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ABSOLUTE POINTS IN $\beta N \setminus N$

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The aim of this paper is the study of the space $N^* = \beta N \setminus N$ in the situation when Continuum Hypothesis (CH) not necessarily holds and Martin's Axiom (MA) is assumed. Now some distinctions of P-points are possible. We introduce a notion of absolute points announced as $P(c)$ points by BOOTH [2] (by CH absolute points coincide with P-points). We prove that there exist 2^c absolute P-points which are minimal in the Rudin-Keisler ordering. Although this result can be obtained in a way analogous to that of BLASS [1] (the existence of 2^c minimal P-points), we get the mentioned result from some theorems of the Baire Category type (Lemmas 2 and 3). These theorems allow to obtain further results concerning the structure of N^*. Namely, we prove that each cover of N^* by means of nowhere dense subsets is of the cardinality greater than c. In other words, the Novák number (introduced in § 3) of N^* is greater than c. It is known to the authors from Professor NOVÁK's oral communication that, without any extra set-theoretical assumptions, the cardinality of any cover of N^* by disjoint nowhere dense closed subsets is greater than $\aleph_1$.

1. Basic Lemmas. A family $T = \{T_\alpha : \alpha < \beta\}$ of closed-open subsets of N^*, where α and β are ordinals, is a *β-tower* (HECHLER [4]) if for all ordinals $\alpha < \gamma < \beta$ we have $T_\gamma \nsubseteq T_\alpha$. A tower T is said to be *maximal* if it is maximal with respect to the length of T, i.e., if $\bigcap T$ is a nowhere dense set (Hechler calls such a tower complete). Hechler [4] proved that if MA holds, then each maximal tower has the cardinality $2^{\aleph_0}$.

It is natural to ask whether there exist P-points which are maximal towers, i.e., P-points with linearly ordered (with respect to the inclusion) base in N^*. It is obvious that if CH holds, then each P-point in N^* is a tower.

In the sequel, we use the usual convention that a cardinal is an initial ordinal, c is the cardinal of the continuum and free ultrafilters on N are regarded as points of N^*.

Lemma 1 (MARTIN, SOLOVAY). *If MA holds and B is a base for a free filter on N such that* card $B < c$, *then there exists an infinite subset T of N such that $T \setminus Y$ is finite, for each $Y \in B$.*

Note. This Lemma follows from the $S_{\aleph}$ hypothesis which is implied by MA (see Martin, Solovay [5]). Because the Lemma is crucial when applying MA, we give here a direct proof (cf. Both [2]).

Proof. Let $P = \{(F, Y) : F$ is finite and $Y \in B\}$ and for (F, Y), $(F', Y') \in P$ put $(F, Y) \leqq (F', Y')$ iff $F \subset F' \subset F \cup Y$ and $Y' \subset Y$. It is obvious that $\leqq$ establishes a partial ordering on P. Note that, if (F, Y) and (F, Y') are in P and have the same first element and $Y_0 \in B$ is such that $Y_0 \subset Y \cap Y'$, then $(F, Y_0) \in P$ and (F, Y_0) is an upper bound for them both. Therefore, if $L \subset P$ is an antichain, then elements of L have different first members, hence L is countable. It is to verify that the sets $D_n = \{(F, Y) \in \in P$: there exists $m \in F$ with $m > n\}$ and $D_A = \{(F, Y) \in P : Y \subset A\}$ are dense subsets of P for all $n \in N$ and $A \in B$. Put $\Delta = \{D_n : n \in N\} \cup \{D_A : A \in B\}$. Thus Δ is a family of dense subsets of P and card Δ = card $B < c$. Let $T = \bigcup\{F : (F, Y) \in G\}$, where G is a generic set for Δ.

T is an infinite subset of N, because for each n there exist $(F, Y) \in G \cap D_n$ and $m \in \in F \subset T$ such that $m > n$. So T is an unbounded subset of N.

$T \setminus A$ is finite for each $A \in B$. In fact, let $(F, Y) \in G \cap D_A$, let (F', Y') be an arbitrary element from G and let $(F'', Y'') \in G$ be greater or equal to (F, Y) and (F', Y'). Since $F' \subset F'' \subset F \cup Y$ and (F', Y') is an arbitrary element from G hence $T \subset F \cup Y$. This implies that $T \setminus A \subset (F \cup Y) \setminus A \subset F$, because $Y \subset A$. This completes the proof.

