Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 26 (1976), No. 2, 273–279

Persistent URL: http://dml.cz/dmlcz/101399

Czechoslovak Mathematical Journal, 26 (101) 1976, Praha

EMBEDDING TREES INTO BLOCK GRAPHS

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(Received November 8, 1974)

Block graphs were studied in various papers and books, e.g. [1], [2], [3], [6]. A block graph is an undirected graph such that each of its blocks is a clique.

Here we shall study only block graphs consisting of exactly two blocks. If k and n are positive integers, $2 \leqq k \leqq [\frac{1}{2}(n + 1)]$, then $G_n(k)$ will denote a block graph with n vertices and two blocks, one of which has k vertices.

An undirected graph with n vertices is called completely separable, if and only if it can be embedded into $G_n(k)$ for each $k = 2, \ldots, [\frac{1}{2}(n + 1)]$. L. NEBESKÝ (oral communication) has suggested the problem of characterizing completely separable graphs. Here we shall consider completely separable trees.

We take $k \geqq 2$, because a block of a connected graph has at least two vertices, and $k \leqq [\frac{1}{2}(n + 1)]$, because otherwise the family of $G_n(k)$ would include isomorphic graphs; we should have $G_n(k) \cong G_n(n - k + 1)$ for each k, $2 \leqq k \leqq n - 1$.

First we present some remarks on branches and medians of trees.

Let a be a vertex of a tree T. We can define a binary relation E on the set of vertices of T which are distinct from a such that $(x, y) \in E$ if and only if the vertex a does not separate x from y in T (this means that the path connecting x and y in T does not contain a). The relation E is evidently an equivalence. The subtree of T induced by the union of one class of E with the one-element set $\{a\}$ is called a branch of T with the knag a.

Now if a tree T with n vertices is embedded into $G_n(k)$ so that a is mapped onto the cut-vertex of $G_n(k)$, then each branch of T with the knag a is mapped into some of the blocks of $G_n(k)$. We obtain a partition of the set of all branches of T with the knag a into two classes such that two branches belong to the same class if and only if they are mapped both into the same block of $G_n(k)$. Conversely, let T have n vertices, let us have a partition of the set of all branches of T with the knag a into two classes. For each class of this partition consider the union of all branches of this class. If the numbers of vertices of these two unions are k and $n - k + 1$, while $2 \leqq k \leqq$ $\leqq [\frac{1}{2}(n + 1)]$, then evidently T can be embedded into $G_n(k)$ so that a is mapped onto the cut-vertex of $G_n(k)$.

In particular, let $k = [\frac{1}{2}(n + 1)]$. This means $k = \frac{1}{2}(n + 1)$ for n odd and $k = \frac{1}{2}n$ for n even. Then $n - k + 1 = \frac{1}{2}(n + 1) = k$ for n odd and $n - k + 1 = \frac{1}{2}n + 1 = k + 1$ for n even.

Let a be a vertex of a tree T with n vertices. Let $\mathscr{P} = \{\mathfrak{C}_1, \mathfrak{C}_2\}$ be a partition of the set $\mathfrak{B}(a)$ of all branches of T with the knag a into two classes. Let C_1 (or C_2) be the union of all branches from $\mathfrak{C}_1$ (or $\mathfrak{C}_2$), let c_1 (or c_2) be the number of vertices of C_1 (or C_2 respectively. Let $h(\mathscr{P}) = |c_1 - c_2|$. In the case when $\mathscr{P}$ is the partition corresponding to the embedding of T into $G_n(k)$, where $k = [\frac{1}{2}(n + 1)]$, we have $h(\mathscr{P}) = 0$ for n odd and $h(\mathscr{P}) = 1$ for n even. This is evidently also the minimal value of $h(\mathscr{P})$ (if T has an even number of vertices, we cannot have $h(\mathscr{P}) = 0$) which can be obtained.

We are interested in the minimum of $h(\mathscr{P})$ on a given tree T; if this minimum is greater than zero at n odd or greater than one at n even, the tree T cannot be embedded into $G_n(k)$, where $k = [\frac{1}{2}(n + 1)]$, and is not completely separable.

