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L_p-THEORY FOR A CLASS OF SINGULAR ELLIPTIC DIFFERENTIAL OPERATORS, II

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1. INTRODUCTION

This paper is a continuation of [7]. We consider special properties for the singular elliptic operators introduced there: distribution of eigenvalues, density of the linear hull of the associated eigenvectors in appropriate function spaces. In sect. 2 we recall the necessary definitions (differential operators, function spaces). Sect. 3 contains the results. The proofs are given in sect. 4.

2. DEFINITIONS

2.1. The weight function $\varrho(x)$. We recall the definition given in [7]. Let Ω be an arbitrary connected (bounded or unbounded) domain in the n-dimensional Euclidean space R_n. Its boundary is denoted by $\partial\Omega$. As usual, $C^\infty(\Omega)$ is the set of all complex infinitely differentiable functions defined on Ω. We consider weight functions $\varrho(x)$ with the following properties:

1.

(1) $$\varrho(x) \in C^\infty(\Omega)\,, \quad \varrho(x) > 0 \quad \text{for} \quad x \in \Omega\,.$$

2. For all multiindices γ there exist $C_\gamma > 0$ such that

(2) $$|D^\gamma \varrho(x)| \leqq C_\gamma\, \varrho^{1+|\gamma|}(x)\,, \quad x \in \Omega\,.$$

3. For all $K > 0$ there exist $\varepsilon_K > 0$ and $r_K > 0$ such that

(3) $$\varrho(x) > K \quad \text{if either} \quad d(x) \leqq \varepsilon_K \quad \text{or} \quad |x| \geqq r_K \; (x \in \Omega)\,,$$

where $d(x)$ denotes the distance of the point $x \in \Omega$ from the boundary $\partial\Omega$.

4. There exists $a \geqq 0$ such that

(4) $$\varrho^{-a}(x) \in L_1(\Omega)\,.$$

Examples of weight functions of this type are given in [7]. In contrast to [7], we add the assumption (4) from the very beginning. Some results in [7] are true without this hypothesis. Nevertheless, we discussed in [7] the importance of (4) in the theory developed there. In particular see Lemma 4.1 in [7]. For our purpose here it is useful to include (4) in the definition.

2.2. Differential operators. Again we follow [7]. Let m be an integer; $m = 1, 2, \ldots$. Let μ and ν be real numbers; $\nu > \mu + 2m$. Let

$$\varkappa_l = \frac{1}{2m}\left(\nu(2m - l) + \mu l\right); \quad l = 0, 1, \ldots, 2m . \tag{5}$$

Then A is said to be an operator of type $A^{(m)}_{\mu,\nu}$, if it can be represented in the form

$$Au = \sum_{l=0}^{m} \sum_{|\alpha|=2l} \varrho^{\varkappa_{2l}}(x)\, b_\alpha(x)\, D^\alpha u + \sum_{|\beta|<2m} a_\beta(x)\, D^\beta u , \tag{6}$$

where the coefficients satisfy the following hypotheses:

1. $b_\alpha(x)$, $a_\beta(x)$ are infinitely differentiable real functions. $D^\gamma b_\alpha(x)$ are bounded in Ω for all γ and all α, where $|\alpha| = 2l$ $(l = 0, \ldots, m)$. Further,

$$D^\gamma a_\beta(x) = o(\varrho^{\varkappa_{|\beta|} + |\gamma|}(x)) \tag{7}$$

for all γ and all β, where $|\beta| < 2m$. ((7) means that for every $\varepsilon > 0$ there exists a number $K = K(\varepsilon)$ such that

$$|D^\gamma a_\beta(x)| \leqq \varepsilon\, \varrho^{\varkappa_{|\beta|} + |\gamma|}(x) \quad \text{for} \quad \varrho(x) \geqq K .)$$

2. (Ellipticity-condition.) There exists a positive number c such that for all $\xi = (\xi_1, \ldots, \xi_n) \in R_n$ and for all $x \in \Omega$,

$$(-1)^m \sum_{|\alpha|=2m} b_\alpha(x)\, \xi^\alpha \geqq c|\xi|^{2m}, \quad b_{(0,\ldots,0)}(x) \geqq c \tag{8a}$$

$$(-1)^l \sum_{|\alpha|=2l} b_\alpha(x)\, \xi^\alpha \geqq 0, \qquad l = 1, \ldots, m-1 . \tag{8b}$$

(as usual, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$).