Corollary 1. *Suppose MA holds and R is an infinite family of open subsets of N^* such that $\bigcap R \neq \emptyset$. If* card $R < c$, *then* Int $\bigcap R \neq \emptyset$.

Corollary 2 (Hechler [4]). *If MA holds, then each maximal tower in N^* has the cardinality c.*

Lemma 2. *Suppose MA holds and $\mathscr{A}$ is a family of nowhere dense subsets of N^*. If* card $\mathscr{A} < c$, *then $\bigcup \mathscr{A}$ is a nowhere dense subset of N^*.*

Proof. Let $\mathscr{A} = \{A_\alpha : \alpha < \gamma\}$, where $\gamma < c$, be a well ordering of $\mathscr{A}$ and suppose Int cl $\bigcup \mathscr{A} \neq \emptyset$. Let V be a non-empty closed-open subset of N^* contained in cl $\bigcup \mathscr{A}$. We define, by transfinite induction, a family $\{V_\alpha : \alpha < \gamma\}$ of non-empty closed-open subsets of N^* such that

(i) $V_\beta \subset V_\alpha \subset V$, for $\alpha < \beta < \gamma$,

(ii) $V_\alpha \cap A_\alpha = \emptyset$, for $\alpha < \gamma$.

Let V_1 be an arbitrary, non-empty and closed-open subset contained in V such that $V_1 \cap A_1 = \emptyset$. Assume that we have defined V_α, for $\alpha < \beta$, which fulfil (i) and (ii). In virtue of compactness of N^* and Corollary 1, Int $\bigcap\{V_\alpha : \alpha < \beta\} \neq \emptyset$. Let V_β be a non-empty, closed-open subset contained in Int $\bigcap\{V_\alpha : \alpha < \beta\}$ such that $V_\beta \cap A_\beta = \emptyset$. In virtue of Corollary 1 again, we infer $G = \text{Int} \bigcap\{V_\beta : \beta < \gamma\} \neq \emptyset$. This contradicts to our assumption, G being a non-empty open set contained in V and disjoint with $\bigcup \mathscr{A}$.

Lemma 3. *Suppose MA holds and $\mathscr{A}$ is a family of nowhere dense subsets of N^*. If* card $\mathscr{A} = c$, *then $N^* \setminus \bigcup \mathscr{A}$ is a dense subset of N^* of cardinality 2^c.*

Proof. Let $\mathscr{A} = \{A_\alpha : \alpha < c\}$ be a well ordering of $\mathscr{A}$. For each ordinal $\alpha < c$ we define, by transfinite induction, a family R_α of disjoint closed-open subsets of N^* which fulfil the following conditions:

(i) $\bigcup R_\alpha \cap A_\alpha = \emptyset$,

(ii) the family R_β refines the family R_α for $\alpha < \beta < c$, i.e., for every $V \in R_\beta$ there exists $U \in R_\alpha$ such that $V \subset U$,

(iii) if $\alpha < \beta < c$ and $U \in R_\alpha$, then card $\{V \in R_\beta : V \subset U\} \geqq 2$,

(iv) if $\gamma < c$ is a limit ordinal and L is a γ-tower consisting of elements of all families R_α for $\alpha < \gamma$, then $\bigcap L$ contains at least two elements of the family R_γ.

Let R_0 be a family consisting of two disjoint, non-empty, closed-open sets which are also disjoint with A_0.

Assume that we have defined the families R_α for $\alpha < \beta$.

If $\beta = \alpha + 1$, then for every $U \in R_\alpha$ take two disjoint, non-empty closed-open sets contained in U and disjoint with $A_{\alpha+1}$. Let $R_{\alpha+1}$ be the family of all these sets, for each $U \in R_\alpha$.

If β is a limit ordinal and L is a β-tower consisting of elements of all families R_α for $\alpha < \beta$, then, by Corollary 1, Int $\bigcap L \neq \emptyset$. Take for every such β-tower two disjoint non-empty closed-open sets contained in Int $\bigcap L$ and disjoint with A_β. Let R_β be the family of all these sets.

Conditions (i)–(iv) are in both cases obviously fulfilled.

Now, conditions (ii), (iii) and (iv) imply that the cardinality of the family of all c-towers of elements of all families R_α for $\alpha < c$, is 2^c. Moreover, if we take two such different c-towers, then their intersections are non-empty and disjoint. Condition (i) implies that the intersection of such c-tower is disjoint with $\bigcup \mathscr{A}$. The elements of R_0 can be chosen as subsets of an arbitrary open set which implies the density of $N^* \setminus \bigcup \mathscr{A}$.