For each non-terminal vertex a of T let $h_0(a)$ be the minimum of $h(\mathscr{P})$ taken over all partitions $\mathscr{P}$ of $\mathfrak{B}(a)$ into two classes. (For terminal vertices such partitions do not exist.) Further, let $h_0(T)$ be the minimum of $h_0(a)$ taken over all non-terminal vertices a of T.

In [4], the vertex median of a graph is defined. In [5] this concept is studied for trees; in the case of trees we call it only median. A median of a tree T with n vertices is the vertex of T in which the vertex deviation $m_1(a)$ attains the minimum. The vertex deviation

$$m_1(a) = \frac{1}{n} \sum_{x \in V} d(a, x),$$

where V is the vertex set of T and $d(a, x)$ denotes the distance between a and x (the length of the path connecting a and x in T). In [5] it is proved that a tree has either exactly one median, or exactly two medians which are joined by an edge.

Lemma 1. *Let T be a finite tree with n vertices, let a, b be two of its vertices which are joined by an edge. If $m_1(a) < m_1(b)$, then $h_0(a) < h_0(b)$ and vice versa.*

Proof. Let B_1 (or B'_1) be the branch from $\mathfrak{B}(a)$ (or $\mathfrak{B}(b)$) which contains b (or a respectively). Let B_2 (or B'_2) be the union of all branches from $\mathfrak{B}(a) - \{B_1\}$ (or $\mathfrak{B}(b) - \{B'_1\}$ respectively). The symbol $V(X)$, where X is a subtree of T, will denote the vertex set of X. Let $h_0(a) < h_0(b)$. Let $\mathscr{P}$ be a partition of $\mathfrak{B}(a)$ into two classes for which $h(\mathscr{P}) = h_0(a)$. The classes of $\mathscr{P}$ are denoted by $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_1$ is the class containing B_1. Let c_1 (or c_2) be the number of vertices of the union of all branches from $\mathfrak{C}_1$ (or $\mathfrak{C}_2$ respectively). If $c_1 < c_2$, then B_2 has more vertices than B_1, because $B_1 \in \mathfrak{C}_1$. If $c_1 \geqq c_2$, then either B_2 has again more vertices than B_1, or the number of vertices of B_1 is greater than or equal to the number of vertices of B_2 and $\mathfrak{C}_1 = \{B_1\}$. (If $\mathfrak{C}_1$ contained still another branch than B_1, the difference $c_1 - c_2 = h(\mathscr{P})$ would be greater than in this case.) We have

$$(1)\qquad m_1(a) = \frac{1}{n} \sum_{x \in V(T)} d(a, x) = \frac{1}{n} \sum_{x \in V(B_2)} d(a, x) + \frac{1}{n} \sum_{x \in V(B_2')} d(a, x),$$

$$m_1(b) = \frac{1}{n} \sum_{x \in V(T)} d(b, x) = \frac{1}{n} \sum_{x \in V(B_2)} d(b, x) + \frac{1}{n} \sum_{x \in V(B_2')} d(b, x)$$

because evidently each vertex of T belongs to exactly one of the subtrees B_2, B_2'. For $x \in V(B_2)$ we have

