Czechoslovak Mathematical Journal

Anthony W. Hager; Michael David Rice

The commuting of coreflectors in uniform spaces with completion

Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 3, 371–380

Persistent URL: http://dml.cz/dmlcz/101413

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

THE COMMUTING OF COREFLECTORS IN UNIFORM SPACES WITH COMPLETION

ANTHONY W. HAGER, Middletown and MICHAEL D. RICE, Fairfax

(Received March 29, 1973)

In the category of uniform spaces and uniformly continuous maps, let $\mathscr{C}$ be a coreflective subcategory with functor c, $\mathscr{R}$ an epireflective subcategory with functor r for which each reflection map is an embedding. For example, $\mathscr{R}$ can be complete spaces Γ, with r the completion operator γ; or $\mathscr{R}$ can be the "α-complete" spaces defined in § 2. We consider the question of when $rc = cr$ occurs. We give some simple conditions for equality, and show how, under certain hypotheses, the categories and functors can be modified to achieve commuting. The constructs seem of interest.

1. Reflection and coreflection. We shall work in the category $\mathscr{U}$ of separated uniform spaces and uniformly continuous maps. (The definitions and construction below can be given in more general categories, but we omit discussion of this.)

The subcategory $\mathscr{C}$ is coreflective if each object $U \in \mathscr{U}$ has a $\mathscr{C}$-coreflection, i.e., there is $U \leftarrow^{c_U} cU$, $cU \in \mathscr{C}$, such that each $U \leftarrow C$, $C \in \mathscr{C}$, factors uniquely through c_U. c_U is called the *coreflection map*. We shall abuse terminology by saying things like "The coreflection $(\mathscr{C}, c)$". The definition of a reflection $(\mathscr{R}, r)$ is dual. One says "epi-reflection" if each reflection map is epic (i.e., has dense image). We suppose that every subcategory is neither $\emptyset$ nor $\{\emptyset\}$.

The following useful facts are due to KENNISON [8a]. (See [4] for generalization and further discussion.) The subcategory $\mathscr{C}$ is coreflective iff $\mathscr{C}$ is closed under sums and quotients; each coreflection map is one-to-one and onto. (The latter permits this interpretation of the coreflection $\mu X \leftarrow^{c_{\mu X}} c\mu X$: $c_{\mu X}$ is the identity on X, and $c\mu$ is the coarsest $\mathscr{C}$-uniformity finer than μ.) The subcategory $\mathscr{R}$ is epi-reflective iff $\mathscr{R}$ is closed under products and closed subspaces.

The standard examples of coreflective categories are fine spaces and uniformly discrete spaces. Actually, $\mathscr{U}$ abounds with interesting examples; see [2c, e], [7], [9b].

The usual example of an epi-reflection is (Γ, γ), where Γ is complete spaces, and γ is completion. Others are precompact spaces, "separable" spaces, compact totally disconnected spaces, etc. We shall be concerned essentially only with epi-reflections $(\mathscr{R}, r)$ for which each reflection map is an embedding; we then say that $\mathscr{R}$ is embedding. From [2d], $\mathscr{R}$ is embedding iff $\mathscr{R} \supset \Gamma$, and then

$$r\mu X = \bigcap\{R : \mu X \subset R \subset \gamma\mu X,\ R \in \mathscr{R}\} \subset \gamma\mu X .$$

For general $(\mathscr{R}, r)$: if $\mu X \subset R \in \mathscr{R}$, then $\mu X \to r\mu X$ is an embedding (see [5] for the proof of the corresponding fact in Tychonoff spaces; the proof translates to $\mathscr{U}$), and it follows that whenever $\mu X \subset R \in \mathscr{R}$, then

$$r\mu X = \bigcap\{R' : \mu X \subset R' \subset R,\ R' \in \mathscr{R}\} .$$

2. Embedding reflectors. Of course Γ is an example. Another is $\varphi^{-1}\Gamma$, where φ is the fine coreflection. By definition, $\mu X \in \varphi^{-1}\Gamma$ iff $\varphi\mu X \in \Gamma$, i.e., the topological space underlying μX is "topologically complete $\varphi^{-1}\Gamma$ is epi-reflective by 4.1 and 4.2.