Remark. A more detailed version (although without further applications in this paper) of Lemma 3 can be stated: $N^* \setminus \bigcup \mathscr{A}$ contains the space 2^c with the box-topology as a dense subspace.

2. Minimal and absolute points in N^*. Let $\aleph$ be a cardinal and let X be a space. A point $p \in X$ is said to be an *$\aleph$-point* if $\aleph$ is the supremum of all cardinals such that the intersection of each family of the cardinality less than $\aleph$ of neighbourhoods of p is a neighborhood of p.

Let X be a space and let $w(X, x)$ denote the weight of X at the point x. A point x which is $w(X, x)$-point is called an *absolute point* of X.

In the sequel, $F_\aleph$ denotes the set of all $\aleph$-points of N^* and F denotes the set of all absolute points of N^*, i.e., if MA is assumed, the set of all c-points.

It is obvious that absolute points of N^* can be characterized in terms of towers as follows:

a point $p \in N^*$ is an absolute point iff there exists a tower T such that $\{p\} = \bigcap T$.

Note that the set of all non-P-points of N^* coincides with $F_{\aleph_0}$.

Theorem 1. *Suppose MA holds. If $\aleph < c$, then the set $F_\aleph$ can be covered by c closed and nowhere dense subsets of N^*.*

Proof. Let B be a base in N^* consisting of closed-open sets and card $B = c$. If p is an $\aleph$-point, then there exists a family R of neighborhoods of p with card $R = \aleph$ and $p \in \bigcap R \setminus \operatorname{Int} \bigcap R$. We can assume that $R \subset B$.

For each family $R \subset B$ with card $R = \aleph$, let $A_R = \bigcap R \setminus \operatorname{Int} \bigcap R$. The cardinality of the set of all subfamilies of the cardinality $\aleph$ from B is equal to $2^{\aleph_0 \cdot \aleph} = 2^\aleph$. In virtue of MA, we have $2^\aleph = 2^{\aleph_0}$ (see Martin, Solovay [5]). Thus the family of all such sets A_R gives the required cover of $F_\aleph$.

Recall that an ultrafilter $p \in N^*$ is a P-point iff each map $f: N \to N$ is either constant or finite-to-one on an element of p. An ultrafilter $p \in N^*$ is minimal (with respect to the Rudin-Keisler ordering) iff each map $f: N \to N$ is either constant or one-to-one on an element of p. It is obvious that the minimal points of N^* are P-points. The definition implies the following characterization of minimal P-points:

Lemma 4. *A P-point $p \in N^*$ is minimal iff for each finite-to-one map $f: N \to N$ there exists a neighborhood U of p in βN such that $\beta f \mid U$ is a homeomorphism onto $(\beta f)(U)$, where βf is the extension of f onto βN.*

Let $f: N \to N$ be a finite-to-one map. Denote by O_f the family $\{\mathrm{cl}_{\beta N} M \setminus N : M \subset N$ and $f \mid M$ is one-to-one$\}$. It is easy to prove the following

Lemma 5. *If $f: N \to N$ is finite-to-one, then $\bigcup O_f$ is dense and open in N^*.*

Theorem 2. *Suppose MA holds. The set of all non-minimal points of N^* can be covered by c closed and nowhere dense subsets of N^*.*

Proof. Let $\mathscr{F}$ be the set of all finite-to-one maps from N into N. The family $\{N^* \setminus \bigcup O_f : f \in \mathscr{F}\}$ is a family of closed and nowhere dense subsets of N^* which covers the set of all non-minimal P-points. The cardinality of this family is c. In virtue of Theorem 1, the set of all non-P-points is covered by c closed and nowhere dense subsets of N^*. Both these families give the required cover of all non-minimal points of N^*.

Theorem 3. *There exists a dense subset D of N^* of cardinality 2^c consisting of points which are both absolute and minimal, and such that $N^* \setminus D$ can be covered by c closed and nowhere dense subsets of N^*.*

Proof. In virtue of Theorem 1, the set $\bigcup\{F_{\aleph} : \aleph < c\}$ of all non-absolute points of N^* can be covered by a family R_1 of closed and nowhere dense subsets of N^* with card $R_1 = c$. From Theorem 2 it follows that there exists a family R_2 of closed and nowhere dense subsets of N of cardinality c which covers the set of all non-minimal points of N. Thus the family $R = R_1 \cup R_2$ is a family of closed and nowhere dense subsets of N^* of cardinality c and $D = N^* \setminus \bigcup R$ is contained in the set consisting of points which are both absolute and minimal. In virtue of Lemma 3, the set D is a dense subset of N^* of cardinality 2^c.