Beside the class $A^{(m)}_{\mu,\nu}$ we need the subclass $\mathring{A}^{(m)}_{\mu,\nu}$. An operator A belonging to $A^{(m)}_{\mu,\nu}$ is said to be an operator of type $\mathring{A}^{(m)}_{\mu,\nu}$ if there exists $\delta > 0$ such that

$$D^\gamma a_\beta(x) = O(\varrho^{\varkappa_{|\beta|} + |\gamma| - \delta}) . \tag{9}$$

Clearly, (9) is a reinforcement of (7). Examples of operators belonging to $A^{(m)}_{\mu,\nu}$ are given in [7]. These are also examples of operators of type $\mathring{A}^{(m)}_{\mu,\nu}$.

2.3. Function spaces. We need two types of complex function spaces: Sobolev-Slobodeckij spaces with weights, and the nuclear function space $S_{\varrho(x)}(\Omega)$. The space $L_p(\Omega)$; $1 < p < \infty$ has the usual meaning. Let

$$\|f\|_{W_p{}^\eta} = \left(\int_{\Omega\times\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+\eta p}} \, dx \, dy + \|f\|_{L_p}^p\right)^{1/p} ; \quad 0 < \eta < 1 \tag{10}$$

and $\|f\|_{W_p{}^0} = \|f\|_{L_p}$; $1 < p < \infty$. Further, for $s \geqq 0$ we write

$$s = [s] + \{s\} ; \quad [s] \text{ integer} ; \quad 0 \leqq \{s\} < 1 .$$

If σ and τ are real numbers such that $\tau \geqq \sigma + sp$, where $s \geqq 0$, then $W^s_{p,\sigma,\tau}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|f\|_{W^s{}_{p,\varrho,\tau}} = \Big(\sum_{|\alpha|=[s]} \|\varrho^{\varrho/p} D^\alpha f\|^p_{W_p{}^{\{s\}}} + \|\varrho^{\tau/p} f\|^p_{L_p}\Big)^{1/p} \tag{11}$$

(for $s = 0$, we assume $\sigma = \tau$).

The theory of these Sobolev-Slobodeckij spaces with weights is developed in [6], [7]. We do not repeat their properties, here.

Further we need the space $S_{\varrho(x)}(\Omega)$ which is defined by

$$S_{\varrho(x)}(\Omega) = \{f \mid f \in C^\infty(\Omega), \ \|f\|_{l,\alpha} = \sup_{x\in\Omega} \varrho^l(x) \, |D^\alpha f(x)| < \infty \tag{12}$$

$$\text{for all } l = 0, 1, 2, \ldots, \text{ and all multiindices } \alpha\} .$$

Theorem 4.2 of [7] yields that $S_{\varrho(x)}(\Omega)$ is nuclear (F)-space, isomorphic to s, the space of rapidly decreasing sequences.

3. RESULTS

The paper contains two theorems. One deals with self-adjoint operators in $L_2(\Omega)$, the other with general operators in the framework of L_p-theory. The first theorem is needed for the proof of the other one (also this seems to be a little surprising at the first glance).

Theorem 1. *Let A be a formally self-adjoint operator of type $A^{(m)}_{\mu,\nu}$, where $\nu > 0$, with the domain of definition*

$$D(A) = W^{2m}_{2,2\mu,2\nu}(\Omega) . \tag{13}$$

Then A is a self-adjoint operator in $L_2(\Omega)$, bounded from below, having a pure point-spectrum. There exist two positive numbers c_1 and c_2 such that

$$c_1(1 + \lambda^{n/2m}) \leqq 1 + N(\lambda) \leqq c_2(1 + \lambda^{(a+n+(\gamma-\tilde{\mu})n/2m)/\nu}) . \tag{14}$$

Here a has the meaning from (4); $\tilde{\mu} = \min(\mu, 0)$; $N(\lambda) = \sum_{|\lambda_j| \leqq \lambda} 1$ *is the number of the eigenvalues λ_j of A (including their multiplicites) less than or equal to λ by modulus*; $\lambda > 0$. *Further, if $s > 0$ and if A is positive-definite then the domain of definition of the fractional power A^s of A is given by*

$$D(A^s) = W^{2sm}_{2,2s\mu,2s\nu}(\Omega) . \tag{15}$$

The left hand side of (14) has the usual behaviour of $N(\lambda)$ for regular elliptic differential operators in bounded domains. But the right hand side shows the influence of the different parameters. For special cases one can strengthen this estimate.

