are $r_1, s_1 \in V(T)$ such that (i) $V(P_T(r, r_1)) \cap V(P_T(s, s_1)) = \emptyset$, (ii) the vertices r and r_1 belong to distinct components of $C - s - s_1$, and (iii) $r_1 s_1 \in E(C)$. Since $rs \in E(T)$, by Lemma 3 $P_T(r_1, s_1) \cup C$ is not outerplanar, which is a contradiction. Hence the theorem follows.

The following theorem concerns the relationship between the structure of T and that of $\mathscr{C}_\psi(T)$.

Theorem 5. *Let T be a tree of order $p \geqq 4$, and let u_1, u_2, v_1 and v_2 be distinct vertices of T. Then the following statements are equivalent:*

(4) *there is $C \in \mathscr{C}_\psi(T)$ such that the vertices u_1 and u_2 belong to distinct components of the graph $C - v_1 - v_2$;*

(5) $V(P_T(u_1, u_2)) \cap V(P_T(v_1, v_2)) \neq \emptyset$.

Proof. (I) Assume that (4) holds and (5) does not hold. Consider an arbitrary cycle $C \in \mathscr{C}(T)$ with the property that the vertices u_1 and u_2 belong to distinct components of $C - v_1 - v_2$. Since $P_T(u_1, u_2)$ and $P_T(v_1, v_2)$ have no vertex in common, it follows from Lemma 3 that $T \cup C$ is not outerplanar. By Theorem 3, $C \notin \mathscr{C}_\psi(T)$, which is a contradiction.

(II) Let (5) hold. Consider $w \in V(P_T(u_1, u_2)) \cap V(P_T(v_2, v_2))$. Without loss of generality we assume that $w \notin \{u_1, u_2, v_2\}$, and that there are trees T_1 and T_2 such that $V(T_1) \cup V(T_2) = V(T)$, $V(T_1) \cap V(T_2) = \{w\}$, $E(T_1) \cup E(T_2) = E(T)$, $E(T_1) \cap \cap E(T_2) = \emptyset$, and that $u_1, v_1 \in V(T_1)$, $u_2, v_2 \in V(T_2)$.

Let $i \in \{1, 2\}$. Obviously, $|V(T_i)| \geqq 2$. If $|V(T_i)| \geqq 3$, then we consider a cycle $C'_i \in \mathscr{C}_\psi(T_i)$. If $|V(T_i)| = 2$, then we denote by C'_i the tree T_i. We denote by P_1 the $w - u_1$ path in C'_1 with the property that if v_1 lies on P_1, then $v_1 = w$. We denote by P_2 the $w - v_2$ path in C'_2 not containing u_2. Moreover, we denote by w_1 or w_2 the vertex belonging to P_1 or to P_2, respectively, which is adjacent to w.

We denote by C' the graph with $V(C') = V(T)$ and $E(C') = E(C'_1) \cup E(C'_2) \cup \cup \{w_1 w_2\}$. Obviously, C' contains precisely one hamiltonian cycle, say C. It is easily seen that $T \cup C$ is outerplanar. By Theorem 3, $C \in \mathscr{C}_\psi(T)$. Clearly, the vertices u_1 and u_2 belong to distinct components of $C - v_1 - v_2$. Hence the theorem follows.

Let T and T' be trees such that $V(T) = V(T')$. We say that the tree T' is an elementary S-modification of the tree T if there are $u, v \in V(T)$ such that $uv \in E(T)$, and

$$E(T') = (E(T) - \{ur \mid ur \in E(T),\ r \in V(T),\ r \neq v\}) \cup$$
$$\cup \{vs \mid us \in E(T),\ s \in V(T),\ s \neq v\}$$

(hence $uv \in E(T')$). We say that T' is an S-modification of T if either $E(T') = E(T)$ or there is a tree T_0 which is an S-modification of T, and T' is an elementary S-modification of T_0.

Theorem 6. *Let T_1 and T_2 be trees with the same vertex set V such that $|V| \geqq 3$. Then $\mathscr{C}_\psi(T_1) \subseteq \mathscr{C}_\psi(T_2)$ if and only if T_2 is an S-modification of T_1.*

Proof. (I) Let $\mathscr{C}_\psi(T_1) \subseteq \mathscr{C}_\psi(T_2)$. We shall say that a tree T with $V(T) = V$ has the property P if for an arbittary path

$$u_0, u_1, \ldots, u_n$$

in T_1 such that $n \geqq 2$ and $u_0 u_n \in E(T)$ it holds that (1) if $0 < i < n$, then u_i has degree one in T, and (2) there exists m, $1 \leqq m \leqq n$. such that (a) if $1 \leqq j < m$, then $u_0 u_j \in E(T)$, and (b) if $m \leqq k < n$, then $u_k u_n \in E(T)$.