$$d(b, x) = d(a, x) + d(a, b) = d(a, x) + 1,$$

for $x \in V(B_2')$ we have

$$d(b, x) = d(a, x) - d(a, b) = d(a, x) - 1.$$

Thus

$$\sum_{x \in V(B_2)} d(b, x) = |V(B_2)| + \sum_{x \in V(B_2)} d(a, x),$$

$$\sum_{x \in V(B_2')} d(b, x) = \sum_{x \in V(B_2')} d(a, x) - |V(B_2')|.$$

From these equalities and from (1) we obtain

$$(2)\qquad m_1(b) = m_1(a) + \frac{1}{n}(|V(B_2)| - |V(B_2')|).$$

If $|V(B_2)| \geqq |V(B_1)|$, then $|V(B_2)| > |V(B_2')|$, because B_2' is a proper subtree of B_1; thus we have $m_1(b) > m_1(a)$. If $|V(B_1)| > |V(B_2)|$, then $|V(B_1)| = c_1$, and $c_1 \geqq c_2$. Let $\mathscr{P}'$ be a partition of $\mathfrak{B}(b)$ into two classes such that these classes are $\mathfrak{C}_1'$, $\mathfrak{C}_2'$ and $\mathfrak{C}_1' = \{B_2'\}$, $\mathfrak{C}_2' = \mathfrak{B}(b) - \{B_2'\}$. The vertex set of B_2' consists of all vertices of B_1 except for a, therefore the number c_1' of vertices of the union of all branches from $\mathfrak{C}_1'$ satisfies $c_1' = c_1 - 1$. Then the number c_2' of vertices of the union of all branches from $\mathfrak{C}_2'$ fulfills $c_2' = c_2 + 1$, because $c_1 + c_2 = c_1' + c_2' = n + 1$. As $c_1 \geqq c_2$, we have $h_0(a) = c_1 - c_2$. Now $c_1' - c_2' = c_1 - c_2 - 2$. If this number is non-negative, then $h_0(b) \leqq c_1 - c_2 - 2 < h_0(a)$, which is a contradiction. If $c_1 - c_2 - 2 < 0$, then it equals either to -1, or to -2, because $c_1 - c_2 \geqq 0$. If it is equal to -1, we have $1 = |c_1 - c_2| = h_0(a) < h_0(b) \leqq |c_1' - c_2'| = 1$, which is a contradiction. If $c_1' - c_2' = -2$, then $c_1 - c_2 = 0$ and $c_1 = c_2$; this means $|V(B_2)| = c_2 = c_1 > |V(B_2')| = c_1 - 1$ and thus also $m_1(b) > m_1(a)$.

Lemma 2. *Let T be a finite tree, let a be its median. Let b be a vertex of T distinct from a, non-adjacent to a and such that no median distinct from a lies on the path connecting a and b in T. Let $c \neq b$ be a vertex of the path connecting a and b in T. Then $h_0(b) > h_0(c)$.*

Proof. In [7] the following assertion is proved: Let u, v, w be three vertices of a tree T, let v be adjacent to u and w. Then $m_1(v) < \max(m_1(u), m_1(w))$. (This assertion was proved in [7] in a more general form.) Let the vertices of the path P connecting a and b in T be $a = u_1, u_2, \ldots, u_r = b$ and let the edges of this path be u_iu_{i+1} for $i = 1, \ldots, r - 1$. We shall prove that $m_1(u_i) < m_1(u_{i+1})$ for $i = 1, \ldots, r - 1$; the proof will be done by induction. For $i = 1$ this assertion holds. We have $u_1 = a$, which is a median of T; no other median lies on P, thus u_2 is not a median and $m_1(u_1) < m_1(u_2)$. Now let $i \geqq 2$ and let $m_1(u_{i-1}) < m_1(u_i)$. We have $m_1(u_i) < \max(m_1(u_{i-1}), m_1(u_{i+1}))$; as $m_1(u_{i-1}) < m_1(u_i)$, we have $\max(m_1(u_{i-1}), m_1(u_{i+1})) = m_1(u_{i+1})$ and $m_1(u_i) < m_1(u_{i+1})$. Thus we have proved the inequality for $i = 1, \ldots, r - 1$. According to Lemma 1 also $h_0(u_i) < h_0(u_{i+1})$. This implies that $h_0(u_i) < h_0(u_j)$ for $1 \leqq i < j \leqq r$. In particular, $h_0(u_i) < h_0(u_r) = h_0(b)$ for each $i = 1, \ldots, r - 1$. Among the vertices $u_1, \ldots, u_{r-1}$ the vertex c occurs, thus $h_0(c) < h_0(b)$.

Theorem 1. *On a finite tree T, the value $h_0(a)$ attains its minimum at a vertex a_0, if and only if a_0 is a median of T.*