Theorem 2. *Let A be an operator of type $\mathring{A}^{(m)}_{\mu,\nu}$, where $\nu > 0$, with the domain of definition*

$$D(A) = W^{2m}_{p,p\mu,p\nu}(\Omega) , \tag{16}$$

$1 < p < \infty$. *Then A is a closed operator in $L_p(\Omega)$. Its spectrum consists of isolated eigenvalues of finite algebraic multiplicity. The eigenvalues and the associated eigenvectors are independent of p. The associated eigenvectors belong to $S_{\varrho(x)}(\Omega)$; their linear hull is dense in $S_{\varrho(x)}(\Omega)$. Further, their linear hull is also dense in all the spaces $W^s_{q,\varkappa,\tau}(\Omega)$, where $0 \leqq s < \infty$; $1 < q < \infty$; $-\infty < \varkappa + sq \leqq \tau < \infty$ (and hence, in particular, dense in $L_p(\Omega)$).*

This theorem is the counterpart to the theory of F. E. Browder and S. Agmon for general regular elliptic differential operators, see [1]. However the situation here is different, and in some respects easier. Although the theorem is formulated as an L_p-theorem, the proof will show that we use Hilbert space methods. The basic idea is to reduce the last theorem to the criterion of J. C. Gochberg and M. G. Krejn [2], Chapter V, Theorem 10.1.

4. PROOFS

4.1. Proof of Theorem 1. Step 1. It follows from [7], namely from Theorem 5.3 in [7], that $A - \lambda E$ is an isomorphic mapping from $D(A)$ onto $L_2(\Omega)$, provided the real number λ is sufficiently small. Since $C_0^\infty(\Omega)$ is dense in $D(A)$ and since A is formally sef-adjoint, it follows easily that A is a symmetric operator. Together with the first fact one obtains that A is self-adjoint and bounded from below. Further, Theorem 5.3 of [7] yields (15) for $s = k = 1, 2, 3, \ldots$. We assume that A is positive-definite. Concerning the use of the well-known interpolation formula

$$D(A^s) = (L_2(\Omega), D(A^k))_{s/k,2} ; \quad 0 < s < k , \tag{17}$$

see for instance [4], Lemma 6. Using the interpolation Theorem 4.3 of [6] and (15) for $s = k = 1, 2, 3, \ldots$, we obtain (15) for $s > 0$. Now we prove that the embedding from $D(A)$ into $L_2(\Omega)$ is compact. Let

$$\Omega_K = \{x \mid x \in \Omega,\ \varrho(x) < K\}\ ;\quad K > 0 \tag{18}$$

and let $\chi_K(x)$ be the characteristic function of Ω_K. Since Ω_K is bounded and since the embedding from $W_2^{2m}(\Omega_K)$ into $L_2(\Omega_K)$ is compact, it follows that for each positive K

$$M_K = \{\chi_K(x)\, u(x) \mid \|u(x)\|_{W^{2m}{}_{2,2\mu,2\nu}} \leqq 1\}$$

is a precompact subset in $L_2(\Omega)$. On the other hand, using $\nu > 0$, the estimate

$$\int_\Omega |1 - \chi_K(x)|^2\, |u(x)|^2\, \mathrm{d}x \leqq K^{-2\nu} \int_\Omega \varrho^{2\nu}(x)\, |u(x)|^2\, \mathrm{d}x$$

yields that M_K is a precompact ε-net for the image of the unit ball of $D(A)$ in $L_2(\Omega)$, provided $K = K(\varepsilon)$ is sufficiently large. Hence, the embedding from $D(A)$ into $L_2(\Omega)$ is compact. Now it follows by the well-known criterion of F. Rellich (see for instance [5], p. 277) that A is an operator with pure point spectrum.