Remark. It can be proved, using Remark to Lemma 3, that the space 2^c with the box-topology can be embedded as a dense set in the set of points which are both absolute and minimal.

Question 1. *Assume MA. Do there exist $\aleph$-points in N^* for $\aleph_0 < \aleph < c$?*

Question 2. *Are all absolute points of the same type in N^*,* i.e., *does there exist for any absolute points p and q in N^* a homeomorphism f of N^* onto itself such that $f(p) = q$?*

3. Novák Number of subspaces of N^*. *The Novák number nX* of a dense in itself space X is the least infinite cardinal being the cardinal of a covering of X by nowhere dense sets.

In this section we establish the Novák number of some subspaces of N^* and we discuss the cardinality of some special families of nowhere dense subsets of N^*. All theorems in this section depend on Martin's Axiom.

First we state same consequences of previous theorems.

Theorem 4. *$nN^* > c$. If D is a dense subset of N^*, then $nD \geqq c$.*

Proof. The former inequality follows from Lemmas 2 and 3, the latter from Lemma 2.

Theorem 5. *$n(F \cap M) > c$ and $n(N^* \setminus (F \cap M)) = c$, where F denotes the set of all absolute points and M the set of all minimal points of N^*.*

Proof. Let $D \subset F \cap M$ be the same as in Theorem 3. If $n(F \cap M) \leqq c$, then $N^* = (N^* \setminus D) \cup F \cap M$ can be covered by a family of closed and nowhere dense subsets of N^* of cardinality $\leqq c$ (nowhere dense subsets in a subspace are nowhere dense subsets in the whole space). This contradicts Lemma 3.

The set $N^* \setminus F \cap M$ is a dense subset of N^*, so from Theorem 4 it follows that $n(N^* \setminus F \cap M) \geqq c$. In virtue of Theorem 3, $N^* \setminus F \cap M \subset N^* \setminus D$ can be covered by a family of closed and nowhere dense subsets of $N^* \setminus F \cap M$ of cardinality c (nowhere dense subsets in the whole space are nowhere dense in a dense subspace).

A cover $\mathscr{A}$ of X by closed and disjoint nowhere dense subsets of a space X is called *upper semicontinuous* (*lower semicontinuous*) if for every open set $U \subset X$ the set $\bigcup\{A \in \mathscr{A} : A \subset U\}$ (the set $\bigcup\{A \in \mathscr{A} : A \cap U \neq \emptyset\}$) is open.

A cover $\mathscr{A}$ of a space X is said to be *regular* if for each non-empty open set $G \subset X$ there exist disjoint and non-empty open sets U, V contained in G such that the sets $\{A \in \mathscr{A} : A \cap U \neq \emptyset\}$ and $\{A \in \mathscr{A} : A \cap V \neq \emptyset\}$ are disjoint.

Theorem 6. *If $\mathscr{A}$ is an upper semicontinuous cover of a normal space X, then $\mathscr{A}$ is regular. Assume MA. If $\mathscr{A}$ is lower semicontinuous cover of N^*, then $\mathscr{A}$ is regular.*

Proof. Let $\mathscr{A}$ be an upper semicontinuous cover of a normal space X, and let U be a non-empty open subset of X. Since the elements of $\mathscr{A}$ are nowhere dense and $\mathscr{A}$ is a cover, hence there exist $A_1, A_2 \in \mathscr{A}$ such that $A_1 \cap U \neq \emptyset \neq A_2 \cap U$ and $A_1 \neq A_2$. Since A_1 and A_2 are disjoint and closed subsets of a normal space X, hence there exist disjoint open sets V_1 and V_2 such that $A_i \subset V_i$ for $i = 1, 2$. Since $\mathscr{A}$ is upper semicontinuous hence $B_i = \bigcup\{A \in \mathscr{A} : A \subset V_i\}$ for $i = 1, 2$ are non-empty and open. Moreover, since $A_i \subset B_i$ hence $B_i \cap U \neq \emptyset$ for $i = 1, 2$. It is obvious that then $U_i = B_i \cap U$, $i = 1, 2$, are the open subsets of U desired for $\mathscr{A}$ to be regular.