Let $\aleph_\alpha$ be an infinite cardinal, α being the usual index by an initial ordinal $(0, 1, \ldots)$. We construct an embedding reflection $(\Gamma_\alpha, \gamma_\alpha)$:

In the topological space X, a G_α-set is the intersection of $<\aleph_\alpha$ open sets. ($G_0 =$ = open, $G_1 = G_\delta$, etc.) X is α-dense in Y if each non-void G_α-set meets X. The α-closure of X in Y consists of all $p \in Y$ such that each G_α-set containing p meets X . X is α-closed in Y if X coincides with its α-closure.

μX is called *α-complete* if μX is α-closed in $\gamma\mu X$. Let Γ_α be the category of α-complete spaces. Note that $\Gamma_0 = \Gamma$; and $\alpha < \beta$ implies $\Gamma_\alpha \subset \Gamma_\beta$.

2.1. Proposition. *Γ_α is epi-reflective. $\mu X \in \Gamma_\alpha$ iff each Cauchy filter with the $<\aleph_\alpha$-intersection property converges. The reflection $\gamma_\alpha\mu X$ is the α-closure of μX in $\gamma\mu X$, or equivalently, all points of $\gamma\mu X$ which correspond to Cauchy filters on μX with the $<\aleph_\alpha$-intersection property. Thus, $\mu X \in \Gamma$ iff $\mu X \in \Gamma_\alpha$ and each Cauchy filter contains a Cauchy filter with the $<\aleph_\alpha$-intersection property.*

The proof of 2.1 can safely be omitted. Of course, the α-complete spaces are the obvious analogue of Herrlich's α-compact Tychonoff spaces [3], [6] (where the reflection is the α-closure in the Stone-Čech compactification and hence for $\alpha = 1$ is the Hewitt realcompactification). We have a more substantial reason for considering $(\Gamma_\alpha, \gamma_\alpha)$, however: It is shown in [9c] that Γ_1 is the epi-reflective hull of metric uniform spaces, and that $\gamma_1 m = m\gamma_1$, where m is the metric-fine coreflection [2a]. We comment more extensively in 7(c) below.

3. Commuting functors. Suppose that $(\mathscr{C}, c)$ is coreflective, and $(\mathscr{R}, r)$ is epi-reflective; this will be standing notation. A mapping $\mu X \to^f \nu Y$ will be called *$\mathscr{R}$-dense* (or perhaps *$\mathscr{R}$-epic*) if there is no space $R \in \mathscr{R}$ with $f(X) \subsetneqq R \subsetneqq \nu Y$. From the construction of $r\mu X$ in § 1, it is clear that: f is $\mathscr{R}$-dense *iff* $rf(X) = r\nu Y$($f(X)$ being given the relative uniformity); an embedding $\mu X \to^e \nu Y$ is $\mathscr{R}$-dense iff $r\nu Y = r\mu X$.

3.1. Proposition. *Let $\mathscr{R}$ be embedding. These conditions are equivalent*:

(a) $cr = rc$.

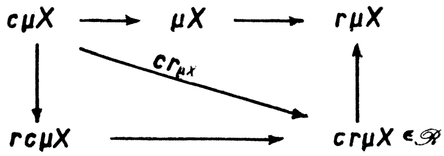
(b) $c\mathscr{R} \subset \mathscr{R}$, *and each* $cr_{\mu X}$ *is an* $\mathscr{R}$*-dense embedding.*

(c) $c\mathscr{R} \subset \mathscr{R}$ *and both*

(b_1) $c\mu X \to^{1x} cr\mu X \mid X$ *is uniformly continuous for each* μX.

(b_2) X *is* $\mathscr{R}$*-dense in* $cr\mu X$ *for each* μX.

Proof. The proof is routine from consideration of the diagram:



3.2. Remarks. (a) Clearly, the conditions in 3.1 are equivalent "locally", i.e., for a particular μX.

(b) The point in 3.1 is what c does to the $\mathscr{R}$-reflection maps $\mu X \to r\mu X$. It is easy to see that every $cr_{\mu X}$ is a $\mathscr{R}$-dense embedding iff c preserves all $\mathscr{R}$-dense embeddings.