Proof. From Lemma 1 and Lemma 2 we obtain that the minimum of $h_0(a)$ can be attained only at a median. Now it remains to deal with the case when T has two medians a and a'; we shall prove that in this case $h_0(a) = h_0(a')$. We use (2); instead of b we write a'. We obtain

$$m_1(a') = m_1(a) + \frac{1}{n}\left(|V(B_2)| - |V(B_2')|\right),$$

where B_2 and B_2' have the same meaning as in the proof of Lemma 1. As a and a' are both medians, we have $m_1(a) = m_1(a')$. This means $|V(B_2)| = |V(B)_2'|$. Each vertex of T belongs either to B_2 or to B_2', therefore $|V(B_2)| = |V(B_2')| = \frac{1}{2}n$. Thus T has an even number of vertices. There exists a partition $\mathscr{P}$ of $\mathfrak{B}(a)$ for which $h(\mathscr{P}) = 1$; one of its classes is $\{B_1\}$. (We use again the notation from the proof of Lemma 1.) There exists also a partition $\mathscr{P}'$ of $\mathfrak{B}(a')$ for which $h(\mathscr{P}') = 1$; one of its classes is $\{B_1'\}$. As T has an even number of vertices we have $h_0(T) \geqq 1$, thus $h_0(T) = 1$ and the minimum is attained at both a and a'.

Proving this theorem we have obtained other two assertions.

Theorem 2. *Let T be a finite tree with two medians. Then T has an even number of vertices.*

Theorem 3. *Let T be a tree with n vertices and with two medians. Then T can be embedded into $G_n(\frac{1}{2}n)$.*

A tree will be called simple, if by deleting all its terminal vertices and terminal edges a simple path is obtained. (We admit also simple paths of the length zero, i.e.

consisting only of one vertex.) If T is a simple tree, then $P(T)$ denotes the path obtained from T by deleting all terminal vertices and terminal edges.

Consider a simple tree T. Let the vertices of $P(T)$ be $u_0, u_1, \ldots, u_m$ and let the edges of $P(T)$ be $u_i u_{i+1}$ for $i = 0, 1, \ldots, m - 1$. The tree T can be determined by a finite sequence $[\alpha_0, \alpha_1, \ldots, \alpha_m]$, where α_i is the number of terminal edges of T incident with u_i for $i = 0, 1, \ldots, m - 1$.

Theorem 4. *Let T be a tree with n vertices, let T contain a simple subtree T' with $[\frac{1}{2}(n + 1)]$ vertices such that one of the terminal vertices of $P(T')$ is a median of T. Then T is completely separable.*

Proof. Let the vertices of $P(T')$ be $u_0, u_1, \ldots, u_m$ and let the edges of $P(T')$ be $u_i u_{i+1}$ for $i = 0, 1, \ldots, m - 1$. Let $[\alpha_0, \alpha_1, \ldots, \alpha_m]$ be the above defined sequence. Let u_m be a median of T. Let k be an integer, $2 \leqq k \leqq [\frac{1}{2}(n + 1)]$. Evidently

$$\sum_{i=0}^{m} (\alpha_i + 1) = [\tfrac{1}{2}(n + 1)] ,$$

because $\alpha_i + 1$ is the number of elements of the set consisting of the vertex u_i and all terminal vertices of T' incident with u_i for $i = 0, 1, \ldots, m$. Thus let j be the maximal number such that

$$\sum_{i=0}^{j} (\alpha_i + 1) \leqq k ,$$

let

$$r = k - \sum_{i=0}^{j} (\alpha_i + 1) .$$

Evidently $r < \alpha_{j+1}$. We embed T into $G_n(k)$ so that u_j is mapped onto the cut-vertex of $G_n(k)$. Choose r terminal vertices of T' which are adjacent to u_j; denote them by $t_1, \ldots, t_r$. Let B_i be the branch from $\mathfrak{B}(u_j)$ consisting of the vertices u_j and t_i and of the edge $u_j t_i$ for $i = 1, \ldots, r$. If $j \geqq 1$, then let B_0 be the branch from $\mathfrak{B}(u_j)$ containing u_0. The partition $\mathscr{P} = \{\mathfrak{C}_1, \mathfrak{C}_2\}$ of $\mathfrak{B}(u_j)$ corresponding to the embedding of T into $G_n(k)$ is such that $\mathfrak{C}_1 = \{B_0, B_1, \ldots, B_r\}$ in the case $j \geqq 1$ and $\mathfrak{C}_1 = \{B_1, \ldots, B_r\}$ in the case $j = 0$.