Step 2. Now we prove the left hand inequality of (14). Assume, without loss of generality, that A is positive-definite. The eigenvalues of A are denoted by λ_j (including their multiplicities), and the approximation numbers of the compact embedding operator I from $D(A)$ into $L_2(\Omega)$ are denoted by s_j. By a suitable choice of an equivalent norm in $D(A)$ ($\|u\|_{D(A)} = \|Au\|_{L_2}$) it holds $s_j = \lambda_j^{-1}$ (see [2] where one has to take into consideration, that I can be written as $I = A^{-1}A$, where A is regarded as unitary operator from $D(A)$ onto $L_2(\Omega)$, and A^{-1} as compact operator in $L_2(\Omega)$). Let K_1 and K_2 be two open balls such that $\overline{K}_1 \subset K_2$ and $\overline{K}_2 \subset \Omega$. We shall use the well-known fact that there exists a linear and bounded extension operator S from $W_2^{2m}(K_1)$ into $\mathring{W}_2^{2m}(K_2)$, and so also from $W_2^{2m}(K_1)$ into $D(A)$. (A description of the extension method may be found, for instance, in [5], p. 377–380). Finally, let R be the restriction operator from $L_2(\Omega)$ onto $L_2(K_1)$, and let $\tilde{I}$ be the embedding operator from $W_2^{2m}(K_1)$ into $L_2(K_1)$. Then

$$\tilde{I} = RIS\,.$$

Denoting the approximation numbers of $\tilde{I}$ by $\tilde{s}_j$, it follows from the ideal properties of the s-number (see [2])

$$\tilde{s}_j \leqq c s_j = c\lambda_j^{-1} \quad (c > 0)\,.$$

On the other hand, the distribution of the numbers $\tilde{s}_j$ is well known,

$$\tilde{s}_j \sim j^{-2m/n}\,.$$

Hence

$$\lambda_j \leqq c' j^{2m/n},$$

where c' is a suitable positive number. This implies the left hand inequality of (14).

Step 3. We prove the right hand inequality of (14). First we have to recall some former notations. If $K = 2^j$, then Ω_K, defined in (18), will be denoted by $\Omega^{(j)}$, see [6]. (For $j \geqq N$, these domains are not empty.) In [6], p. 73, it is shown that there exists a positive number c_1, independently of j, such that

$$\inf_{\substack{x \in \partial\Omega^{(j)} \\ y \in \partial\Omega^{(j+1)}}} |x - y| \geqq c_1 2^{-j}, \quad j = N, N + 1, \ldots. \tag{19}$$

Let λ be a sufficiently large positive number and let $j_\lambda = [\log_2 \lambda^{1/\nu}] + 1$ be the corresponding integer. Then $\Omega^{(j_\lambda)}$ is covered by cubes Q_l of the side – length $d \,.\, 2^{-j}$, where $d > 0$ is chosen (independently of j_λ) such that

$$\Omega^{(j_\lambda)} \subset \bigcup Q_l \subset \Omega^{(j_\lambda + 1)}. \tag{20}$$

It holds

$$|\Omega^{(j_\lambda)}| = \int_{\Omega^{(j_\lambda)}} \varrho^{-a}(x)\, \varrho^a(x)\, dx \leqq c_2\, 2^{j_\lambda a} \leqq c_3 \lambda^{a/\nu}. \tag{21}$$

Let L_λ be the number of the cubes Q_l needed for covering $\Omega^{(j_\lambda)}$ in the way described above. Then it follows from (20) and (21) that

$$L_\lambda \leqq c_4 \lambda^{a/\nu} d^{-n}\, 2^{j_\lambda n} \leqq c_5 \lambda^{a/\nu + n/\nu}. \tag{22}$$

It holds

$$\|u\|^2_{W^{2m}{}_{2,2\mu,2\nu}(\Omega)} = \|u\|^2_{W^{2m}{}_{2,2\mu,2\nu}(\Omega - \bigcup Q_l)} + \sum_{l=1}^{L_\lambda} \|u\|^2_{W^{2m}{}_{2,2\mu,2\nu}(Q_l)}. \tag{23}$$

These norms are considered as quadratic forms over the Hilbert spaces $L_2(\Omega)$, $L_2(\Omega - \bigcup Q_l)$ and $L_2(\Omega_l)$, respectively. These quadratic forms are generated by the corresponding self-adjoint positive-definite operators A_0, A_∞ and A_l, respectively. (See [3], p. 317–318.) It holds

$$L_2(\Omega) = L_2(\Omega - \bigcup Q_l) \oplus \sum_{l=1}^{L_\lambda} \oplus\, L_2(Q_l),$$

$$W^{2m}_{2,2\mu,2\nu}(\Omega) \subset W^{2m}_{2,2\mu,2\nu}(\Omega - \bigcup Q_l) \oplus \sum_{l=1}^{L_\lambda} \oplus\, W^{2m}_{2,2\mu,2\nu}(Q_l).$$