(c) One can formulate a rather trivial dual of 3.1 (b) as follows: Call $\mu X \to^{f} vY$ a $\mathscr{C}$-map if cf is an isomorphism. Then: $cr = rc$ iff $r\mathscr{C} \subset \mathscr{C}$ and each $rc_{\mu X}$ is a $\mathscr{C}$-map. This does not seem useful.

(d) A fairly simple argument shows that (for arbitrary reflection $(\mathscr{R}, r)$) $c\mathscr{R} \subset \mathscr{R}$ iff $r\mathscr{C} \subset \mathscr{C}$ [9a].

3.3. Proposition. (a) $\mu X \to^{f} vY$ *is* Γ_α*-dense iff* $f(X)$ *is* α*-dense in* vY.

(b) *If* $\mathscr{C}$ *is hereditary for* α*-dense sets, then for each* μX, $c\mu X \to^{1x} c\gamma_\alpha \mu X | X$ *is uniformly continuous* (*i.e.* 3.1 (b_1)).

(c) *If* c *preserves topology, then*

X *is* α*-dense in* $c\gamma_\alpha \mu X$ *for each* μX (*i.e.* ,3.1 (b_2)), *and* $c\Gamma_\alpha \subset \Gamma_\alpha$.

(d) *If* $(\mathscr{C}, c)$ *is hereditary for* α*-dense sets and preserves topology, then* $c\gamma_\alpha = \gamma_\alpha c$.

Proof. (a) is routine. (b) and the first part of (c) are obvious. Let $\mu X \in \Gamma_\alpha$, and let $\mathscr{F}$ be a $c\mu$-Cauchy filter with the $\aleph_\alpha$-intersection property. Then γ is μ-Cauchy and converges by 2.1. Then $\mathscr{F}$ converges in $c\mu X$ (because the topology is that of μX). (d) now follows using 3.1.

3.3 (d) for $\alpha = 0$ is noted in [7b, p. 127].

4. A preliminary construction. Suppose given a class $\mathscr{A}$ of spaces with an "operator" $\mathscr{A} \to^{e} \mathscr{U}$ which assigns to each $A \in \mathscr{A}$ a one-to-one onto map $A \leftarrow^{e_A} eA$. Let $\langle \mathscr{A}, e \rangle$

be the class of spaces μX for which each $\mu X \to A$ factors through e_A. Suppose further that each $eA \in \langle \mathscr{A}, e \rangle$. (So e behaves like a coreflector.)

Or dually: Suppose that each $A \to^{e_A} eA$ is epic; define $\mu X \in \langle e, \mathscr{A} \rangle$ iff each $\mu X \leftarrow A$ factors through e_A; suppose that each $e \in \langle e, \mathscr{A} \rangle$. (So e behaves like an epi-reflector.)

4.1. Theorem. (a) *$\langle \mathscr{A}, e \rangle$ is coreflective, with functor extending e.*

(b) *$\langle e, \mathscr{A} \rangle$ is epi-reflective, with functor extending e.*

(b) is a specialization of a theorem of Kennison [8b], and (a) is the dual. A special case was given by Isbell [7a]. (The method has also been studied and used in [2], [9], [1], [10]. For our category $\mathscr{U}$, a simple proof can be given from the Kennison criteria stated above. 4.1 is a useful referent for the main construction in §'s 5,6 below, and will be used in several examples.

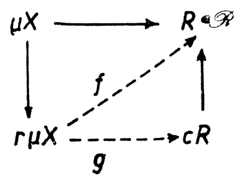
(Recall from 3.2 (d) that $c\mathscr{R} \subset \mathscr{R}$ iff $r\mathscr{C} \subset \mathscr{C}$.)

4.2. Theorem. (a) *If $c\mathscr{R} \subset \mathscr{R}$, then $\langle \mathscr{R}, c \rangle = r^{-1}\mathscr{C}$.*

(b) *If $r\mathscr{C} \subset \mathscr{C}$ then $\langle r, \mathscr{C} \rangle = c^{-1}\mathscr{R}$.*

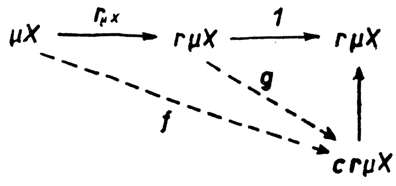
Proof. We shall prove (a). (b) is dual.