Theorem 5. *Let T be a completely separable tree with n vertices and two medians a and a'. Let T_0 be a tree obtained from T by deleting the edge aa' and identifying the vertices a and a'. Then T_0 is completely separable.*

Proof. Let k be an integer, $2 \leqq k \leqq [\frac{1}{2}(n + 1)]$. As T is completely separable, it can be embedded into $G_n(k)$. The graph $G_n(k)$ has two blocks, one of them has k vertices, another $n - k + 1$ vertices. If both medians of T are mapped into the blocks with $n - k + 1$ vertices, then evidently T_0 can be embedded into $G_{n-1}(k)$. (The graph $G_{n-1}(k)$ is then obtained from $G_n(k)$ by identifying the images of vertices a

and a'.) Thus it remains to prove that a and a' are mapped into the block with $n - k + 1$ vertices. At least one of the vertices a and a' is mapped onto a vertex which is not a cut-vertex of $G_n(k)$. Without loss of generality let a be such a vertex. Let the block of $G_n(k)$ into which a is mapped be denoted by B_0. Then a' is mapped also onto a vertex of B_0, because a' is joined by an edge with a. All branches of $\mathfrak{B}(a)$ except for the branch containing a' are mapped into the block B_0. But, as we have shown in the proof in the proo of Theorem 1, the union of these branches has $\frac{1}{2}n$ vertices and this is $[\frac{1}{2}(n + 1)]$, because n is even according to Theorem 2. Thus the block of $G_n(k)$ into which a is mapped has at least $\frac{1}{2}n + 1$ vertices (the vertices of these branches and the vertex a'). As $k \leqq \frac{1}{2}n$, the block B_0 contains $n - k + 1$ vertices.

Theorem 6. *Let T be a tree with $n \geqq 4$ vertices. Then T can be embedded into $G_n(2)$ and $G_n(3)$.*

Proof. Let t be a terminal vertex of T, let u be the vertex of T adjacent to t. Let B be the branch from $\mathfrak{B}(u)$ consisting of the vertices t and u and the edge joining them. Then $\mathscr{P}_2 = \{\{B\}, \mathfrak{B}(u) - \{B\}\}$ is the partition of $\mathfrak{B}(u)$ corresponding to the embedding of T into $G_n(2)$. Now if u is adjacent to a terminal vertex t' of T distinct

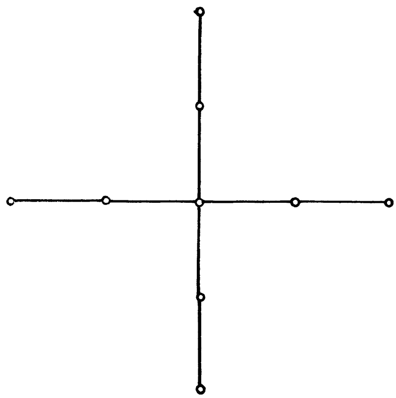

Fig. 1.

from t, let B' be the branch of T consisting of the vertices u and t' and the edge joining them. Then $\mathscr{P}_3 = \{\{B, B'\}, \mathfrak{B}(u) - \{B, B'\}\}$ is a partition of $\mathfrak{B}(u)$ corresponding to the embedding of T into $G_n(3)$. If T does not contain any vertex adjacent at least to two terminal vertices, then consider the tree T' obtained from T by deleting all terminal vertices and terminal edges. As T' is again a finite tree, it has a terminal vertex u_1. The vertex u_1 is adjacent in T to only one terminal vertex t (because this is supposed above) and with only one non-terminal vertex u_2 (because u_1 is a terminal vertex of T' whose vertex set is the set of all non-terminal vertices of T).

Thus we have a branch B_1 consisting of the vertices t, u_1, u_2 and the edges joining these vertices. The partition $\mathscr{P}'_3 = \{\{B_1\}, \mathfrak{B}(u_2) - \{B_1\}\}$ corresponds to an embedding of T into $G_n(3)$.

Fig. 1 shows an example of a tree with nine vertices which is not embeddable into $G_9(4)$.

In the end of the paper we prove a theorem on $h_0(T)$.

Theorem 7. *Let m be a positive integer. Then there exists a finite tree T for which $h_0(T) = m$.*

Proof. Let T be a tree which contains a vertex a and three branches with the knag a while each of these branches is a simple path of the length m. The vertex a is evidently the unique median of T and $h_0(a) = h_0(T) = m$.

Thus we see that $h_0(T)$ can be arbitrarily large.

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