We distinguish two cases:

(A) Assume that T_2 has not the property P. Then it is not difficult to see that there exist distinct vertices $r_1, r_2, s_1, s_2 \in V$ such that $V(P_{T_1}(r_1, r_2)) \cap V(P_{T_1}(s_1, s_2)) \neq \emptyset$ and $V(P_{T_2}(r_1, r_2)) \cap V(P_{T_2}(s_1, s_2)) = \emptyset$. From Theorem 5 it follows that there exists $C \in \mathscr{C}_\psi(T_1) - \mathscr{C}_\psi(T_2)$, which is a contradiction.

(B) Assume that T_2 has the property P. If T is a tree with $V(T) = V$, then we shall denote

$$\|T\| = \sum_{uv \in E(T)} (d_{T_1}(u, v) - 1)\,.$$

If $\|T_2\| = 0$, then T_2 is identical with T_1, and thus T_2 is an S-modification of T_1. Let $\|T_2\| \geqq 1$. Assume that for each tree T' with the properties that $V(T') = V$, $\|T'\| < \|T_2\|$ and with the property P it holds that T' is an S-modification of T_1.

Since $\|T_2\| > 0$ and T_2 has the property P, we have that there are distinct vertices $u, v, w \in V$ such that v belongs to the $u - w$ path in T_1, $uw \in E(T_2)$ $vw \in E(T_1) \cap E(T_2)$, and v has degree one in T_2. We denote by V_0 the set of all vertices $v_0 \neq v$ adjacent to w in T_2 with the property that v belongs to the $v_0 - w$ path in T_1. We denote by T_0 the tree with $V(T_0) = V$ and

$$E(T_0) = (E(T_2) - \{sw \mid s \in V_0\}) \cup \{tv \mid t \in V_0\}\,.$$

It is obvious that $\|T_0\| < \|T_2\|$. Clearly, T_2 is an elementary S-modification of T_0. It is easy to see that T_0 has the property P. This implies that T_0 is an S-modification of T_1. Therefore, T_2 is an S-modification of T_1.

(II) Let T_2 be an S-modification of T_1. The case $E(T_2) = E(T_1)$ is obvious. Assume that there exists a tree T_0 which is an S-modification of T_1 and such that T_2 is an elementary S-modification of T_0 and $\mathscr{C}_\psi(T_1) \subseteq \mathscr{C}_\psi(T_0)$. Let $t, u, v, w \in V$. It is easy to see that if $V(P_{T_0}(t, v)) \cap V(P_{T_0}(u, w)) \neq \emptyset$, then $V(P_{T_2}(t, v)) \cap V(P_{T_2}(u, w)) \neq \emptyset$. Thus we have $\mathscr{C}_\psi(T_0) \subseteq \mathscr{C}_\psi(T_2)$, which completes the proof.

Corollary 4. *Let T_1 and T_2 be trees with the same vertex set V such that $|V| \geqq 3$. Then $\mathscr{C}_\psi(T_1) = \mathscr{C}_\psi(T_2)$ if and only if there exists an isomorphism $f : T_1 \to T_2$*

such that for every $v \in V$ it holds that

(6) *if $f(v) \neq v$, then v and $f(v)$ are adjacent in T_1, and one of the vertices v and $f(v)$ has degree one in T_1.*

Proof. (I) It is clear that if $\mathscr{C}_\psi(T_1) = \mathscr{C}_\psi(T_2)$, then there exists an isomorphism $f : T_1 \to T_2$ fulfilling (6) for each $v \in V$.

(II) Assume that there exists an isomorphism $f : T_1 \to T_2$ which fulfils (6) for each $v \in V$.

Let $r \in V$. Denote $s = f(r)$. If $s = r$, then $f(f(r)) = r$. Assume that $s \neq r$. Denote $t = f(s)$. Since $s \neq r$, we have that $t \neq s$. If $t = r$, then $f(f(r)) = r$. Let $t \neq r$. Then both r and t have degree one in T_1. If $f(t) = t$, then $f(s) = s$, which is a contradiction. Thus $f(t) \neq t$. From (6) it follows that $f(t) = s$. Since $f(t) = f(r)$, it is $t = r$, which is a contradiction.

We have proved that $f(f(v)) = v$ for each $v \in V$. It is not difficult to see that T_2 is an S-modification of T_1 and that simultaneously T_1 is an S-modification of T_2. From Theorem 6 it follows that $\mathscr{C}_\psi(T_1) = \mathscr{C}_\psi(T_2)$.

Remark. This paper originated in the author's studies of applying the graph theory to linguistics.

Added on 5th December 1975: (I) For connected graphs G of order at least three, there exists a relationship between cycles $C \in \mathscr{C}_\psi(G)$ and closed walks of minimum length. Such walks were studied in S. E. Goodman and S. T. Hedetniemi: On Hamiltonian walks in graphs. SIAM J. Comput. *3* (1974), 214–221.

(II) The author thanks to Professor M. Sekanina for his critical remark to the original version of the first part of the proof of Theorem 6.

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