Let $\mu X \in r^{-1}\mathscr{C}$, i.e. $r\mu X \in \mathscr{C}$. Consider:



where first f exists because r is a reflection, then g exists because $r\mu X \in \mathscr{C}$. Thus $\mu X \in \langle \mathscr{R}, c \rangle$.

Let $\mu X \in \langle \mathscr{R}, c \rangle$. Consider:



where first f exists because $\mu X \in \langle \mathscr{R}, c \rangle$, then g exists because $cr\,\mu X \in \mathscr{C}$. Since g is the identity on the set $r\mu X$, it follows that $r\mu = cr\mu$, i.e. that $r\mu X \in \mathscr{C}$.

Vilímovský [10] has proved 4.2 (a) for $\mathscr{C}$ = fine space and $\mathscr{R} = \Gamma$.

5. The modification of $\mathscr{C}$. Given $(\mathscr{C}, c)$ and $(\mathscr{R}, r)$, we look for the least coreflective category containing $\mathscr{C}$ whose functor commutes with r. We cannot show that this category always exists. However, under certain hypotheses we prove existence via an explicit description. Assuming that $\mathscr{R}$ is embedding, set $\bar{c}\,\mu X = cr\,\mu X/X$; $\bar{c}$ depends on $\mathscr{R}$, of course. (This is designed to accomplish 3.1 (b_1).)

5.1. Theorem. *Let $\mathscr{R}$ be embedding, and let $c\mathscr{R} \subset \mathscr{R}$. Then: $\mu X \in \langle \mathscr{R}, c\rangle$ iff $\mu X = \bar{c}\,\mu X$; each $\mu X \leftarrow A \in \langle \mathscr{R}, c\rangle$ factors uniquely through $\bar{c}\mu X \to^{1x} \mu X$.*

5.1 does not assert that $\bar{c}\mu X$ is the coreflection into $\langle \mathscr{R}, c\rangle$. Write $\mathscr{C}^*$ for $\langle \mathscr{R}, c\rangle$, and c^* for the functor. (These depend on $(\mathscr{R}, r)$.)

5.2. Corollary. *Let $\mathscr{R}$ be embedding, $c\mathscr{R} \subset \mathscr{R}$, and suppose that:*

(∗) *For each μX, X is $\mathscr{R}$-dense in $\bar{c}r\,\mu X$.*

Then

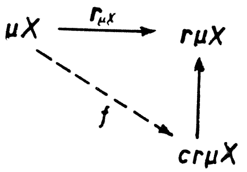
(a) *Each $\bar{c}\mu X \in \mathscr{C}^*$; hence $\bar{c} = c^*$.*
(b) *$c^*r = rc^*$.*
(c) *$cr\mu X = rc\mu X$ iff $c^*\mu X = c\mu X$ (i.e. $c\mu X = cr\mu X$).*
(d) *If $(\mathscr{C}', c')$ is coreflective, $\mathscr{C}' \supset \mathscr{C}$, and $c'r = rc'$, then $\mathscr{C}' \supset \mathscr{C}^*$.*
(e) *$\mathscr{C}^*$ consists exactly of the $\mathscr{R}$-dense subspaces of spaces in $\mathscr{C}$.*
(f) *(∗) holds if for each μX, X is $\mathscr{R}$-dense in $cr\mu X$.*

(We note (f) because of the desirability of having (∗) in terms of the data given a priori.)

5.2 immediately yields.

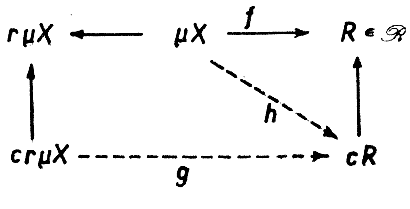
5.3. Corollary. *Suppose that $c\Gamma_\alpha \subset \Gamma_\alpha$ and that X is α-dense in $c\gamma_\alpha\,\mu X$ for each μX. Then $\mathscr{C}^*$ consists exactly of the α-dense subspaces of spaces in $\mathscr{C}$.*

Proof of 5.1. Let $\mu X \in \langle \mathscr{R}, c\rangle$, and consider:



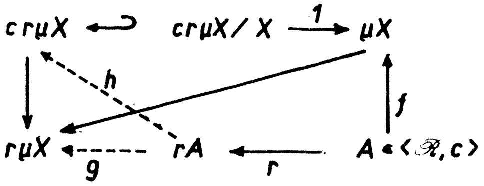
Since $r_{\mu X}$ is an embedding, so is f; thus $cr\mu X/X = \mu X$.