If $N_C(\lambda)$ denotes the distribution of eigenvalues for the self-adjoint operator C, Courant's variation principle implies

$$N_{A_0}(\lambda) \leqq N_{A_\infty}(\lambda) + \sum_{l=1}^{L_\lambda} N_{A_l}(\lambda). \tag{24}$$

The choice of j_λ yields

$$\|u\|^2_{W^{2m}{}_{2,2\mu,2\nu}(\Omega-\bigcup Q_l)} \geqq 2^{2j_\lambda \nu}\|u\|^2_{L_2(\Omega-\bigcup Q_l)} \geqq (\lambda^2+\varepsilon)\|u\|^2_{L_2(\Omega-\bigcup Q_l)}$$

where ε is a sufficiently small positive number. Hence, $N_{A_\infty}(\lambda) = 0$. By virtue of $\tilde{\mu} = \min(\mu, 0)$ it follows

$$\|u\|^2_{W^{2m}{}_{2,2\mu,2\nu}(Q_l)} \geqq c_6 2^{2\tilde{\mu} j_\lambda}\|u\|^2_{W^{2m}{}_2(Q_l)} \geqq c_7\lambda^{2\tilde{\mu}/\nu}\|u\|^2_{W^{2m}{}_2(Q_l)},$$

where c_6 and c_7 are suitable positive numbers. Let B be the operator belonging to the quadratic form $\|u\|^2_{W^{2m}{}_2(Q_l)}$ (with respect to the Hilbert space $L_2(Q_l)$). Then (22), (24), and the last estimate yield

$$N(\lambda) \leqq c_8\lambda^{a/\nu+n/\nu}N_{c_9\lambda^{\tilde{\mu}/\nu}B}(\lambda) \leqq c_{10}\lambda^{a/\nu+n/\nu}N_B(c_{11}\lambda^{1-\tilde{\mu}/\nu}). \tag{25}$$

Let Q be the unit cube, and let D be the operator belonging to $\|u\|^2_{W^{2m}{}_2(Q)}$. We use the well-known fact

$$N_B(\eta) \leqq c_{12}\, N_D(\eta), \quad \eta \geqq 1.$$

This assertion can be proved by mapping Q_l onto Q with the aid of a linear transformation of coordinates and comparing the corresponding quadratic form with $\|u\|^2_{W^{2m}{}_2(Q)}$. Then one obtains the last assertion from Courant's principle. However, the distribution of the eigenvalues for D is known. It holds $N_D(\eta) \leqq c_{13}\eta^{n/2m}$ (see also the second step, $\tilde{s}_j \sim j^{-2m/n}$). Hence

$$N(\lambda) \leqq c_{14}\lambda^{a/\nu+n/\nu}\lambda^{n(1-\tilde{\mu}/\nu)/2m}.$$

This proves Theorem 1.

4.2. Proof of Theorem 2. Step 1. Assume $p = 2$. Let

$$Au = \tfrac{1}{2}\sum_{l=0}^{m}\sum_{|\alpha|=2l}[\varrho^{\varkappa_{2l}}(x)\,b_\alpha(x)\,D^\alpha u + D^\alpha(\varrho^{\varkappa_{2l}}(x)\,b_\alpha(x)\,u)] + Bu = \mathring{A}u + Bu.$$

Now, using the method developed in [7] (in particular, formula (10) in [7]) we obtain that $\mathring{A}$ belongs to $\mathring{A}^{(m)}_{\mu,\nu}$, while B is a perturbation operator of order $2m-1$, whose coefficients have the property (9). Since $\mathring{A}$ is formally self-adjoint, Theorem 1 yields that $\mathring{A}$, $D(\mathring{A}) = W^{2m}_{2,2\mu,2\nu}(\Omega)$, is self-adjoint in $L_2(\Omega)$. Using again Theorem 1, in particular (15), we obtain that $\mathring{A}$ with the domain of definition

$$D(\mathring{A}) = W^{2(k+1)m}_{2,2(k+1)\mu,2(k+1)\nu}(\Omega)$$

is a self-adjoint operator in the Hilbert space $H = W^{2km}_{2,2k\mu,2k\nu}(\Omega)$ (after a suitable choice of an equivalent norm). Here $k = 0, 1, 2, \ldots$. This operator has a pure point spectrum. For a suitable number $\eta > 0$ it holds