Assume MA. Let $\mathscr{A}$ be a lower semicontinuous cover of N^* and let U be a non-empty open subset of N^*. Let us suppose, on the contrary, that for any open sets $V_1, V_2 \subset U$ there is

$$(\bigcup\{A \in \mathscr{A} : A \cap V_1 \neq \emptyset\}) \cap (\bigcup\{A \in \mathscr{A} : A \cap V_2 \neq \emptyset\}) \neq \emptyset .$$

The last assumption implies that for each open set $V \subset U$ the set $D_V = \bigcup\{A \in \mathscr{A} : A \cap V \neq \emptyset\}$ is a dense and open subset of U. Hence for each open $V \subset U$ we have that $U \setminus D_V$ is a nowhere dense subset of N^*. Now, let B be a base in N^* consisting of closed-open sets and card $B = c$. The family $R = \{U \setminus D_W : W \in B,\ W \subset U\}$ is a family of nowhere dense subsets of N^* of cardinality c. R is a cover of U. To see this, let $A \in \mathscr{A}$ be such that $A \cap U \neq \emptyset$. Since A is a nowhere dense subset of N^* hence there exists $W \in B$ such that $W \subset U$ and $W \cap A = \emptyset$. This means $A \cap U \subset U \setminus D_W$ and hence $U \subset \bigcup R$. The last inclusion contradicts Lemma 3.

Theorem 7. *If $\mathscr{A}$ is a regular cover of N^*, then* card $\mathscr{A} = 2^c$.

Proof. For each $\alpha < c$ we define a family R_α of closed-open subsets of N^* which are disjoint and which fulfil the following conditions:

(i) if $U, V \in R_\alpha$ and $U \neq V$, then the sets $\{A \in \mathscr{A} : A \cap U \neq \emptyset\}$ and $\{A \in \mathscr{A} : A \cap V \neq \emptyset\}$ are disjoint,

(ii) if $\alpha < \beta < c$, then R_β refines R_α,

(iii) if $\alpha < c$ and $L = \{V_\gamma : \gamma < \alpha\}$ is an α-tower consisting of elements of families R_γ for $\gamma < \alpha$, then card $\{V \in R_\alpha : V \subset \bigcap L\} \geqq 2$.

Let R_0 consist of two arbitrary disjoint, closed-open sets U, V which fulfil (i) (the existence is implied by regularity of $\mathscr{A}$). Assume that we have defined the families R_γ for each $\gamma < \alpha$ which fulfil conditions (i)–(iii). Take an arbitrary α-tower L consisting of elements of families R_γ for $\gamma < \alpha$. In virtue of MA we have $\operatorname{Int} \bigcap L \neq \emptyset$. Since $\mathscr{A}$ is regular hence there exist closed-open and disjoint sets U, V contained in $\operatorname{Int} \bigcap L$ such that the sets $\{A \in \mathscr{A} : A \cap U \neq \emptyset\}$ and $\{A \in \mathscr{A} : A \cap V \neq \emptyset\}$ are disjoint. Put R_α to be the family of all such U, V for all α-towers consisting of elements of all families R_γ for $\gamma < \alpha$. It is obvious that $\{R_\gamma : \gamma \leqq \alpha\}$ fulfils conditions (i)–(iii).

Now, conditions (ii) and (iii) imply that the set of all c-towers consisting of elements of all families R_α for $\alpha < c$ has cardinality 2^c. For such a c-tower L, denote by A_L the element of $\mathscr{A}$ such that $A_L \cap \bigcap L \neq \emptyset$ (such an element A_L exists because $\bigcap L \neq \emptyset$ and $\mathscr{A}$ is a cover of N^*). Moreover, if L and L' are distinct such c-towers, then there exists an ordinal $\beta < c$ and sets $U_\beta \in L \cap R_\beta$, $V_\beta \in L' \cap R_\beta$ such that $U_\beta \cap V_\beta = \emptyset$. In virtue of condition (i), we have $A_L \neq A_{L'}$. Hence card $\mathscr{A} = 2^c$.

Added in proof. Question 1 was answered positively by the second author, On the existence of $P(\aleph)$-points for $\aleph_0 < \aleph < c$, Colloquium Mathematicum (in print).

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