Conversely, let $cr\mu X/X = \mu X$. Consider



where $g = c(rf)$, and h is the restriction of g to the subspace μX. Thus, $\mu X \in \langle \mathscr{R}, c \rangle$.

Finally, consider:



where $g = rf$; and then $h = c(rf)$, which exists because $rA \in \mathscr{C}$. Since f takes values in the set X, $h \circ r_A$ takes values in the set X. Thus, $A \to^{h \circ r_A} cr\, \mu X/X$ is a map.

Proof of 5.2. (a). From $(*)$ and $c\mathscr{R} \subset \mathscr{R}$, $r\bar{c}\mu X = cr\mu X$ follows. Thus, $\bar{c}\mu X \in \in r^{-1}\mathscr{C} = \mathscr{C}^*$ (by 4.2).

(b) By (a) and 3.1 applied to c^*.

(c) By $\bar{c}\mu X = cr\mu X/X$ and $rc\mu X/X = c\mu X$.

(d) If $c'r = rc'$, then $c'\mu X = c'r\mu X/X$, by 3.1. Thus, $c'(\bar{c}\,\mu X) = c'r(\bar{c}\,\mu X)/X = = c'(cr\mu X)/X$ (from (a)), which is $cr\mu X/X = \bar{c}\mu X$ because $\mathscr{C}' \supset \mathscr{C}$. So $c'/\mathscr{C}^*$ is the identity, and $\mathscr{C}' \supset \mathscr{C}^*$.

(e) Each $\bar{c}\mu X$ is $\mathscr{R}$-dense in $cr\mu X \in \mathscr{C}$. Conversely, if μX is $\mathscr{R}$-dense in $C \in \mathscr{C}$, then $r\mu X = rC$. But $rC \in \mathscr{C}$ because $c\mathscr{R} \subset \mathscr{R}$, and hence $r\mathscr{C} \subset \mathscr{C}$ by 3.2 (d). Then $\mu X \in r^{-1}\mathscr{C} = \mathscr{C}^*$.

(f) is obvious.

5.4. Remark. From 3.3 and 5.3, when c preserves topology and $\mathscr{R} = \Gamma$ $(\alpha = 0)$, then $\gamma c^* = c^*\gamma$ where $\mathscr{C}^* =$ the dense subspaces of $\mathscr{C}$-spaces. This applies with $\mathscr{C} =$ fine spaces $\mathscr{F}$ or metric-fine spaces $\mathscr{M}$ (§ 7), and the categories $\mathscr{C}^*$ look interesting. From [2c], on the category $\mathscr{S}$ of complete separable subfine spaces, the passage $\mu X \mapsto U(\mu X)$ (the family of real uniformly continuous functions) is a categorical isomorphism onto the category of function algebras with countable composition. Now separable $\mathscr{M} \subset \mathscr{S}$ [2a], and for $\mu X \in \mathscr{M}$, $U(\mu X)$ has the stronger property of inversion. Thus $U(\mathscr{M}^* \cap \mathscr{S})$ is a category of function algebras possessing some property between countable composition and inversion; it would be interesting to see an algebraic description of this property. Similar remarks apply to $U(\mathscr{F}^*)$ (but it is not clear what $U(\mathscr{F})$ is). See [2e].