$$(26)\quad \|Bu\|_H = \|Bu\|_{W^{2km}_{2,2k\mu,2k\nu}(\Omega)} \leqq c\left(\sum_{|\beta|\leqq 2km+2m-1}\int_\Omega \varrho^{\varkappa_{|\beta|}-\eta}(x)\,|D^\beta u|^2\,\mathrm{d}x\right)^{1/2},$$

$$\varkappa_j = \frac{2(k+1)\,\mu}{2(k+1)\,m}\,j + \frac{2(k+1)\,\nu}{2(k+1)\,m}\,(2(k+1)\,m - j) = \frac{\mu}{m}\,j + \frac{\nu}{m}\,(2km + 2m - j)\,.$$

(Here we use again the method developed in the proof of Lemma 3.1 in [7]. One has to take into consideration that A is not only of type $A^{(m)}_{\mu,\nu}$ but even of type $\mathring{A}^{(m)}_{\mu,\nu}$.) Setting $\eta = 2\nu s$, we have

$$\varkappa_j - \eta = \frac{2(k+1-s)\,\mu}{2(k+1-s)\,m}\,j + \frac{2(k+1-s)\,\nu}{2(k+1-s)\,m}\,(2(k+1-s)\,m - j)\,.$$

Without loss of generality we may assume $0 < s < 1/2m$. Now we use Lemma 3.2 of [7]. This lemma is formulated for $l = 1, 2, \ldots$ (see the notation introduced there). But the lemma, as well as the proof, are true, for arbitrary positive numbers l. Then (26) yields

$$(27)\qquad \|Bu\|_H \leqq c\,\|u\|_{W^{2(k+1-s)m}_{2,2(k+1-s)\mu,2(k+1-s)\nu}(\Omega)}\,.$$

For a moment we denote by I the embedding operator from

$$W^{2(k+1)m}_{2,2(k+1)\mu,2(k+1)\nu}(\Omega)\quad\text{into}\quad W^{2(k+1-s)m}_{2,2(k+1-s)\mu,2(k+1-s)\nu}(\Omega)\,.$$

But now it is an easy consequence of Theorem 1, in particular of (14) and (15), that I belongs to the ideal S_r, where r is a suitable number, $1 < r < \infty$. (For the definition of S_r see [2]. The original definition of S_r is restricted to compact operators acting from one Hilbert space into itself. Nevertheless, of course, there is no difference when considering compact operators acting from one Hilbert space into another one.) Let λ be a real number, not an eigenvalue of A. Then $B(\mathring{A} - \lambda E)^{-1}$, viewed as an operator from H into H, can be represented as

$$(28)\qquad B(\mathring{A} - \lambda E)^{-1} = B\,I(\mathring{A} - \lambda E)^{-1}\,,$$

where (27) yields that B on the right hand side is a bounded operator acting from $W^{2(k+1-s)m}_{2,2(k+1-s)\mu,2(k+1-s)\nu}(\Omega)$ into H. It follows from the ideal property of S_r that $B(\mathring{A} - \lambda E)^{-1}$ belongs to S_r (viewed as an operator from H into H). But now we can apply the important criterion of I. C. Gochberg and M. G. Krejn for the density of associated eigenvectors, ee [2], Chapter V, Theorem 10.1. Applying this theorem we obtain that A with the domain of definition $W^{2(k+1)m}_{2,2(k+1)\mu,2(k+1)\nu}(\Omega)$ is a closed operator in the Hilbert space H, its spectrum consists of isolated eigenvalue of finite algebraic multiplicity, and the linear hull of the associated eigenvectors is dense in H.