6. The modification of $\mathscr{R}$. Given $(\mathscr{C}, c)$ and $(\mathscr{R}, r)$ we look for explicit description (and proof of existence) of the least epi-reflective category containing $\mathscr{R}$ whose functor commutes with c. Assume $\mathscr{R}$ embedding, and set $\bar{r}\,\mu X =$ the image in $r\,\mu X$ of $rc\,\mu X$ under $r(r_{\mu X} \circ c_{\mu X})$. This construct may be described in terms of filters as follows:

Each Cauchy filter $\mathscr{F}$ in μX "converges" to a point p of $\gamma\mu X$. If $p \in rc\mu X$, call γ "Cauchy ($\mathscr{R}$)". Then

$$\bar{r}\mu X = \{p \in r\mu X\colon \text{There is a } c\mu\text{-Cauchy } (\mathscr{R}) \text{ filter on } \mu X \text{ with } \mathscr{F} \to p\}\,.$$

6.1. Theorem. *Let $\mathscr{R}$ be embedding, and let $r\mathscr{C} \subset \mathscr{C}$. Then: $\mu X \in \langle r, \mathscr{C}\rangle$ iff $\mu X = \bar{r}\mu X$; each $\mu X \to A \in \langle r, \mathscr{C}\rangle$ factors uniquely through $\mu X \to \bar{r}\mu X$.*

Let $\mathscr{R}^* = \langle r, \mathscr{C}\rangle$ *with r^* the functor. These depend on $(\mathscr{C}, c)$.*

6.2. Theorem. *Let $\mathscr{R}$ be embedding, $r\mathscr{C} \subset \mathscr{C}$, and suppose that:*

$(*)$ *For each μX, $c\mu X \subset c\bar{r}\mu X$.*

Then:

(a) *Each $\bar{r}\mu X \in \langle r, \mathscr{C}\rangle$; hence $\bar{r} = r^*$.*

(b) $cr^* = r^*c$.

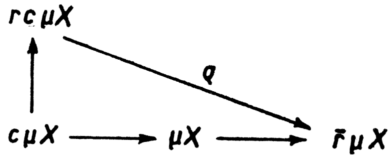
(c) *$cr\mu X = rc\mu X$ iff $r^*\mu X = r\mu X$ (i.e., $c\mu X$ is $\mathscr{R}^*$-dense in $cr\mu X$).*

(d) *If $(\mathscr{R}'r')$ is epi-reflective, $\mathscr{R}' \supset \mathscr{R}$, and $cr' = r'c$, then $\mathscr{R}' \supset \mathscr{R}^*$.*

(f) *$(*)$ holds if for each μX, $c\mu X = cr\mu X$.*

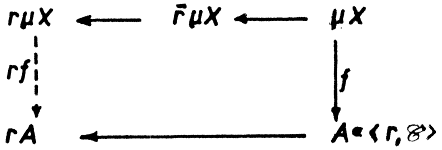
One could add this Proposition as the dual of 6.1 (e) in the sense of 3.2 (c): 6.2 (e). $\mathscr{R}^*$ consists exactly of the images of spaces in $\mathscr{R}$ under $\mathscr{C}$-maps. The proof is obvious, indeed, virtually the definition of $\mathscr{C}$-map. Again, this seems uninteresting.

Proof of 6.1. Consider the diagram:



where ϱ is $r(r_{\mu X} \circ c_{\mu X})$, *onto* the image $\bar{r}\mu X$. If $\mu X \in \langle r, \mathscr{C}\rangle$, then $c\mu X \in \mathscr{R}$ by 4.2, and $c\mu X = rc\mu X$; thus ϱ maps $c\mu X$ onto $\bar{r}\mu X$, and so $\mu X = \bar{r}\mu X$. If $\mu X = \bar{r}\mu X$, then ϱ maps $rc\mu X$ onto μX, and $c\mu X = rc\mu X$ follows; i.e. $c\mu X \in \mathscr{R}$, so $\mu X \in$ $\in c^{-1}\mathscr{R} = \langle r, \mathscr{C}\rangle$.

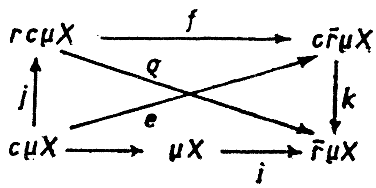
To prove the "extension property" of $\bar{r}\,\mu X$, we can use either description of $\bar{r}\,\mu X$. The filter description seems preferable: Consider



We are to show that $rf/\bar{r}\,\mu X$ takes values in A. Let $p \in \bar{r}\,\mu X$, and let $\mathscr{F} \to p$ with $\mathscr{F}$ $c\mu$-Cauchy ($\mathscr{R}$) on μX. Let $\mathscr{G} \equiv f(\mathscr{F})$, i.e., $\mathscr{G} \equiv \{G \subset A$: There is $F \in \mathscr{F}$ with $f(F) \subset G\}$. Upon replacing f by cf here, we see that $\mathscr{G}$ is Cauchy in cA (as well as in A).

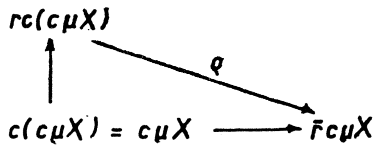
Now $f(\mathscr{G}) \to f(p)$ (a property of γf). Since $\gamma f/r\,\mu X = rf$ and rf takes values in rA, $f(\mathscr{G}) \to r\,f(p) \in rA$. This shows that $f(\mathscr{G})$ is Cauchy ($\mathscr{R}$), so that $r\,f(p) \in \bar{r}A = A$.

Proof of 6.2. (a). We shall prove that $rc\mu X = c\bar{r}\mu X$; then $\bar{r}\mu X \in c^{-1}\mathscr{R} = \langle r, \mathscr{C}\rangle$. Consider:



where ϱ is $r(r_{\mu X} \circ c_{\mu X})$, onto $\bar{r}\,\mu X$; $e = ci$, and by $(*)$ is an embedding; $f = c\varrho$, which exists because $rc\mu X \in \mathscr{C}$. Evidently, f is onto. Now, $kfj = ke$; since k is monic, $fj = e$. Thus f is an embedding (because our category is uniform spaces!). So, $rc\mu X = c\bar{r}\mu X$.

(b) Consider the diagram



(as in 6.1, but for $c\mu X$ instead of μX). Here, ϱ is an embedding; and onto. So $rc\mu X = (rc(c\mu X) =)\ \bar{r}c\mu X$.

Now using (a), $c\bar{r}\mu X = rc\mu X = \bar{r}c\mu X$.

(c) If $r\mu X = \bar{r}\mu X$, then $cr\mu X = c\bar{r}\mu X$, which by (a) is $rc\mu X$. If $cr\mu X = rc\mu X$, then $c\bar{r}\mu X = cr\mu X$, so in the diagram of (a), k maps onto $r\mu X$, and so does ϱ. That is, $r\mu X = \bar{r}\mu X$.

(d) is routine and (f) is obvious.

7. Some remarks. (a) Assume $\mathscr{R}$ embedding, and $c\mathscr{R} \subset \mathscr{R}$ $(= r\mathscr{C} \subset \mathscr{C})$. We have no counterexamples to the assertions that $\bar{c}\mu X \in \langle \mathscr{R}, c\rangle$ and $\bar{r}\mu X \in \langle r, \mathscr{C}\rangle$ without assuming 5.1 $(*)$ and 6.1 $(*)$). These questions are not important to the present paper, because one must have the conditions $(*)$ to get commuting. But the questions seem worthwhile in view of the importance of the $\langle \mathscr{A}, e\rangle$ and $\langle e, \mathscr{A}\rangle$ constructions.

(b) Here is an example of $(\mathscr{C}, c)$ with $c\Gamma \not\subset \Gamma$. Let $\mathscr{P}$ be precompact spaces, and for $P \in \mathscr{P}$ let eP have basis of all finite covers. eP is the precompact reflection of the discrete uniformity on P; $e = pd$. The functor for $\langle \mathscr{A}, e \rangle$ is $\mu X \mapsto (\mu \vee pd\mu) X$. If K is infinite and compact, then $K \in \Gamma$, but $eK \notin \Gamma$ (for γeK is the Stone-Čech compactification of discrete K).