Step 2. Let again $p = 2$. It follows from the first step that the linear hull of the associated eigenvectors of the operator A, where $D(A) = W^{2m}_{2,2\mu,2\nu}(\Omega)$, is dense in $L_2(\Omega)$. Now we want to show that

$$(29) \qquad D(A^\infty) = \bigcap_{j=0}^{\infty} D(A^j) = \bigcap_{j=0}^{\infty} W^{2jm}_{2,2\mu j,2\nu j}(\Omega) = S_{\varrho(x)}(\Omega)\,.$$

Theorem 1 yields that only the last assertion as not evident (the first one is definition). Let $u \in C_0^\infty(\Omega)$. Then it follows

$$(30) \quad \|u\|_{W^{2jm}{}_{2,2\mu j,2\nu j}(\Omega)} = \left[\int_\Omega \left(\sum_{|\alpha|=2jm} \varrho^{2\mu j}(x)\,|D^\alpha u|^2 + \varrho^{2\nu j}(x)\,|u|^2\right) \mathrm{d}x\right]^{1/2} \leqq$$

$$\leqq \sum_{|\alpha|=2jm}\left[\sup_{x\in\Omega} \varrho^{\mu j + a/2}(x)\,|D^\alpha u(x)| \left(\int_\Omega \varrho^{-a}(y)\,\mathrm{d}y\right)^{1/2} + \right.$$

$$\left. + \sup_{x\in\Omega} \varrho^{\nu j + a/2}(x)\,|u(x)|^2 \left(\int_\Omega \varrho^{-a}(y)\,\mathrm{d}y\right)^{1/2}\right]$$

and, with the aid of Sobolev's embedding theorem and Lemma 3.2 of [7],

$$(31) \qquad \sup_{x\in\Omega} \varrho^l(x)\,|D^\alpha u(x)| \leqq c\|\varrho^l(x)\,D^\alpha u(x)\|_{W^n{}_2(\Omega)} \leqq c'\|u\|_{W^{2jm}{}_{2,2\mu j,2\nu j}(\Omega)}\,.$$

Here, $l = 0, 1, 2, \ldots, \alpha$ multiindex, and $j = j(l, \alpha)$ is a sufficiently large positive integer. $C_0^\infty(\Omega)$ is a dense subset both in $S_{\varrho(x)}(\Omega)$ and $W^{2jm}_{2,2\mu j,2\nu j}(\Omega)$. Hence it follows by completion that (30) and (31) are also true for $u \in S_{\varrho(x)}(\Omega)$ and $u \in W^{2jm}_{2,2\mu j,2\nu j}(\Omega)$, respectively. But then (30) and (31) yield (29), where the equality is to be understood in the topological sense. Hence, the associated eigenvectors belong to $S_{\varrho(x)}(\Omega) = D(A^\infty)$. This implies that these associated eigenvectors are the same as the corresponding associated eigenvectors for the operator A, considered in the Hilbert space $H = W^{2km}_{2,2k\mu,2k\nu}(\Omega)$. However, one obtains now from (29) that the linear hull of the associated eigenvectors of the operator A is dense in $S_{\varrho(x)}(\Omega)$.

Step 3. Let $1 < p < \infty$. Repating the argument of the first step of 4.1, it follows that the embedding from $D(A) = W^{2m}_{p,p\mu,p\nu}(\Omega)$ into $L_p(\Omega)$ is compact. Further, Theorem 5.3 of [7] yields that for suitable λ the operator $A - \lambda E$ gives an isomorphic mapping from $W^{2m}_{p,p\mu,p\nu}(\Omega)$ onto $L_p(\Omega)$. Hence $(A - \lambda E)^{-1}$ is a compact operator acting in $L_p(\Omega)$. It follows (as a consequence of the Riesz-Schander-theory for compact operators in Banach spaces) that A is a closed operator having a pure point spectrum, which consists of isolated eigenvalues of finite algebraic multiplicity. Now, using again Theorem 5.3 of [7] and repeating the argument of the second step we obtain also in this case

$$D(A^\infty) = S_{\varrho(x)}(\Omega)\,.$$

Hence the associated eigenvectors belong to $S_{\varrho(x)}(\Omega)$. But then they are the same as in the case $p = 2$. Consequently, they are independent of p (and so are also the eigenvalues, including their multiplicities). Finally, let $0 \leqq s < \infty$; $1 < q < \infty$; $-\infty < \varkappa + sq \leqq \tau < \infty$. The counterpart to (30) yields

$$S_{\varrho(x)}(\Omega) \subset W^s_{q,\varkappa,\tau}(\Omega) .$$

Since $C_0^\infty(\Omega)$ is dense in $W^s_{q,\varkappa,\tau}(\Omega)$, so $S_{\varrho(x)}(\Omega)$ is also dense. Hence we obtain that the linear hull of the associated eigenvectors of the above operator is dense in $W^s_{q,\varkappa,\tau}(\Omega)$.

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