(c) Let $\mathscr{A}$ be metric spaces, e = the fine coreflection. The members of $\langle \mathscr{A}, e \rangle = \mathscr{M}$ are called metric-fine [2a], and the coreflection is called m. This example, with results from [9c], motivated at least our treatment of the categories Γ_α: The basic facts here are: m preserves topology, and $\mathscr{M}$ is l-densely hereditary. (See [2a], [9d]. These are actually easy. For the latter: if μX is l-dense in $\nu Y \in \mathscr{M}$, and $\mu X \to^f \varrho M \in \mathscr{A}$, then there is the extension $\nu Y \to^{\bar{f}} \gamma \varrho M$. Now ϱM is l-dense in $\gamma \varrho M$ iff $\varrho M = \gamma \varrho M$. So $\bar{f}$ takes values in ϱM, $\nu Y \to^{\bar{f}} e\varrho M$ is a map, and so is $\bar{f}/\mu X$.)

By 3.3 (d), then, $m\gamma_1 = \gamma_1 m$, a result from [9c]. Consider 6.2, with $\mathscr{C} = \mathscr{M}$, $\mathscr{R} = \Gamma$. The condition in 6.2 (f) fails; but 6.2 (*) holds! The proof of this uses the description of $m\mu X$ given independently in [1c] and [9(d)] (see also [1a] and [9(*b*)]), which description further implies that $\gamma^* = \gamma_1$. See [9c]. (d) There is this obvious question: if $\mathscr{R}$ is embedding, is there coreflective $\mathscr{C}$ such that 6.2 (*) holds for $\mathscr{C}$ and Γ, and $\Gamma^* = \mathscr{R}$?

References

[1] *Z. Frolík:* (a) Interplay of measurable and uniform spaces. Proc. Second Budva Symp. 1972, Beograd, 1973, 96—99. (b) Measurable uniform spaces. Pac. J. Math., *55* (1974), 93—105. (c) A note on metric-fine spaces. *46* (1974), 111—119.

[2] *A. W. Hager:* (a) A class of nearly fine uniform spaces. Proc. London Math. Soc., *28* (1974), 517—546. (b) Measurable uniform spaces. Fund. Math. *77* (1972), 51—73. (c) Three classes of uniform spaces. Proc. Third Prague Symp., 1971, Academia, Prague, 1972. (d) Perfect maps and epi-reflective hulls. *27* (1975), 11—24. (e) Vector lattices of uniformly continuous functions and some categorical methods in uniform spaces. Proc. Second Pittsburgh Symp. 1972, Springer-Verlag Lecture Notes 378.

[3] *H. Herrlich:* Fortsetzbarkeit stetiger Abbildungen und Kompactheiltsgrad topologischer Räume. Math. Zeit. *96* (1967), 64—72.

[4] *H. Herrlich* and *G. Strecker:* (a) Coreflective subcategories. Trans. Amer. Math. Soc. *157* (1971), 205—226. (b) Coreflective subcategories II — applications in general topology, to appear.

[5] *H. Herrlich* and *J. van der Slot:* Properties which are closely related to compactness. Proc. Kon. Nederl. Akad. v. Weten. *70* (1967), 524—529.

[6] *M. Hušek:* Categorial methods in topology, Proc. Second Prague Symp. 1966, Prague—N.Y., 1967.

[7] *J. R. Isbell:* (a) Structure of categories. Bull. Amer. Math. Soc. *2* (1966), 619—655. (b) Uniform Spaces (Amer. Math. Soc.) Providence, 1964.

[8] *J. F. Kennison:* (a) Reflective functors in general topology and elsewhere. Trans Amer. Math. Soc. *118* (1965), 303—315. (b) A note on reflection maps. Ill. J. Math. *11* (1967), 404—409.

[9] *M. D. Rice:* (a) Covering and function — theoretic properties of uniform spaces. Thesis, Wesleyan University, 1973. (b) Covering and function — theoretic properties of uniform spaces. Bull. Amer. Math. Soc. *80* (1974), 159—163. (c) Subcategories of uniform spaces. Trans. Amer. Math. Soc. *201* (1975), 305—314. (d) Metric-fine uniform spaces. Proc. London Math. Soc. *11* (1975), 53—64.

[10] *J. Vilímovský:* (a) O jedné třídě koreflektorů (On a class of coreflectors). Thesis, Charles University, 1972. (b) Generation of coreflections in categories. Comm. Math. Univ. Carolinae, *14* (1973), 305—323.

Authors' addresses: A. W. Hager, Wesleyan University, Charles University and the Czechoslovak Academy of Sciences, M. D. Rice, Wesleyan University and George Mason University.